Operator Theory and Noncommutative Polynomial Optimization in Quantum Information Theory

Subject:

Quantum Information Theory (QIT) is a rapidly growing field, in which quantum correlations play a central role, with far stemming foundational results (e.g., the Bell theorem) and applications (e.g., quantum cryptography and computing).

Characterization of quantum correlations is challenging. Given the mathematical formalism of QIT, existing mathematical and algorithmic approaches are based on operator theory, C*-Algebras and non-commutative polynomial optimization. The standard approach is called the NPA hierarchy [NPA07, NPA08]. This method is the noncommutative counterpart [H02] of the Lasserre-Parillo hierarchy [AL11, GP04], a sum of squares method that relaxes these NP-hard problems into a hierarchy of SemiDefinite programs. However, it is not adapted to some recent theoretical, experimental and technological developments in quantum physics.

While the NPA hierarchy is mathematically proven to converge to the set of quantum correlations in the standard Bell scenario (Fig. 1.(a)), no method has been mathematically proven to converge to more general quantum network correlations (Fig. 1.(b) and Fig. 2). Existing methods are only mathematically characterized in very restrictive cases. Moreover, their numerical implementations are only able to tackle the most elementary problems.

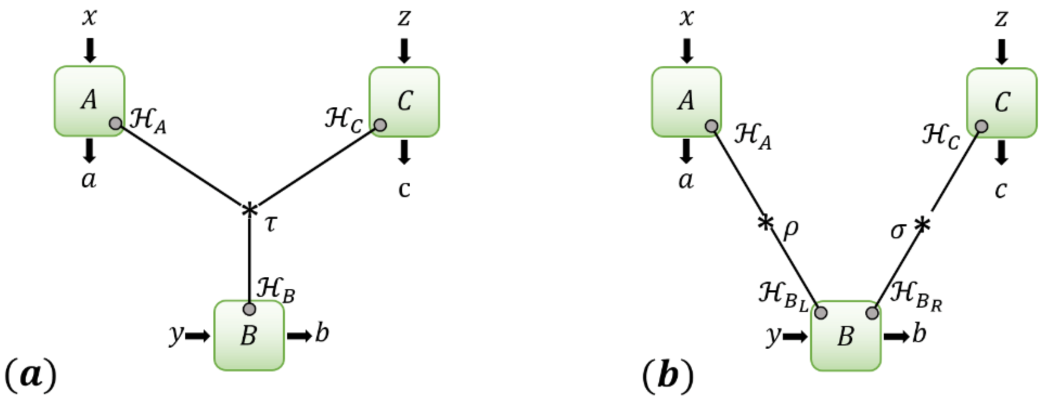

Figure 1. (a) *Standard three-party Bell scenario*. A three-particles quantum state, mathematically represented by a projector over a pure state $\tau \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ (a positive operator such that $\mathrm{Tr}(\tau) = 1$ and $\tau^2 = \tau$), is distributed such that each particle is sent to one of three separated parties A, B, C. A measures the received particle according to some input x, obtaining an output a, mathematically represented by a PVMs $A_{a|x} \in \mathcal{B}(\mathcal{H}_A)$ (a set of positive operators such that $\sum_a A_{a|x} = \mathbb{1}$), and B, C do the same. The behavior of the experiment is described by a probability distribution $\vec{P} = \{p(abc|xyz)\}$ with the $p(abc|xyz) = \mathrm{Tr}\big(\tau(A_{a|x} \otimes B_{b|y} \otimes C_{c|z})\big)$.
(b) *Bilocal scenario*. Two two-particles quantum state (mathematically represented by two projectors over pure states $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_{B_L}), \sigma \in \mathcal{B}(\mathcal{H}_{B_R} \otimes \mathcal{H}_C)$ such that $\mathrm{Tr}(\rho) = \mathrm{Tr}(\sigma) = 1$ and $\rho^2 = \rho, \sigma^2 = \sigma$) are created, A (resp. C) receiving one particle from ρ (resp. σ) and B one of each state, as depicted. A, B, C measurement operators are mathematically represented by positive operators $A_{a|x} \in \mathcal{B}(\mathcal{H}_A), B_{b|y} \in \mathcal{B}(\mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R}), C_{c|z} \in \mathcal{B}(\mathcal{H}_C)$ such that $\sum_a A_{a|x} = \sum_b B_{b|y} = \sum_c C_{c|z} = \mathbb{1}$). The behavior of the experiment is described by a probability distribution $\vec{P} = \{p(abc|xyz)\}$ with $p(abc|xyz) = \mathrm{Tr}\big((\rho \otimes \sigma)(A_{a|x} \otimes B_{b|y} \otimes C_{c|z})\big)$.

Two potential directions are envisioned for the PostDoc:

- A first pure mathematics direction: focus on the C* formalism of QIT, and delve into the understanding of how quantum correlations can be characterized based on non-commutative polynomial optimization.

- A second (numerical) optimization direction: develop new algorithms able to achieve significantly more efficient characterization of quantum network correlations (e.g., based on symmetry reduction) and perform numerical implementations of these algorithms.

Depending on the direction taken by the project, it will involve collaborations with the groups of Victor Magron (LASS Toulouse), Igor Klep (Lubjana University) Antonio Acin (ICFO Barcelona), David Gross (Cologne University)

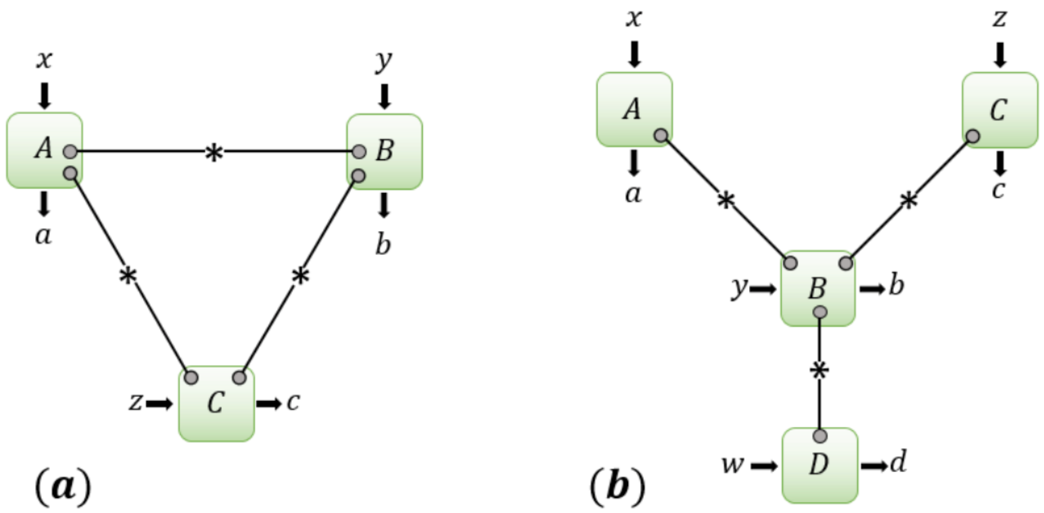

Figure 2. Examples of other general network scenarios: *the triangle scenario* (a) and the *four party star network* (b)

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[NPA08] Miguel Navascues, Stefano Pironio, Antonio Acín, A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations, New Journal of Physics, 10 (2008)