A VU-Point of View of Nonsmooth Optimization

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Program

1. Yesterday morning:
   Introduction to nonsmooth convex optimization
2. Yesterday afternoon:
   Models and the proximal point algorithm
3. Today morning:
   Bundle methods and the Moreau-Yosida regularization
4. Today afternoon:
   Beyond first order: VU-decomposition methods

Introduction to nonsmooth convex optimization

Let's start with a question

What is the

fast*est* method

you know

to solve $F(x) = 0$,

a nonlinear system of equations?

Answer provided by Isaac Newton

source: Alan Rust

Answer: Newton's method

from J-Ch. Gilbert's classes

$$\begin{aligned} 0 &= F(x^*) \\ &\approx F(x^k) + F'(x^k)d \end{aligned}$$

fast **convergence** $x^{k+1} = x^k + d^*$

Answer: Newton's method

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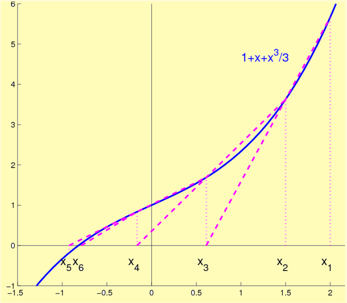

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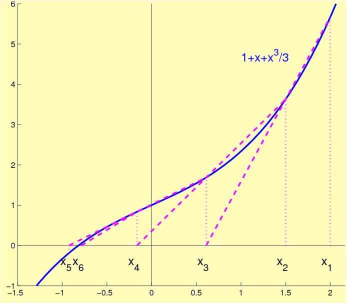
fast **convergence** $x^{k+1} = x^k + d^*$

one linear system per iteration



Newton method is accurate

$$F(x) = 1 + x + x^3/3$$

1	2.00000000000000	0
2	0.86666666666667	1
3	-0.32323745064862	1
4	-0.92578663808031	1
5	-0.82332584261905	2
6	-0.81774699537697	5
7	-0.81773167400186	9
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Newton

From J-Ch. Gilbert's classes

Le résultat de base :

Si · $F(x_*) = 0$,

· F est $C^{1,1}$ dans un voisinage de x_*,

· $F'(x_*)$ est inversible,

alors il existe un voisinage V de x_* tel que si $x_1 \in V$, l'algorithme de Newton est bien défini et génère une suite $\{x_k\} \subset V$ qui converge *quadratiquement* vers x_* : il existe une constante C telle que

$$\|x_{k+1} - x_*\| \leq C\|x_k - x_*\|^2, \quad \forall k \geq 1.$$

- Si F est seulement C^1, la convergence n'est que superlinéaire :

$$\frac{\|x_{k+1} - x_*\|}{\|x_k - x_*\|} \to 0, \quad \text{pour } k \to \infty.$$

And Newton's method for optimization?

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fast **convergence**

In optimization

$$F(x) = \nabla f(x)$$

for an objective f

And Newton's method for optimization?

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\begin{aligned}
0 &= \nabla f(x^*) \\
&\approx \nabla f(x^k) + \nabla^2 f(x^k) d
\end{aligned}
$$

fast **convergence**

In optimization

$$F(x) = \nabla f(x)$$

$$x^{k+1} = x^k - [\nabla^2 f(x^k)]^{-1} \nabla f(x^k)$$

for an objective f

And Newton's method for optimization?

$$\begin{aligned} 0 &= \nabla f(x^*) \\ &\approx \nabla f(x^k) + \nabla^2 f(x^k) d \end{aligned}$$

fast **convergence**

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In optimization

$$F(x) = \nabla f(x)$$

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$$\min f \approx \min f\textbf{-model}$$

$$\min_d f(x^k) + \langle \nabla f(x^k), d \rangle + \frac{1}{2} \langle \nabla^2 f(x^k) d, d \rangle$$

From J-Ch. Gilbert's classes

- Convergence :

Si $\cdot$ $\nabla f(x_*) = 0$,

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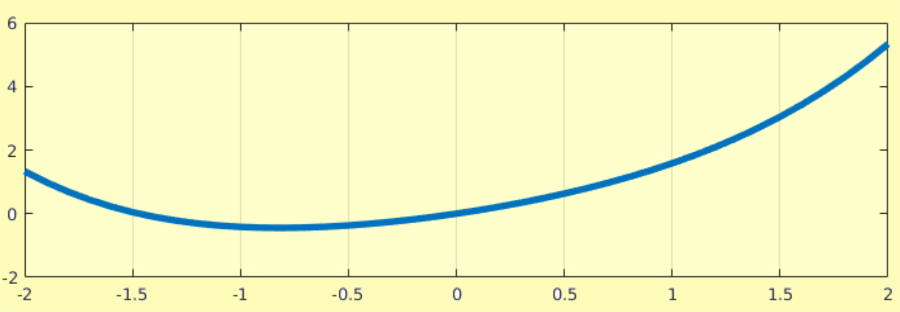
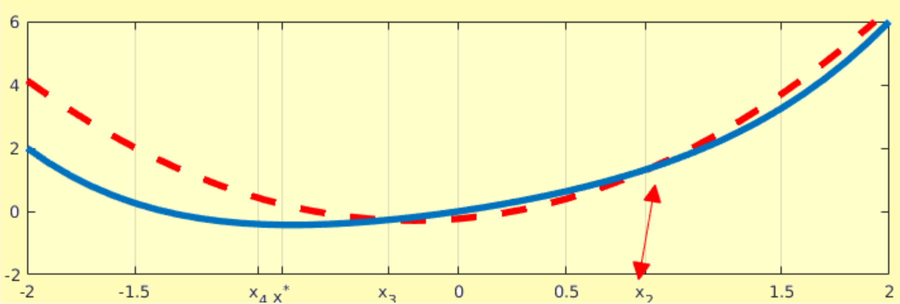
- $f \in C^2 \Longrightarrow$ superlinear convergence

Newton iterates for optimization



$$F(x) = 1 + x + x^3/3 \quad \Longrightarrow f(x) = x + x^2/2 + x^4/12$$

Newton iterates for optimization



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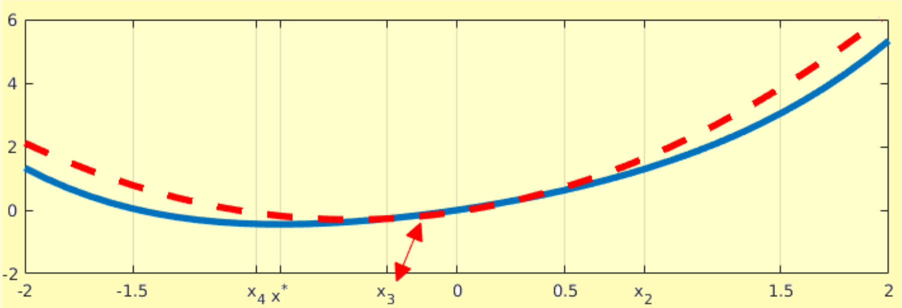

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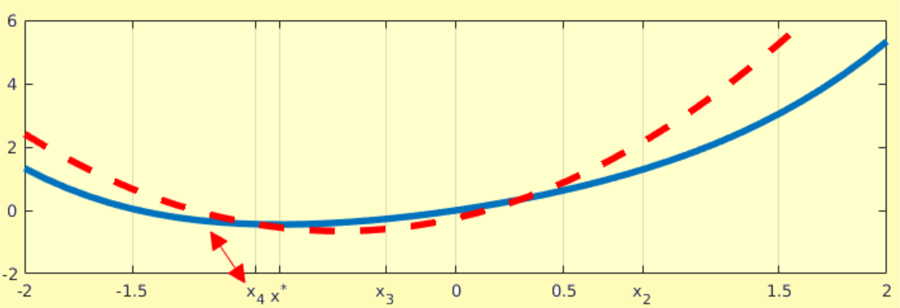

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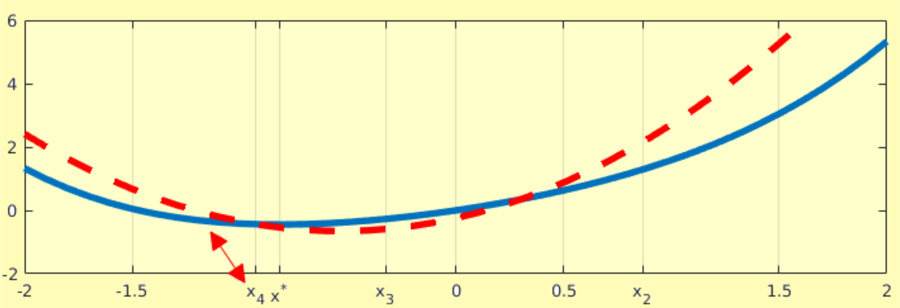
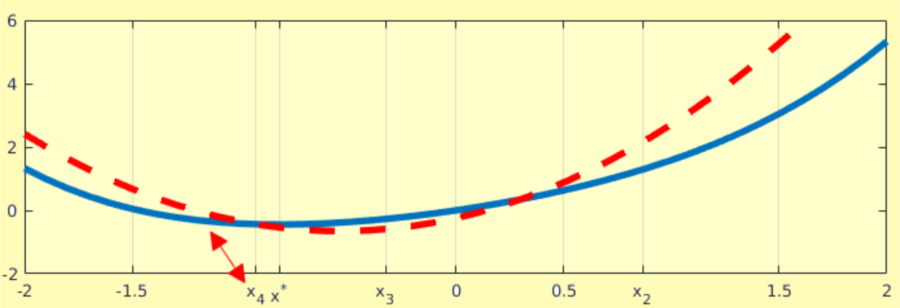


Can we avoid computing the Hessian matrix?

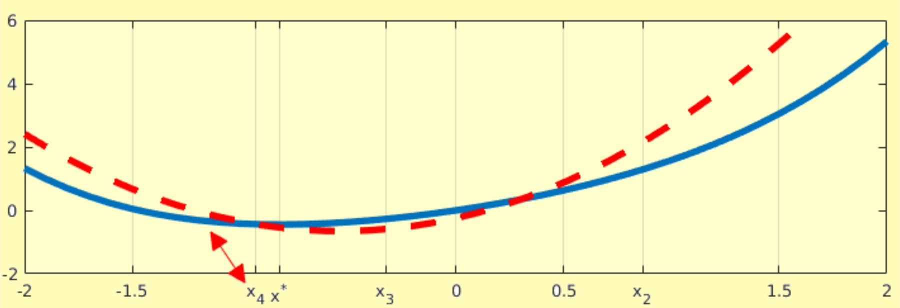
Newton iterates for optimization



Can we avoid computing the Hessian matrix? **YES!**

$$\min_{d} f(x^k) + \left\langle \nabla f(x^k), d \right\rangle + \frac{1}{2} \left\langle \boxed{M^k} d, d \right\rangle$$

Newton iterates for optimization



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quasi-Newton matrix

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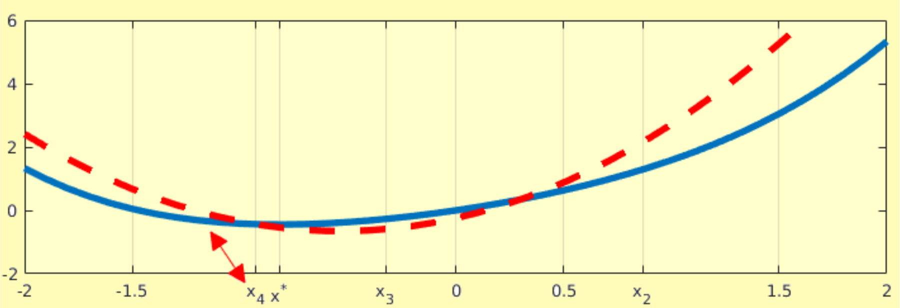

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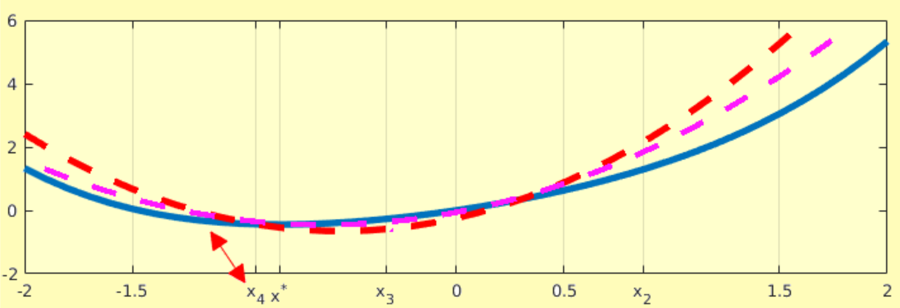
quasi-Newton matrix

$$0 = \nabla f(x^k) + M^k d$$

quasi-Newton iterates for optimization



Eventually, the true Hessian curvature is estimated only along the generated directions (secant equation)

quasi-Newton methods are **accurate** too!

1	2.00000000000000	0
2	1.50000000000000	0
3	0.61224489795918	1
4	-0.16202797536640	1
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Newton

Dennis & Moré Criterion

from J-Ch. Gilbert's classes

IF

- $\{x^{k+1} = x^k + M^k d^k\}$ **converges to** x^*
- $\nabla^2 f(x^*)$ **is non singular**
- $\nabla f(x^*) = 0$

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THEN

the convergence is superlinear if and only if

$$\Big(M^k - \nabla^2 f(x^*)\Big)(x^{k+1} - x^k) = o(\|x^{k+1} - x^k\|)$$

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What about 1st-order methods?

The same game, this time using a **gradient** method $x^{k+1} = x^k - t_k \nabla f(x^k) = x^k - t_k F(x^k)$ for $t_k > 0$ a sufficiently small stepsize (note easy calculation)

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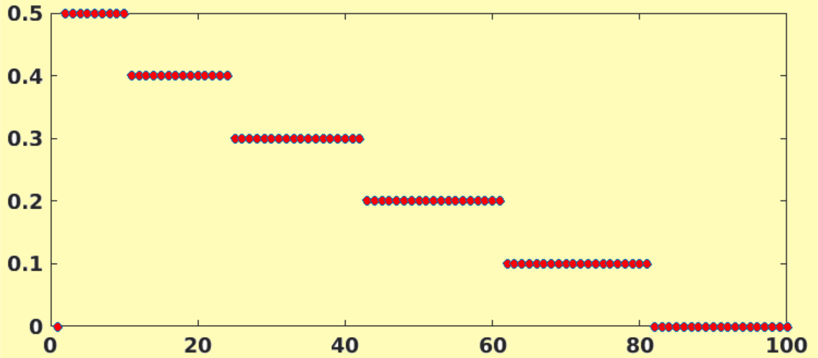
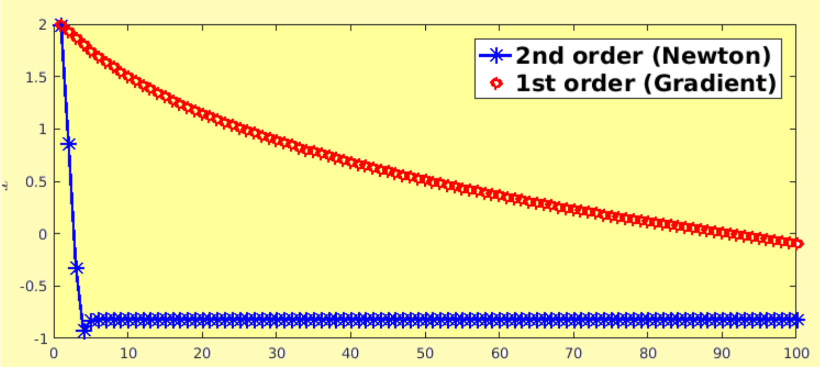


$x^{100} = -0.088237488857720$

($x^* = -0.81773167386824$)

Comparison of x-values



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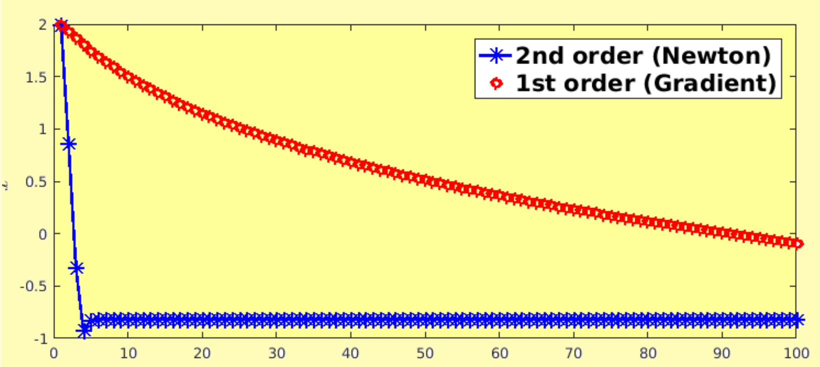

Regarding F-values,

- **with gradient method, after 100 iterations,** $F(x^100) = 0.9115$
- **with Newton's method, after 9 iterations,** $F(x^9) = -8.0491 \times 10^{-16}$

Take away message from the smooth world

- Newton-like methods are *fast*

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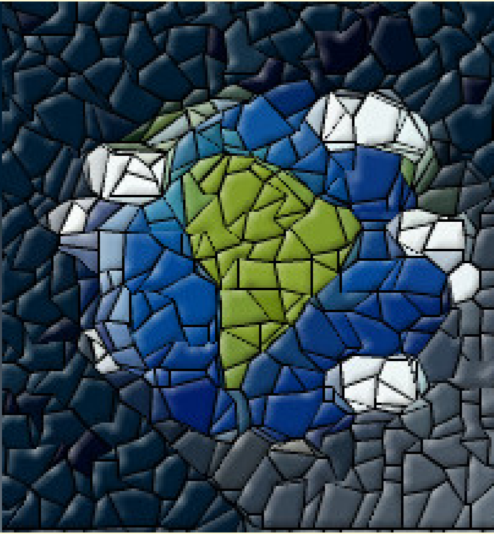
for functions that are C^2

and have an invertible Hessian at x^*

Moving to the nonsmooth world



Computational NSO: what do we mean?

For the unconstrained problem

$$\boxed{\min f(x)},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex but not differentiable at some points Algorithms defined according on how much information is provided by certain *oracle*

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an **informative oracle**

x

$f(x)$

the full $\partial f(x)$

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ONE $g(x) \in \partial f(x)$

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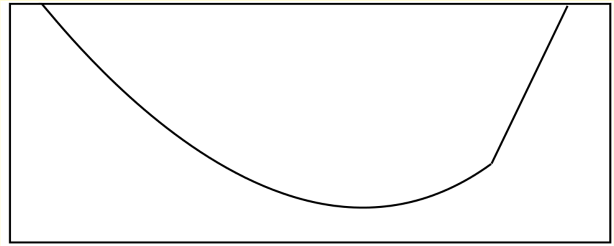
A quick overview of Convex Analysis

An example of a convex nonsmooth function



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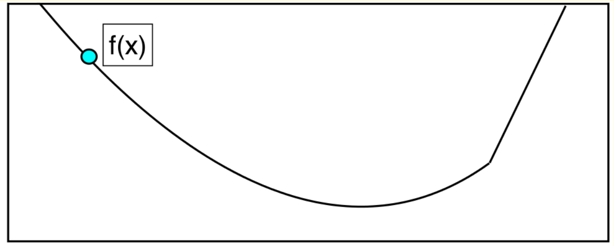

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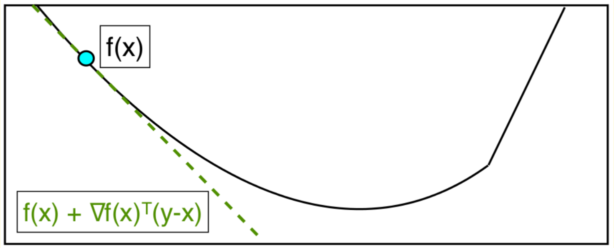

$$\begin{aligned}\partial f(x) &= \{\nabla f(x)\} \\ &= \{\text{slopes of linearizations supporting } f\text{, tangent at } x\}\end{aligned}$$

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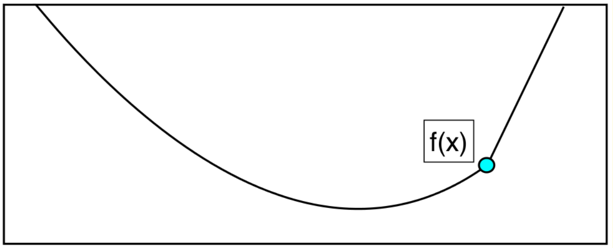

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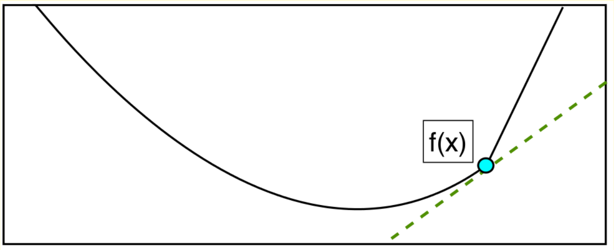

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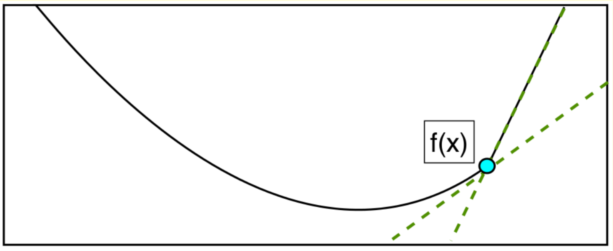

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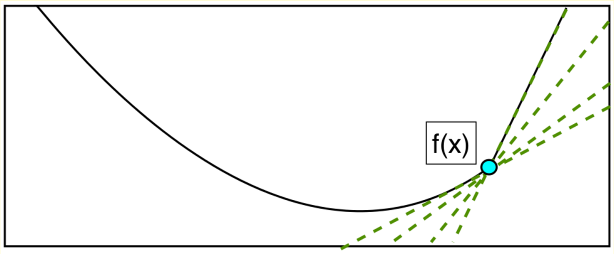

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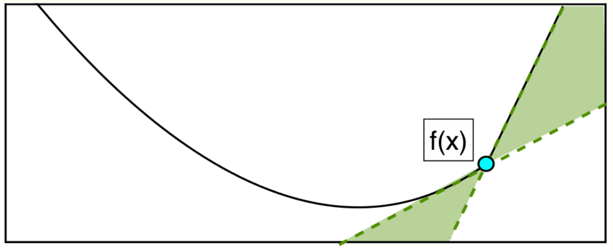

$$\partial f(x) = \{g \in \mathbb{R}^n : f(y) \geq f(x) + g^\top (y - x) \text{ for all y}\}$$

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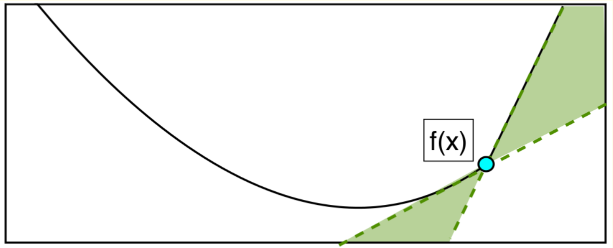

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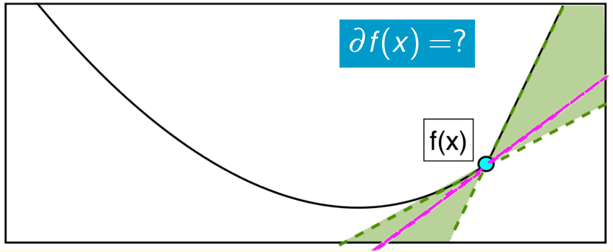
A very useful calculus rule for subdifferentials

A finite max-function $f(x) := \max_{j \in I} f^j(x)$, with $f^j : \mathbb{R}^n \to \mathbb{R}^n$ convex and differentiable



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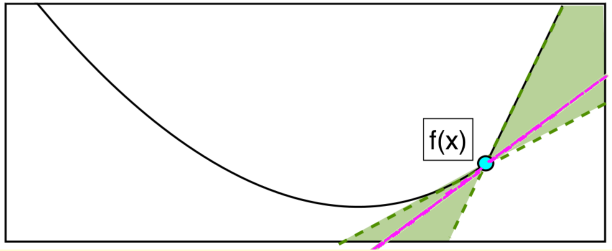

$$\begin{aligned} \partial f(x) &= \{g \in \mathbb{R}^n : f(y) \geq f(x) + g^\top (y - x) \text{ for all y}\} \\ &= \boxed{conv\{\nabla f^j(x) : j \in I(x)\}} \end{aligned}$$

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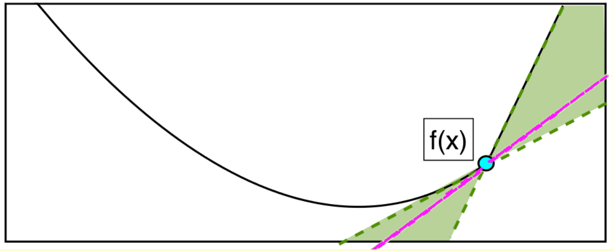

only active indices!

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$$I(x) := \left\{ j \in I : f(x) = f^j(x) \right\}$$

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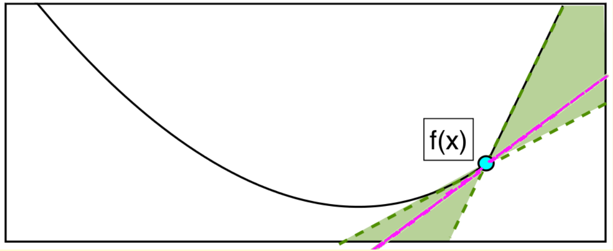

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What is the subdifferential of $f(x) = |x|$?

Extension to compact set I and nonsmooth f^j

A sup-function $f(x) := \sup_{j\in I} f^j(x)$, with $f^j : \mathbb{R}^n \to \mathbb{R}^n$ convex,

- I compact
- $j \mapsto f^j(x)$ is upper-semicontinuous

$$\partial f(x) = cl\Big(conv \cup_{j\in I(x)} \partial f^j(x)\Big)$$

where the active set is defined as before:

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Given n symmetric matrices of order m, consider

$$A(x) := x_1 A_1 + x_2 A_2 + \ldots x_n A_n$$

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A sup-function $f(x) := \sup_{j \in I} f^j(x)$, with $f^j : \mathbb{R}^n \to \mathbb{R}^n$ convex,

- I compact
- $j \mapsto f^j(x)$ is upper-semicontinuous

$$\partial f(x) = cl\Big(conv \cup_{j \in I(x)} \partial f^j(x)\Big)$$

where the active set is defined as before:

$$I(x) := \left\{ j \in I : f(x) = f^j(x) \right\}$$

Given n symmetric matrices of order m, consider

$A(x) := x_1 A_1 + x_2 A_2 + \ldots x_n A_n$

What is the subdifferential of $f(x) = \lambda_{\max}(A(x))$?

(the maximum-eigenvalue)

Computational NSO: what do we mean?

For the unconstrained problem

$$\boxed{\min f(x)},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex but not differentiable at some points. Algorithms defined according on **how much** information is provided by certain *oracle* a **“black-box”**

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$f(x)$

ONE $g(x) \in \partial f(x)$

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x

$f(x)$

ONE $g(x) \in \partial f(x)$

How common are nonsmooth objective functions?

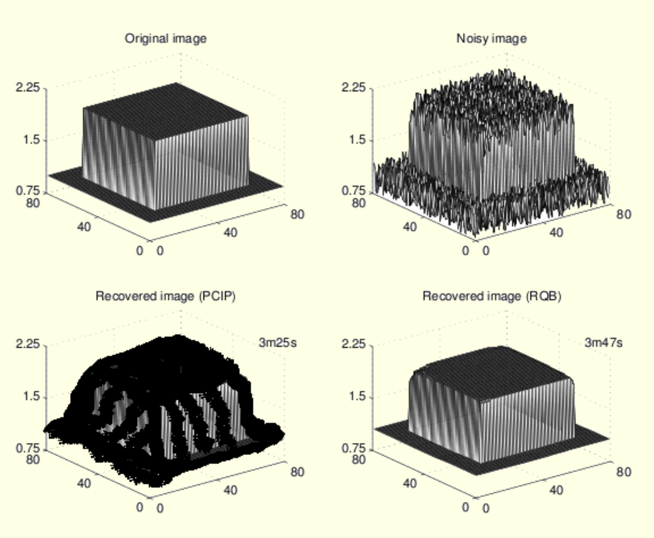
When does nonsmoothness appear?

- * if the **nature** of the problem imposes a nonsmooth model; or
- * if **sparsity** of the solution is a concern; or
- * in problems difficult to solve,
  - because they are large scale
  - because they are heterogeneous

sometimes the **solution method** induces nonsmoothness

Example of NS model

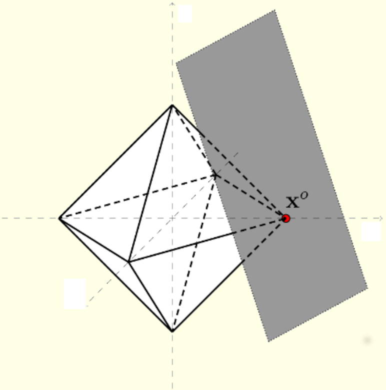
Recovery of **blocky** images (ℓ_1-regularization of TV)



Example of sparse optimization

$$\min\{\|x\|_1 : Ax = b\}$$

Basis pursuit: find least 1-norm point on the affine plane
Tends to return a sparse point (sometimes, the sparsest)



ℓ_1 ball touches the affine plane

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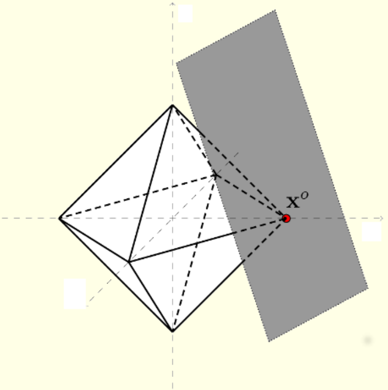

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LASSO denoises basis pursuit

$$\min\left\{\|Ax - b\|_2^2 : \|x\|_1 \le \tau\right\}$$

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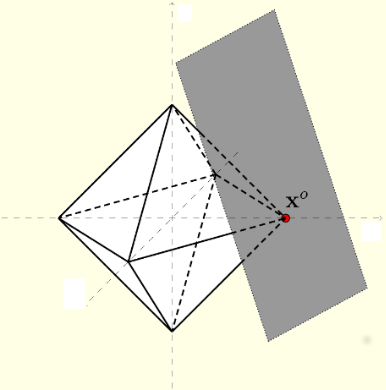

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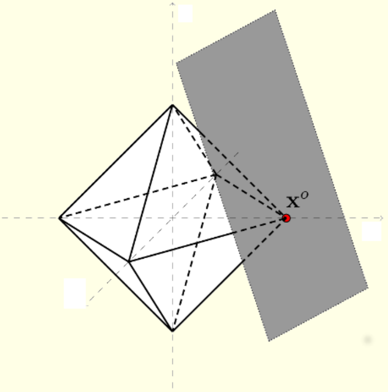

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or

$$\min\left\{\|x\|_1 : \|Ax - b\|_2^2 \le \sigma\right\}$$

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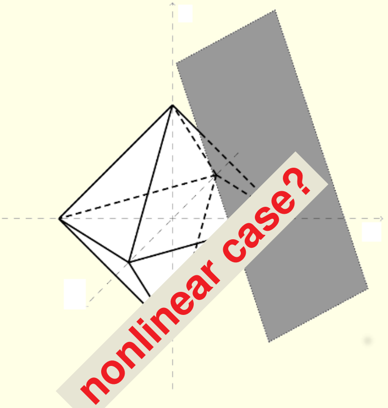

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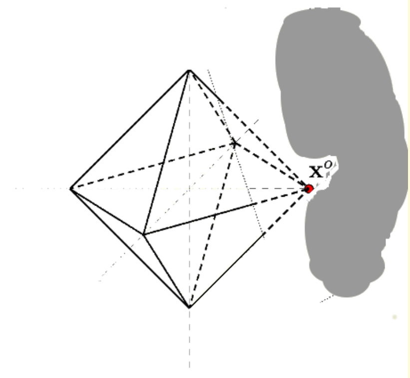
Basis pursuit: find least 1-norm point on a **nonlinear set**

Tends to return a sparse point (sometimes, the sparsest)



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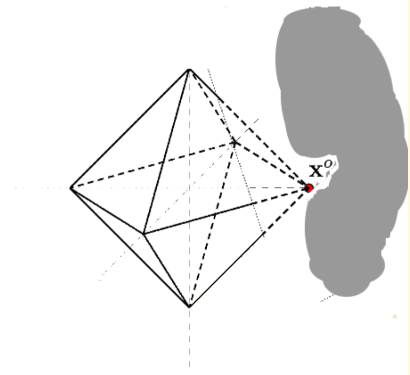

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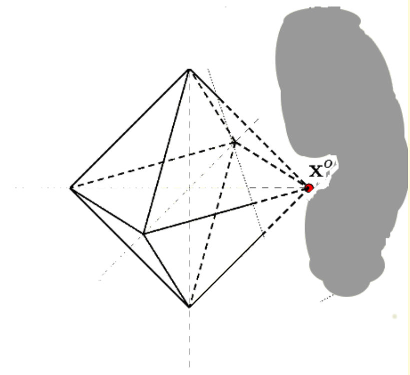

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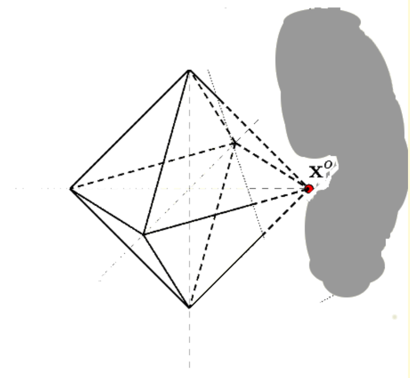

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Lagrangian Relaxation Example

Real-life optimization problems

$$(\texttt{primal}) \quad \left\{ \begin{array}{ll} \min & \displaystyle\sum_{j\in J} \mathcal{C}^j(p^j) \\ & \text{for } j \in J\text{: } p^j \in \mathcal{P}^j \\ & \displaystyle\sum_{j\in J} g^j(p^j) = Dem \end{array} \right.$$

Lagrangian Relaxation Example

Real-life optimization problems

$$
(\texttt{primal}) \quad \left\{
\begin{array}{ll}
\max & \sum_{j \in J} -\mathcal{C}^j(p^j) \\
& \text{for } j \in J\text{: } p^j \in \mathcal{P}^j \\
& \boxed{\sum_{j \in J} g^j(p^j) = Dem} \quad \boxed{\leftarrow x}
\end{array}
\right.
$$

Lagrangian Relaxation Example

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have a (dual) with separable structure:

Lagrangian Relaxation Example

Real-life optimization problems

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\texttt{(primal)} \quad \begin{cases} \max & \sum_{j \in J} -\mathcal{C}^j(p^j) \\ & \text{for } j \in J\text{: } p^j \in \mathcal{P}^j \\ & \boxed{\sum_{j \in J} g^j(p^j) = Dem} \quad \leftarrow x \end{cases}
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have a `(dual)` with separable structure:

$$
\min_x \quad f(x) := \quad f_0(x) \quad + \sum_{j \in J} \quad f^j(x)
$$

$$
\min_x \quad -\langle x, Dem \rangle \quad + \sum_{j \in J} \begin{cases} \max & -\mathcal{C}^j(p^j) + \langle x, g^j(p^j) \rangle \\ & p^j \in \mathcal{P}^j \end{cases}
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Similar for Benders Decomposition

Computing subgradients: how difficult is it?

1. $f(x) = |x|$, for $n = 1$
2. A linear Lasso function,
   $f(x) = \|x\|_1 + \frac{\mu}{2}\|Ax - b\|_2^2$
3. A nonlinear Lasso function, $h \in C^1$,
   $f(x) = \|x\|_1 + \frac{\mu}{2}\|\Big(h(x) - b\Big)^+\|_2^2$
4. One of the Lagrangian subproblems,
   $$f^j(x^k) := \begin{cases} \max & -\mathcal{C}^j(p^j) + \langle x^k, g^j(p^j) \rangle \\ & p^j \in \mathcal{P}^j \end{cases}$$
5. The max-eigenvalue case
   $f(x^k) = \max\{y^\top A(x)y : \|y\|_2 \leq 1\}$

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What about computing the full $\partial f(x^k)$?

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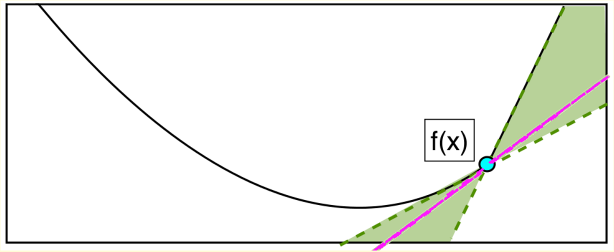
What about computing the full $\partial f(x^k)$?

NSO methods in general are designed

for oracles delivering 1-subgradient only

What can be done with the oracle output?

An example of a convex nonsmooth function



$$\begin{aligned}\partial f(x) &= \{g \in \mathbb{R}^n : f(y) \geq f(x) + g^\top(y - x) \text{ for all y}\} \\ &= \{\text{slopes of linearizations supporting } f\text{, tangent at } x\}\end{aligned}$$

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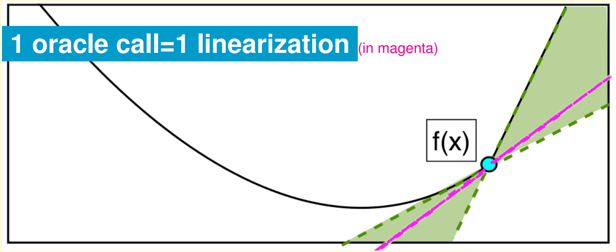

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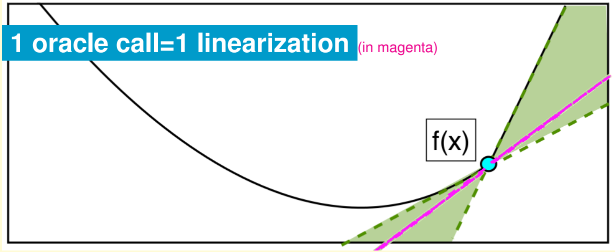
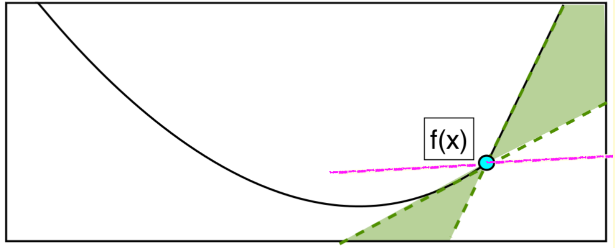


Linearization bad if oracle output is bad
wrong *g* can give a bad linearization



Using oracle subgradients in the stopping test

Algorithms for unconstrained **smooth** optimization use as optimality certificate Fermat's rule

$$0 = \nabla f(\bar{x})$$

and generate a *minimizing sequence*:

$$\{x^k\} \to \bar{x} \text{ such that } \nabla f(x^k) \to 0\,.$$

If $f \in C^1$, then $\nabla f(\bar{x}) = 0$

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In NSO things are less straightforward...

Using oracle subgradients in the stopping test

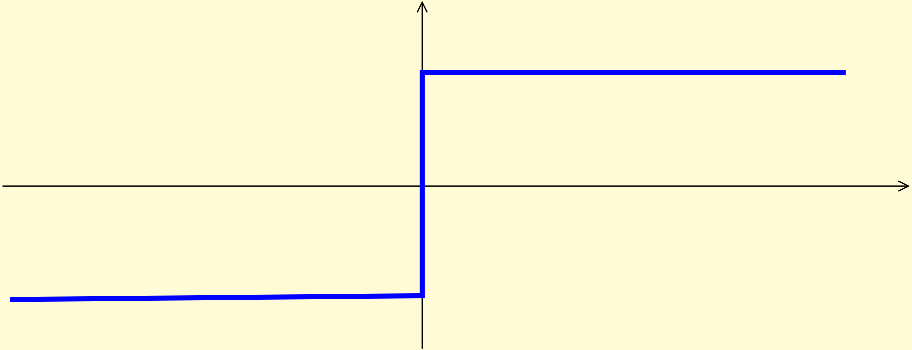
For the absolute value function, $f(x) = |x|$ and

$$\partial f(x) = \begin{cases} -1 & x < 0 \\ [-1, 1] & x = 0 \\ 1 & x > 0 \end{cases}$$

Using as optimality certificate the inclusion

$$0 \in \partial f(\bar{x})$$

requires knowing the whole subdifferential!



NSO pitfalls

NSO pitfalls

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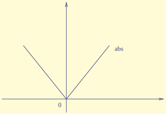
NSO pitfalls

Why special NSO methods?

Smooth optimization techniques **do not work**



$$\begin{aligned} f(x) &= |x| \\ |\nabla f(x^k)| &= 1,\ \forall x^k \neq 0 \quad \partial f(0) = [-1, 1] \end{aligned}$$

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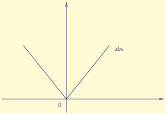

$$\begin{aligned} f(x) &= |x| \\ |\nabla f(x^k)| &= 1,\ \forall x^k \neq 0 \quad \partial f(0) = [-1, 1] \end{aligned}$$

- Smooth stopping test **fails**:
  $|\nabla f(x^k)| \leq \text{TOL} \qquad (\leftrightarrow |g(x^k)| \leq \text{TOL})$

Why special NSO methods?

Smooth optimization techniques **do not work**

- Finite differences **fail**

  For $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x) = \max(x_1, x_2, x_3)$

  $\partial f(0) =?$

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  For $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x) = \max(x_1, x_2, x_3)$

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Forward finite difference $\frac{f(x+\Delta x)-f(x)}{\Delta x}$

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  $\partial f(0) = ?$

Forward finite difference $\frac{f(x+\Delta x)-f(x)}{\Delta x}$

Central finite difference $\frac{f(x+\Delta x)-f(x-\Delta)}{2\Delta x}$

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  For $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $f(x) = \max(x_1, x_2, x_3)$

  $\partial f(0) =$unit simplex

Forward finite difference $\frac{f(x+\Delta x)-f(x)}{\Delta x}$ $= (1, 1, 1)$

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none of them in the subdifferential!

Why special NSO methods?

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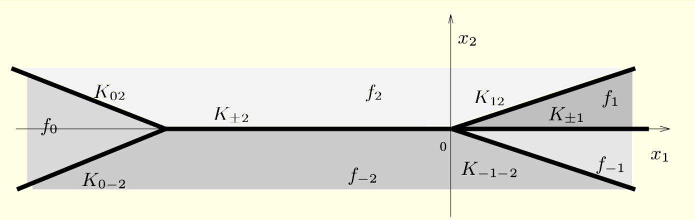
- Linesearches get trapped in kinks and **fail**



Example 9.1, “Instability of steepest descent”

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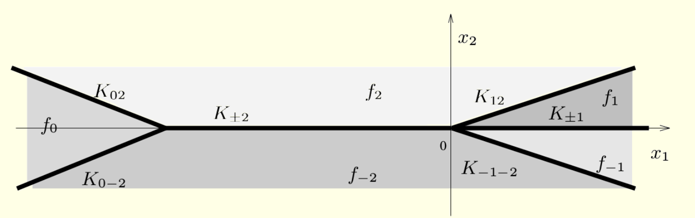

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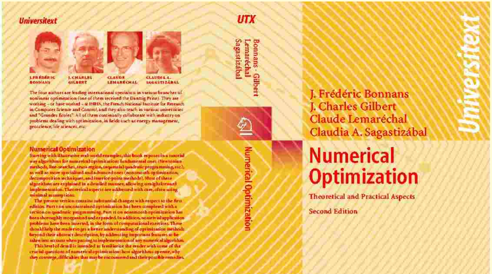

Why special NSO methods?

Smooth optimization techniques **do not work** (suite)

$-g(x^k)$ may **not** provide descent

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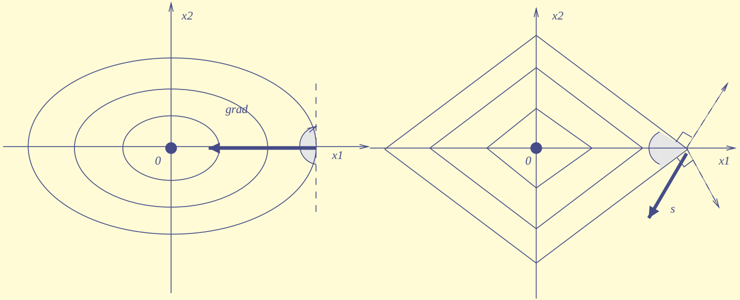


Direction opposite to a subgradient may **increase** the functional values

Why special NSO methods?

Smooth optimization techniques **do not work**

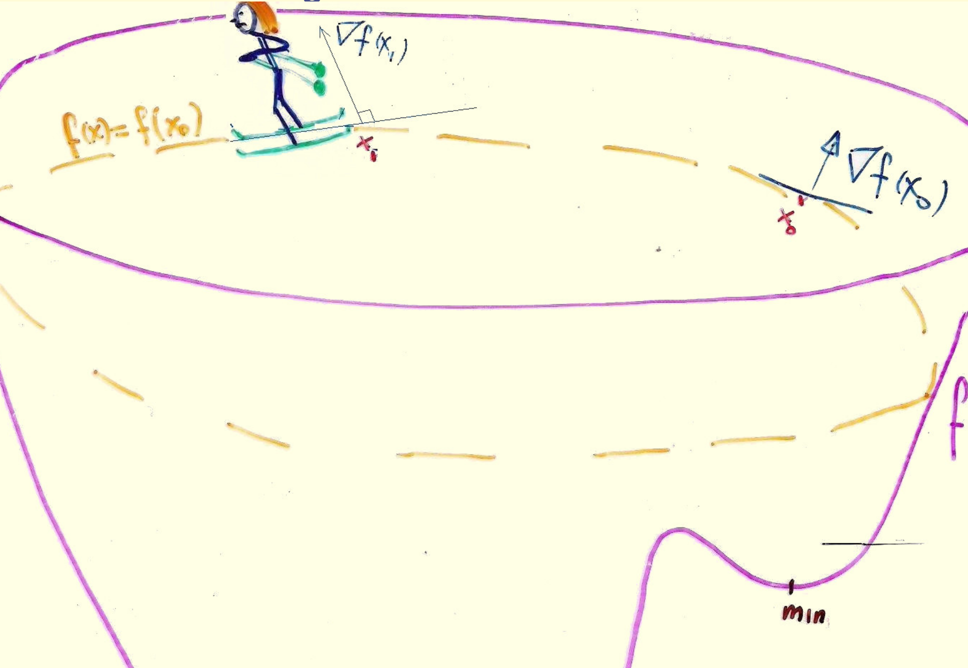
Smooth stopping test **fails**

Finite difference approximations **fail**

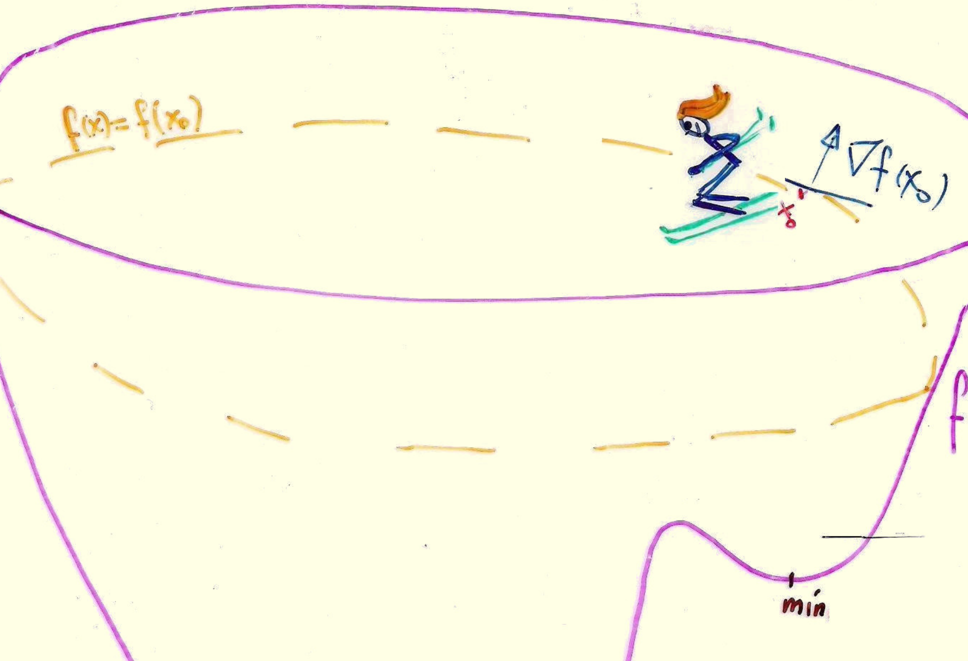
Linesearches get trapped in kinks and **fail**

Direction opposite to a subgradient may **increase** the functional values

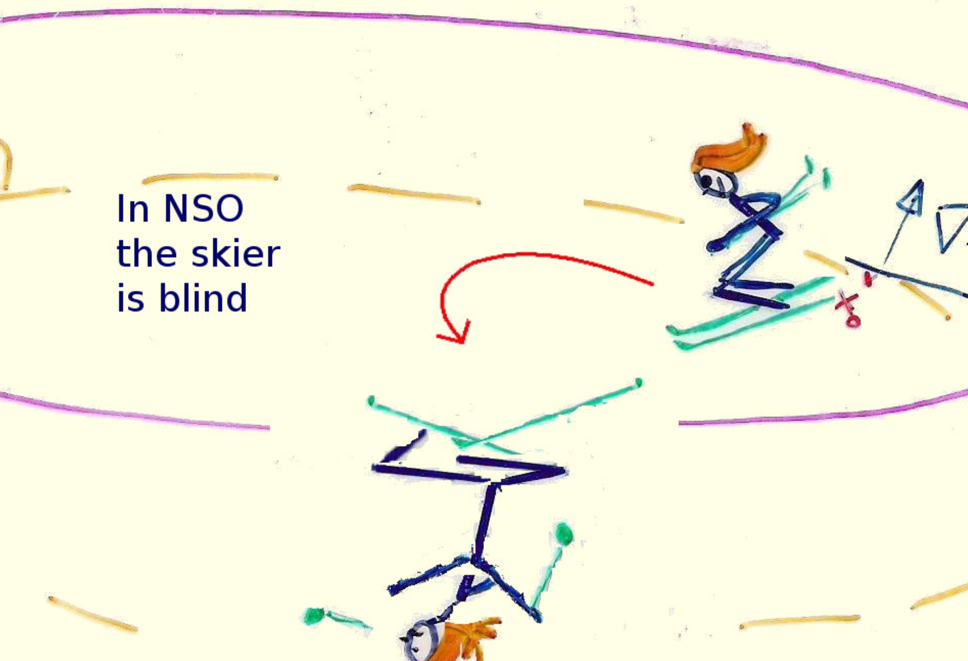
Nonsmooth skiing



Nonsmooth skiing



Nonsmooth skiing



Looking for sound optimality certificates in NSO

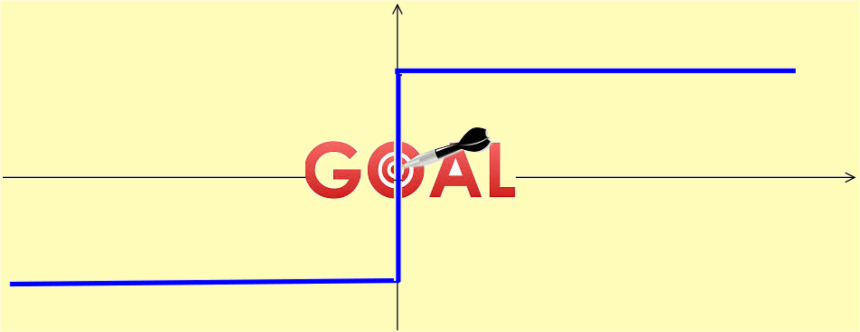
For the absolute value function, $f(x) = |x|$ and

$$\partial f(x) = \begin{cases} -1 & x < 0 \\ [-1, 1] & x = 0 \\ 1 & x > 0 \end{cases}$$

Using as optimality certificate the inclusion

$$0 \in \partial f(\bar{x})$$

requires knowing the whole subdifferential!



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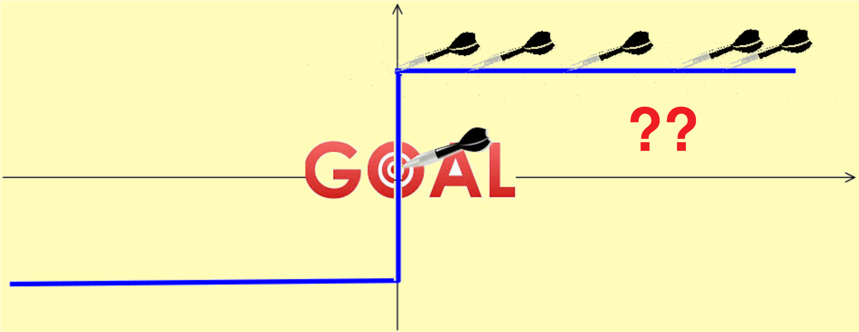

What happens with the stopping test?

In **nonsmooth** optimization the inclusion

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fails as optimality certificate

- As a set-valued mapping $\partial f(x)$ is osc:

$$\left(x^k, g(x^k) \in \partial f(x^k)\right) : \begin{cases} x^k \to \bar{x} \\ g(x^k) \to \bar{g}\,. \end{cases} \Longrightarrow \bar{g} \in \partial f(\bar{x})$$

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We use instead

$$\bar{g} \in \partial_{\bar{\varepsilon}} f(\bar{x}) \quad \text{for } \|\bar{g}\| \text{ and } \bar{\varepsilon} \text{ small}$$

where the ε-subdifferential contains the slopes of linearizations supporting f **up to ε**, tangent at x:

$$\partial_\varepsilon f(x) = \{g \in \mathbb{R}^n : f(y) \geq f(x) + g^\top (y - x) - \varepsilon \text{ for all y}\}$$

The ε-subdifferential

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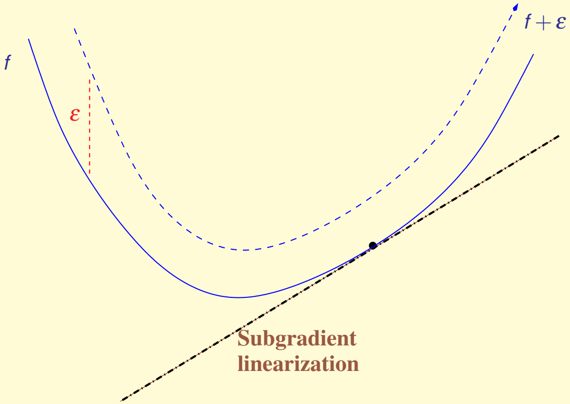

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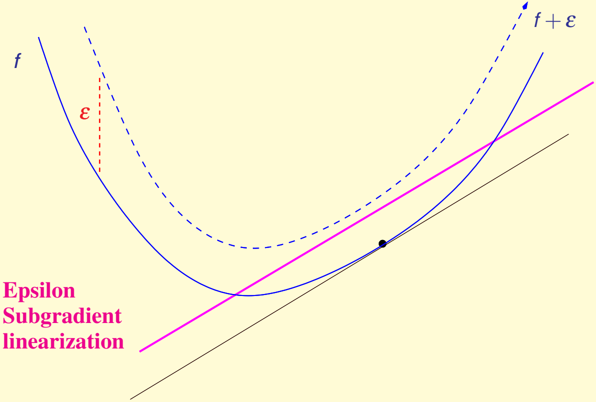

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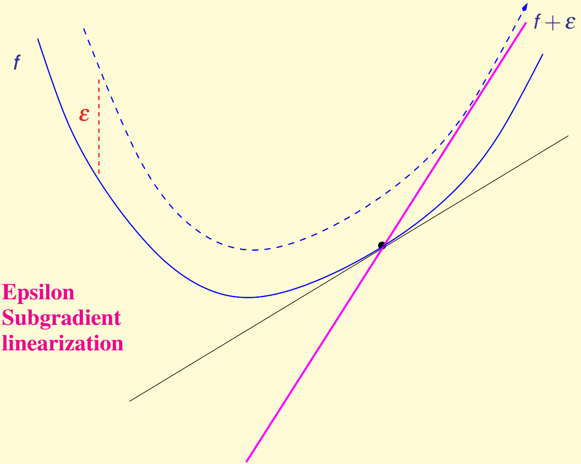

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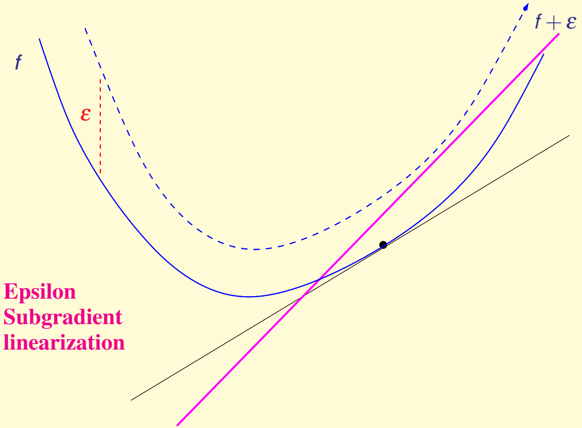

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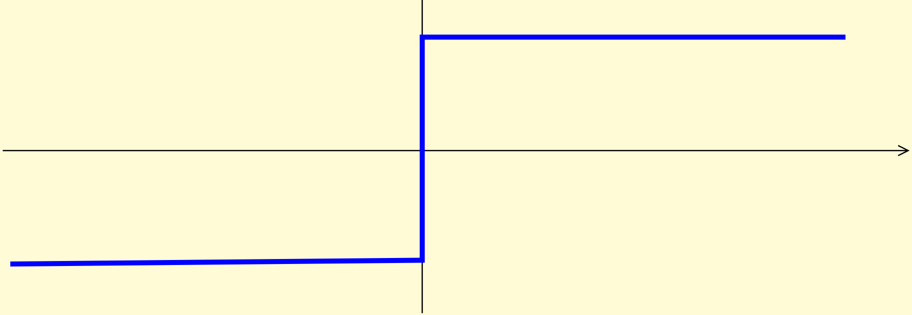

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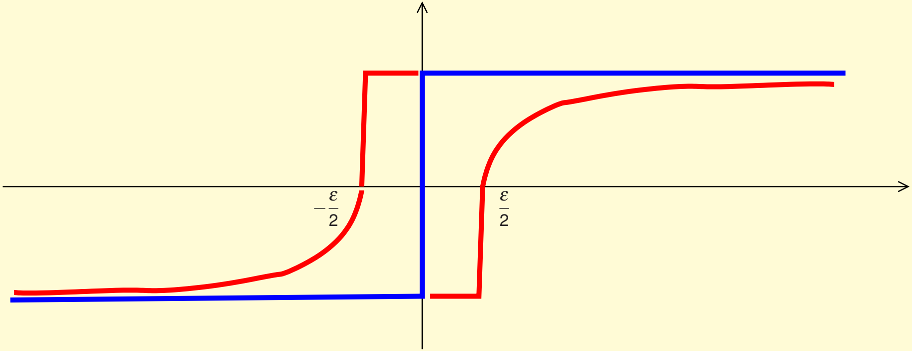

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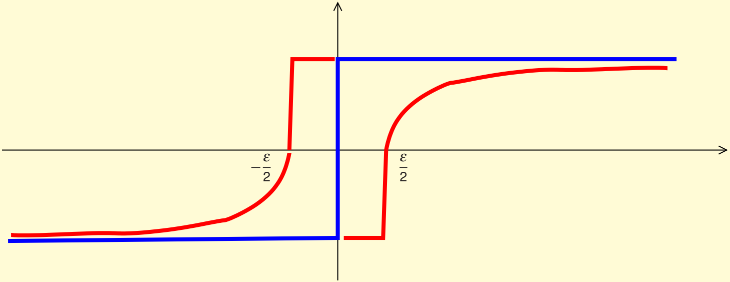

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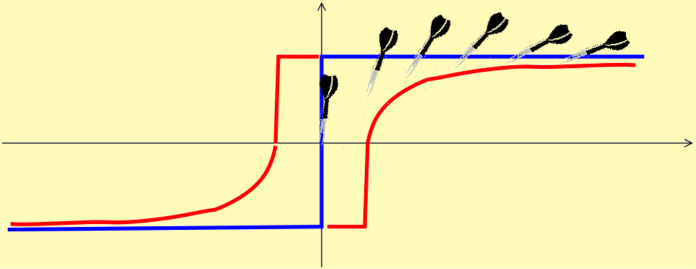

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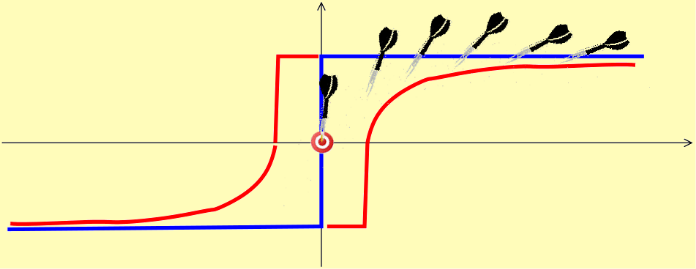

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The ε-subdifferential and bundle methods

Generate iterates so that for a **subsequence $\{\hat{x}^k\}$**

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with $\bar{\varepsilon} \approx 0$ and $\|\bar{g}\| \approx 0$

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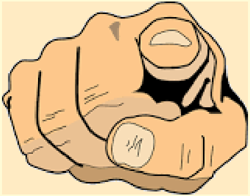
Building up ε-subgradients in bundle methods

You told us



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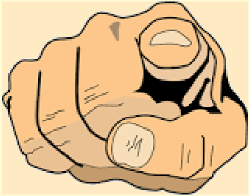
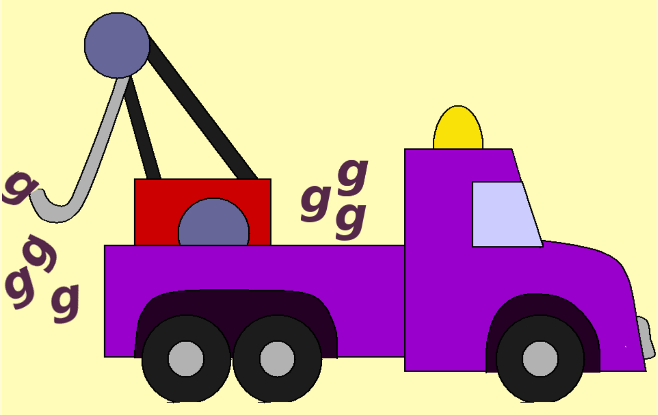


we were going to use subgradient information provided by a black-box because knowing the full subdifferential is too much of a requirement.

Now you want to use the ε-subdifferential, an even larger set!

The transportation formula



Or how to express subgradients at x^i as ε-subgradients at $\hat{x}^k$

The transportation formula

Consider $g^i \in \partial f(x^i)$

The inclusion holds if and only if, for all $y \in \mathbb{R}^n$

$$f(y) \quad \geq \quad f(x^i) + g^{i\top}(y - x^i)$$

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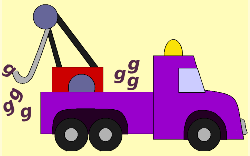

$$\Longrightarrow \boxed{g^i \in \partial_{e^i(\hat{x}^k)} f(\hat{x}^k)}$$

for $e^i(\hat{x}^k) := f(\hat{x}^k) - f(x^i) - g^{i\top}(\hat{x}^k - x^i) \geq 0$

The ε-subdifferential and bundle methods

We collect the black-box

 output at past iterations $x^i, i = 1, 2, \ldots, k$,

so that at iteration k we can define a **bundle** of information, centered at a special iterate $\hat{x}^k \in \{x^i\}$

$$\mathcal{B}^k := \begin{pmatrix} e^i(\hat{x}^k) = f(\hat{x}^k) - f(x^i) - g^{i\top}(\hat{x}^k - x^i) \\ g^i \in \partial_{e^i(\hat{x}^k)} f(\hat{x}^k) \end{pmatrix}$$

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A suitable convex combination

$$\varepsilon^k := \sum_{i \in \mathcal{B}^k} \alpha^i e^i(\hat{x}^k) \text{ and } G^k := \sum_{i \in \mathcal{B}^k} \alpha^i g^i$$

will eventually satisfy the optimality condition!

Back to Computational NSO

For the unconstrained problem

$$\boxed{\min f(x)},$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex but not differentiable at some points, we shall define algorithms based on information provided by an *oracle* or “black box”  endowed with reliable **stopping tests**

How is the oracle information used?

We look for algorithms based on information provided

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Subgradient methods

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1 Call the oracle at x^k.

2 Compute $x^{k+1} = x^k - t_k g(x^k)$ for a suitable stepsize $t_k > 0$.

3 Make $k = k + 1$ and loop to 1.

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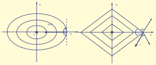

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- Generate non-monotone functional values

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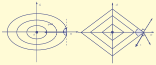

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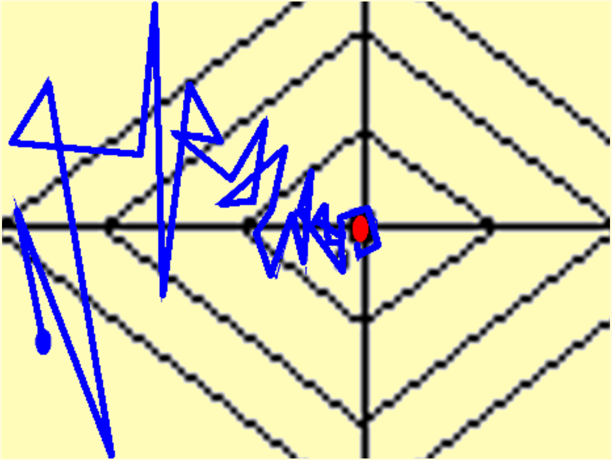

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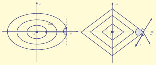

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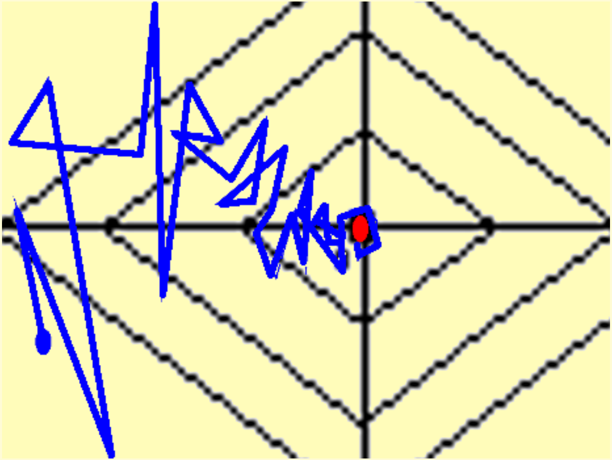

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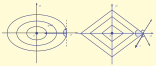

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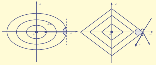

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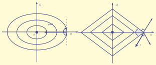

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…**does not use all available information!**

f(x)??

$g(x) \in \partial f(x)$

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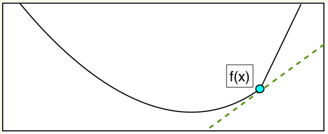
Black box information defines linearizations



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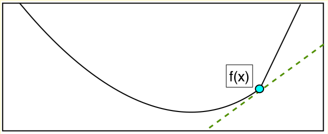

that put together create a **model M** of the function f.

The model is used to define iterates and to put in place a stopping test

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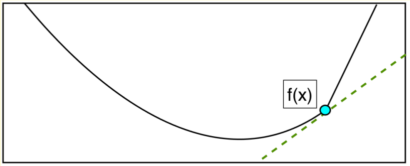

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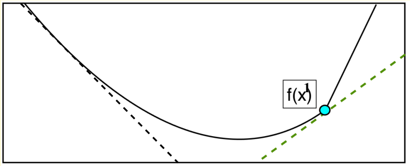

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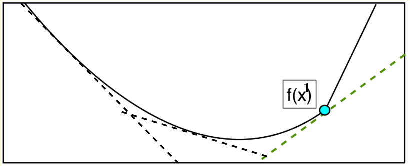

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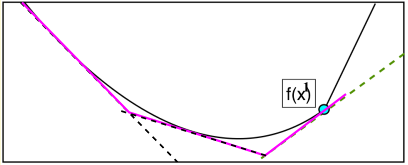

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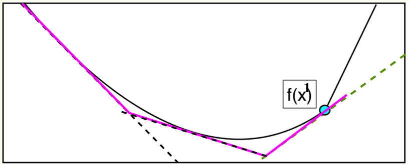

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For future use: $\partial \mathbf{M}(x) = conv\{g^i : i \in I(x)\}$

Cutting-plane methods

To minimize f (unavailable in an explicit manner),

minimize its model $\boxed{\mathbf{M}(x) = \max_i \left\{ f^i + g^{i\top}(x - x^i) \right\}}$

Improve the model at each iteration

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Instead of $x^* \in \arg\min f(x)$ **at one shot**
$x^k \in \arg\min \mathbf{M}_{k-1}(x)$ **iteratively**

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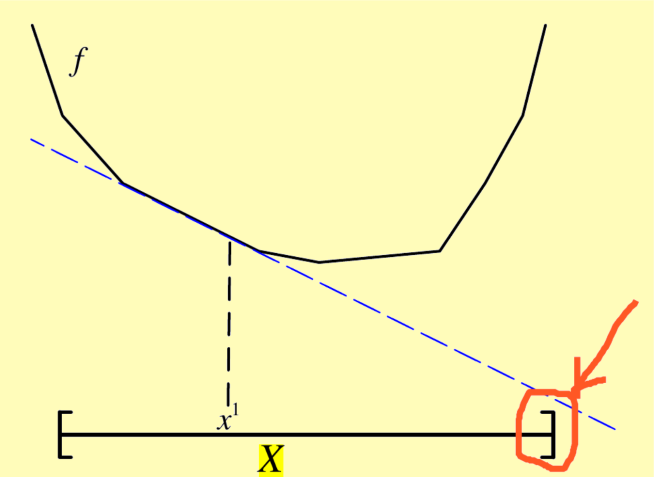
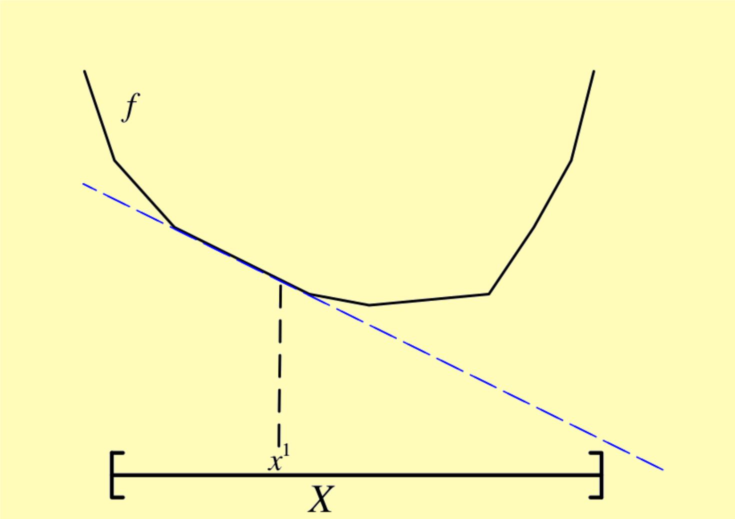


Artificial bounding at least for the first iterations

Cutting-plane methods



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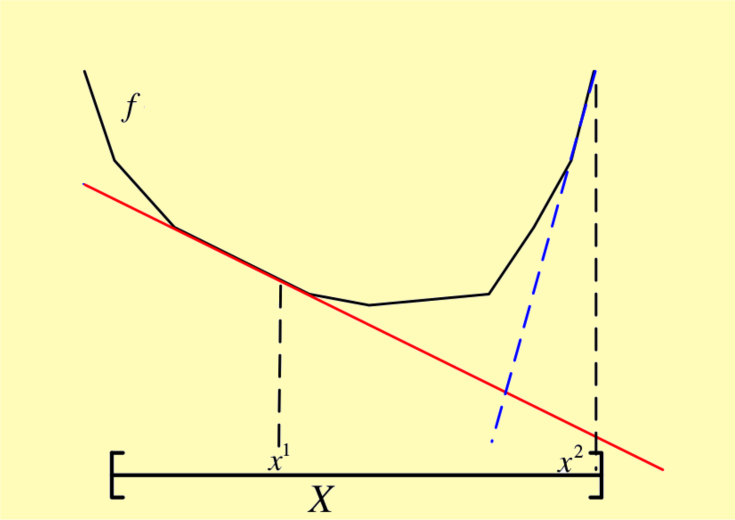

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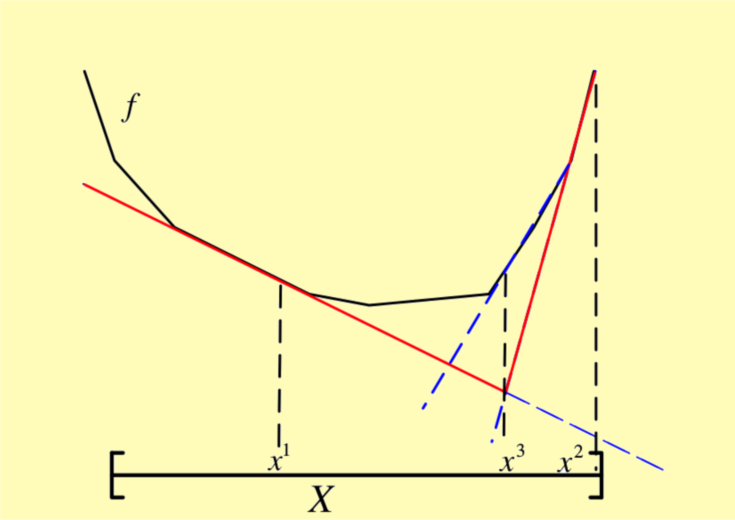

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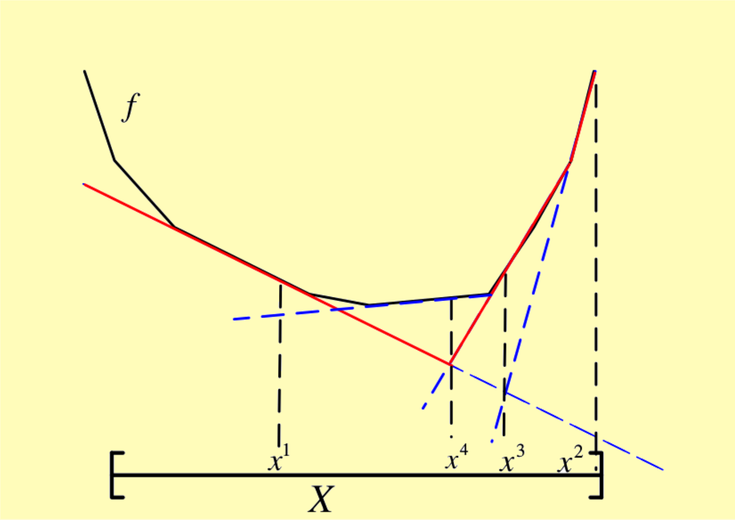

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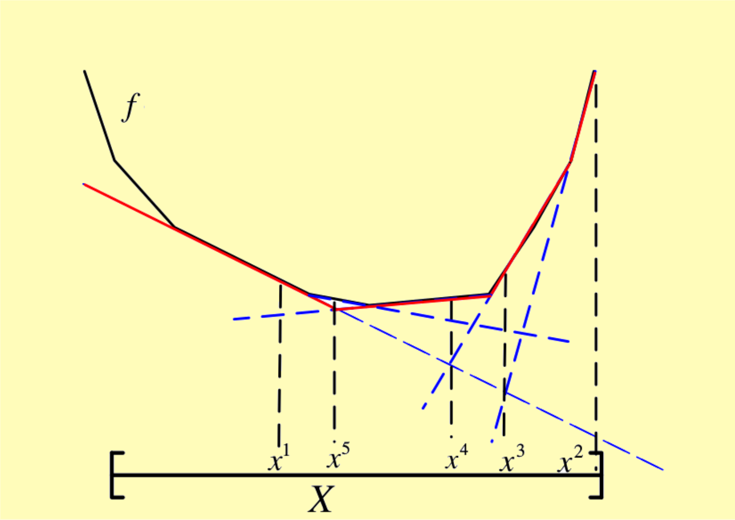

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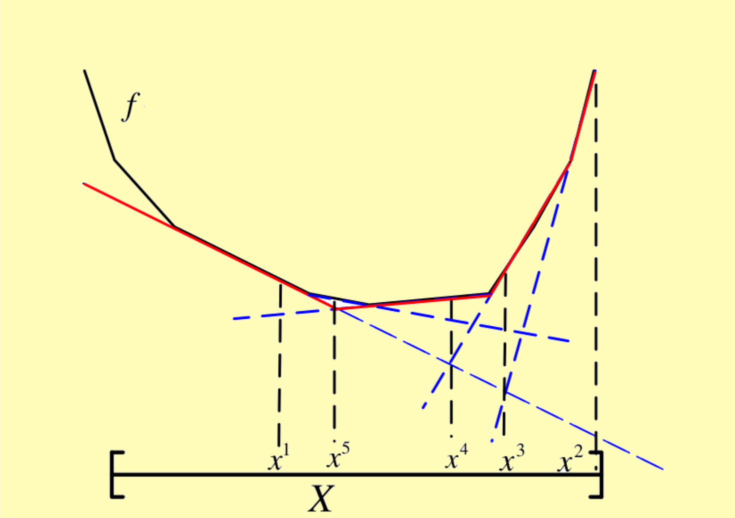

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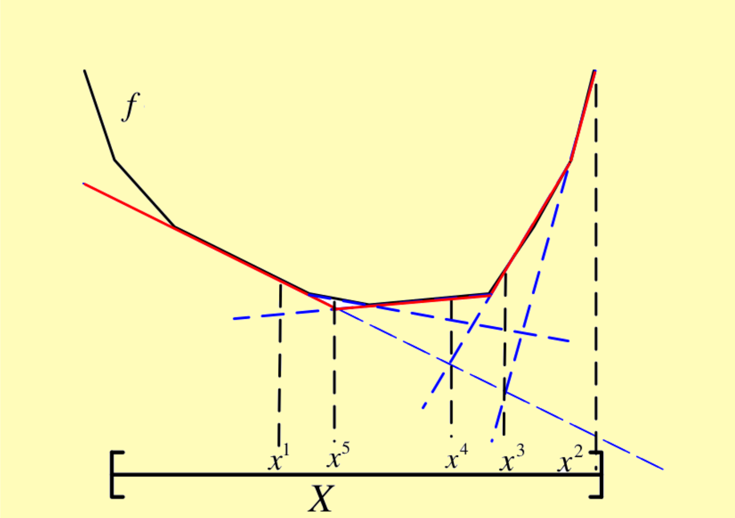
Cutting-plane methods



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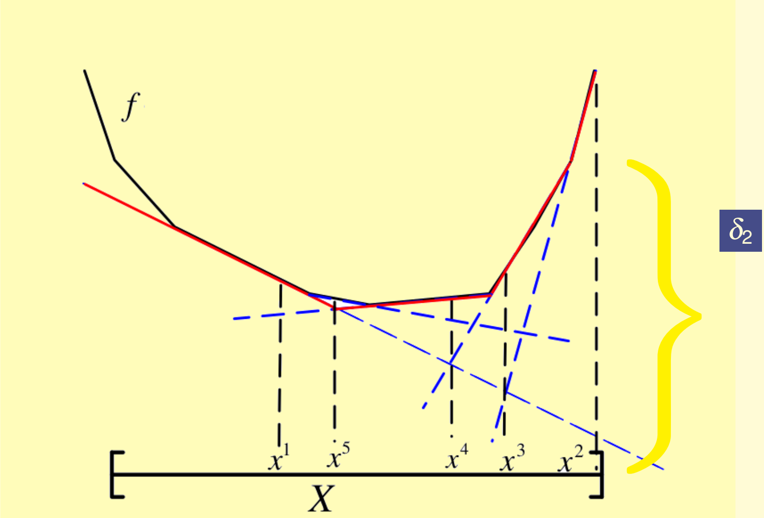

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Optimality certificate checks if $\delta_k := f(x^k) - \mathbf{M}_{k-1}(x^k)$ is small

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$\delta_k = f(x^k) - \mathbf{M}_{k-1}(x^k) \leq tol$

2 Given $\mathbf{M}_k(\cdot) = \max\Big(\mathbf{M}_{k-1}(\cdot), f^k + g^{k\top}(\cdot - x^k)\Big)$

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In 2, $\mathbf{M}_k(x) = \max_{i \leq k}\{f^i + g^{i\top}(x - x^i)\}$ and X polyhedral.
$\Longrightarrow$ 2 $\equiv$ to solving a linear programming problem

$$\begin{cases} \min & r \\ \text{s.t.} & r \in \mathbb{R}, x \in X \\ & r \geq f^i + g^{i\top}(x - x^i) \text{ for } i \leq k \end{cases}$$

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- One LP solve per iteration
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- LP problem has more and more constraints, and eventually numerical errors prevail

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- CP methods bring in the concept of a model, which gives a stopping test
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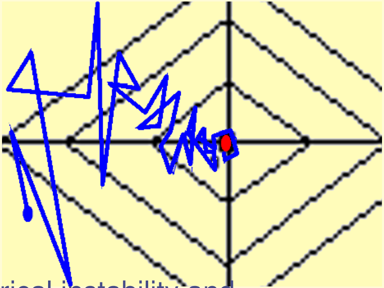
- CP methods still non-monotone



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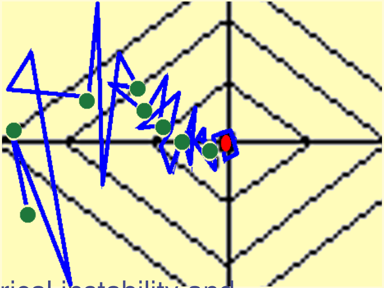

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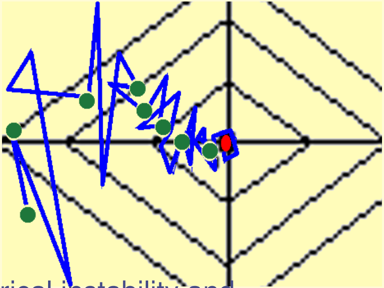

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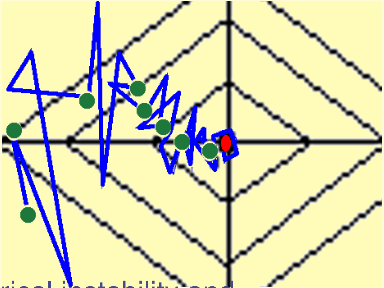

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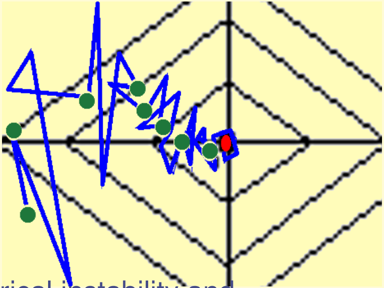

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$$f(x^{k+1}) \leq f(\hat{x}^k) - m\delta_k \Longrightarrow \hat{\mathbf{x}}^{\mathbf{k+1}} := \mathbf{x}^{\mathbf{k+1}}$$

Limit points of the **serious-step** subsequence $\{\hat{\mathbf{x}}^{\mathbf{k}}\}$ minimize f

Bundle methods

0 Choose x^1, $t_1 > 0$, and set $\hat{x}^1 = x^1$, $k = 1$.

1 Given $\hat{x}^k$, $\mathbf{M}_k$ and t_k, compute x^{k+1} and $\delta_{k+1} := f(\hat{x}^k) - \mathbf{M}_k(x^{k+1})$

2 Call the oracle at x^{k+1}. If $\delta_{k+1} \leq tol$ **STOP**

3 `(Descent Rule)`

$$f(x^{k+1}) \leq f(\hat{x}^k) - m\delta_k? \quad \begin{cases} \text{yes} & \text{SS: } \hat{x}^{k+1} = x^{k+1} \\ \text{no} & \text{NS: } \hat{x}^{k+1} = \hat{x}^k \end{cases}$$

4 Choose a new model and stepsize.

5 Set $k = k + 1$, loop to 1.

Bundle methods: the power of the descent rule

The serious-step subsequence satisfies the descent rule

$$f(\hat{x}^{k+1}) \leq f(\hat{x}^k) - m\delta_k \qquad (\texttt{DR})$$

Rearranging terms, for k serious,

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As long as $\delta_k \geq 0$ in `(DR)`,

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Without even specifying how iterates x^k are computed, if the descent rule holds for infinitely many iterates, the subsequence of serious optimality certificates (δ_k) goes to zero (and $\{\hat{x}^k\}$ will be minimizing).

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The analysis of the situation when there is a last serious iterate, $\hat{x}$, followed by infinitely many null steps, is related to the proximal point operator

A question for you:

Ever heard of the proximal point mapping?

Answer provided by Jean-Jacques Moreau

The Moreau-envelope of f is a $C^{1,1}$-smoothing of f

$$F_\mu(x) := \min\left\{ f(y) + \frac{1}{2}\mu \|y - x\|^2 \right\}$$

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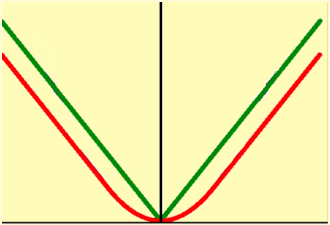

- the unique minimizer is the proximal point mapping $p^f_\mu(x)$

- the envelope's gradient is $\nabla F_\mu(x) = \mu\Big(x - p\mu(x)\Big)$

How to compute the prox?

$$
\begin{aligned}
p = p_t^f(x) \quad &\Longleftrightarrow \quad p = \arg\min f(y) + \frac{1}{2t}\|y - x\|_2^2 \\
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note 1: $\Longleftrightarrow \; p \in x - t\partial f(p)$ **akin to a subgradient method**

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note 2: $\bar{x}$ **minimizes** $f \; \Longleftrightarrow \; p = \bar{x} \; \Longleftrightarrow \; 0 \in \partial f(p) = \partial f(\bar{x})$

Proximal point algorithm (PPA) (Accel. Nesterov, FISTA)

Given starting point x^1 and prox-stepsize $t_1 0$,

$$x^{k+1} = p^f_{t_k}(x^k)$$

$$\iff$$

$$x^{k+1} = \arg\min f(y) + \frac{1}{2t_k}\|y - x^k\|_2^2$$

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Given starting point x^1 and prox-stepsize $t_1 0$,

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Proximal point: calculus rules

- separable sum:
  $f(x, y) = g(x) + h(y) \Longrightarrow$
  $p_t^f(x) = \Big(p_t^g(x), p_t^h(y) \Big)$
- scalar factor ($\alpha \neq 0$) and translation ($v \neq 0$):
  $f(x) = g(\alpha x + v) \Longrightarrow$
  $p_t^f(x) = \frac{1}{\alpha} \Big(p_t^{\alpha^2 g}(\alpha x + v) - v \Big)$
- "perspective" ($\alpha > 0$):
  $f(x) = \alpha g(\frac{1}{\alpha} x) \Longrightarrow p_t^f(x) = \alpha p_t^{g/\alpha}(\frac{x}{\alpha})$

Proximal point: special functions

- + linear term ($v \neq 0$):
  $f(x) = g(x) + \langle v, x \rangle \Longrightarrow p_t^f(x) = p_t^g(x - v)$
- + convex quadratic term ($t > 0$):
  $f(x) = g(x) + \dfrac{1}{2t}\|x - v\|^2 \Longrightarrow$
  $p_t^f(x) = p_t^{\lambda g}(\lambda x + (1 - \lambda)v)$ for $\lambda = \dfrac{t}{t+1}$
- composition with linear term such that $A^\top A = \frac{1}{\alpha} I$, ($\alpha \neq 0$):
  $f(x) = g(Ax + v) \Longrightarrow$
  $p_t^f(x) = (I - \alpha A^\top A)x + \alpha A^\top \left[p_t^{g/\alpha}(Ax + v) - v \right]$

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Mathematical Programming

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Convergence of some algorithms for convex minimization

Rafael Correa, Claude Lemaréchal

the proximal point subgradient

$$p = p_t^f(x) \iff p = \arg\min f(y) + \frac{1}{2t}\|y - x\|_2^2$$
$$\iff 0 \in \partial f(p) + \frac{1}{t}(p - x)$$

Take $g(p) := \frac{1}{t}(x - p) \in \partial f(p)$ satisfying the OC

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\begin{aligned}
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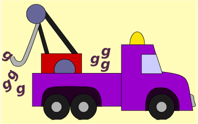

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For the PPA, this means that

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Use the bundle ideas

to compute (ap)proximal points

Bundle methods prove most useful

In situations

- when the objective function is not available explicitly

and/or

- when we do not have access to the full subdifferential

and/or

- when calculations need to be done with high precision

Bundling to approximate the p

WANT: $$p = p_t^f(\hat{x}) = \arg\min f(y) + \frac{1}{2t}\|y - \hat{x}\|_2^2$$

or, equivalently, finding p such that $\frac{1}{t}(\hat{x} - p) \in \partial f(p)$

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HAVE: $\mathbf{M}$, a model of f for which we do have full knowledge of the subdifferential (recall note for future use):

$$q = p_t^{\mathbf{M}}(\hat{x}) = \arg\min \mathbf{M}(y) + \frac{1}{2t}\|y - \hat{x}\|_2^2 \qquad (\mathtt{QP})$$

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$$G := \frac{1}{t}(\hat{x}-q) = \sum_i \alpha^i g^i$$

is a convex combination of active subgradients, computed for free when solving (QP)

Bundling to approximate the p

Theorem[CL93] Suppose the models satisfy

- $M_k(y) \leq f(y)$ for all k and y
- $M_{k+1}(y) \geq f(q^k) + g(q^k)^\top (y - q^k)$
- $M_{k+1}(y) \geq M_k(q^k) + G^{k\top}(y - q^k)$

If $0 < t_{\min} \leq t_{k+1} \leq t_k$, then

$$\lim_{k\to\infty} q^k = \hat{x} \quad \text{and} \quad \lim_{k\to\infty} M_k(q^k) = f(\hat{x})$$

the bundle subgradient

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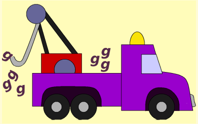

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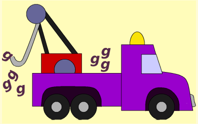

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Illustration of oracles and models

The cutting-plane model is not the only one satisfying the [CL93] conditions

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There are many different models

CP models are the most straightforward ones. There are other possibilities, that exploit structural knowledge

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For $x \in \mathbb{R}^n$, given matrices $A \succeq 0$, $B \succ 0$,

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The oracle information for f can be available in different forms

Models for the half-and-half function

STRUCTURE	$f(x)$	
none	$\sqrt{x^\top A x} + x^\top B x$	
sum	$f_1(x) + f_2(x)$	$f_1(x) = \sqrt{x^\top A x}$ $f_2(x) = x^\top B x$ f_2 is smooth
compo sition	$(h \circ c)(x)$	$c(x) = (x, x^\top B x) \in \mathbb{R}^{n+1}$ c $h(C) = \sqrt{C_{1:n}^\top A C_{1:n}} + C_{n+1}$ h

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Models for the half-and-half function

STRUCTURE	$f(x)$	
none	$\sqrt{x^\top Ax} + x^\top Bx$	$f^k := f(x^k),\ g^k \in \partial f(x^k)$
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none	$\sqrt{x^\top A x} + x^\top B x$	$f^k := f(x^k),\ g^k \in \partial f(x^k)$
sum	$f_1(x) + f_2(x)$	$f_1(x) = \sqrt{x^\top A x}$ $f_2(x) = x^\top B x$ $f_1^k, g_1^k, f_2^k, \nabla f_2(x^k)$
compo sition	$(h \circ c)(x)$	$c(x) = (x, x^\top B x) \in \mathbb{R}^{n+1}$ $c^k = c(x^k), c'(x^k)$ $h(C) = \sqrt{C_{1:n}^\top A C_{1:n}} + C_{n+1}$ $h^k, g^k \in \partial h(c^k)$

Take away messages

- PPA has good properties for minimizing convex nonsmooth functions
- If computing exactly the prox at each iteration is too demanding, a bundling approach can be put in place
- the theory in [CL93]] allows for great generality in how the approximation is done
- Bundle methods enter into play to decide when the approximal point is sufficiently good `(DR)`.
- If rough approximations without optimality certificate are enough: SG
- CP methods are more reliable (stopping test) but LP grows indefinitely, need bounding set X, and can be unstable
- Bundle methods stabilize CP, have reliable stopping test, solve one QP of controllable size per iteration

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Beware all of the above depends on having exact oracle information