Some aspects of the making of mathematical texts.

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Introduction

> *The mathematician's understanding of the history of his subject is frequently sharply anti-sociological. But as frequently it is a 'rational reconstruction' of a development governed by universal laws - those of mathematics itself. In the unfolding of eternal mathematical laws through time there is no room for stories. (...) We have to construe mathematics as both a body of knowledge and a field of social practice at the same time.*

(Herbert Mehrtens, Social History of Mathematics, 1981, 259-265)

- Mathematics exists and is developed as part of a social environment;
- An example of how context influences mathematical work;
- An example of the (possible) contrast between research writing and publication writing.

Contexts and developments of mathematics

Catherine Goldstein, Long term history and ephemeral configurations, in B. Sirakov, P. Ney de Souza and M. Viana (eds.), Proceedings of the International Congress of Mathematicians- Rio de Janeiro, 2018, World Scientific, 2019, vol. 1, p. 487-522. See `https://www.youtube.com/watch?v=vcoQQzb7yls`

- Ephemeral configurations: interactions between actors, at different scales, including intellect, actions, and reciprocal relationships.
- In the history of math « configurations organise texts and persons » (CG).
- How to articulate a long term history and the "ephemeral configurations"?
- Social aspects to be taken into account in understanding the development of mathematical knowledge, for example:

*The **unification of Italian states during the nineteenth century and the cultural Risorgimento which accompanied it favoured a flourishing of mathematics**, in particular a strong renewal of interest in geometry in all its forms, with Luigi Cremona, Corrado Segre, Guido Castelnuovo or Eugenio Beltrami and their followers (Bottazzini, 1994; Bottazzini and Nastasi, 2013; Casnati et al., 2016). The **Meiji Restoration in Japan witnessed a multifaceted confrontation between the then extremely active, traditional Japanese mathematics (wasan) and its Western counterparts** (Horiuchi, 1996). The **First World War**, a "war of guns and mathematics" as one soldier described it, did not just kill hundreds of mathematicians on the battlefields, among many millions of others: it also **launched entire domains on a vast new scale**, such as fluid mechanics or probability theory, and completely **reconfigured international mathematical exchanges** (for instance fostering a development of set theory, logic and real analysis in newly independent Poland) (Aubin and Goldstein, 2014). One might also think of the variety of national circumstances which preceded the creation of **mathematical societies** in the late nineteenth and early twentieth centuries (Parshall, 1995) or the various **reforms in mathematical education** (Gispert and Schubring, 2011; Karp and Schubring, 2014).* (Ibid.)

- *This is, by definition, ephemeral in the sense that it implies links with social situations which have their own time scale and are most certainly not "eternal truths."* (Ibid.)
- How mathematics is made "public" and circulate, for example letters, journals, editorial policies, academies, teaching...
- "Aspects ... not necessarily linked to public and noisy debates, but are part and parcel of ordinary mathematical activities": mathematical practices, habits and choices of the mathematician at work, what is interesting, for and by whom and why...

A case study: set theory in France

Hélène Gispert, La théorie des ensembles en France avant la crise de 1905 : Baire, Borel, Lebesgue... et tous les autres, *Revue d'histoire des mathématiques,* 1 (1995), p. 39–81.

- How French did mathematicians receive Georg Cantor's works in the 19th century?
- Foundations of analysis were not in favor in French – unlike in Germany, where many mathematicians studied these questions, including Cantor.
- Gaston Darboux (1842-1917) published several papers on discontinuous functions in the 1870s: no reaction by French mathematicians.
- So little interest in the foundations of analysis at that time, that...

Baire, jeune étudiant, redécouvre la différence, pour une fonction de deux variables, entre la continuité et la continuité séparée par rapport à chacune des variables, propriété mise en évidence à l'aide de contre-exemples par Thomae [1870] et Schwarz [1872], et à laquelle s'était intéressé Darboux dans un article du Bulletin des sciences mathématiques [1872].

(Gispert, p. 47)

In 1883, Mittag-Leffler asked Charles Hermite to translate Cantor's works in French. Hermite wrote about

> *l'impression désolante que nous produisent les mémoires de Cantor [qui] tant qu'on n'en aura point déduit quelque chose, résulte[nt] de considérations tellement arbitraires, que l'auteur aurait mieux fait de [les] garder et d'attendre. (...) J'ajoute que nous jugeons absolument contraire à notre esprit mathématique d'exposer pour n'en rien faire, un appareil compliqué de nouvelles notations et de nouvelles dénominations.*

René Baire (1874-1932) has to write a paper on set theory for the Frenc *l'Encyclopédie des sciences mathématiques* and wrote to Émile Bore (1871-1956) that it was a « corvée assommante » :

> *En écrivant mes Leçons sur les fonctions discontinues, j'avais les coudées franches, j'exposais de la manière qui me paraissait la plus claire les théories dont j'avais l'intention de me servir. Je ne suis plus ici dans les mêmes conditions, je n'ai plus le droit de faire dévier la pensée de G. Cantor. Il me faut bon gré, mal gré, parler de l'addition, de la multiplication des types ordinaux, etc., choses dont je ne connais pas la moindre application. Je ne peux tout de même pas en inventer. [Lettres, p. 83].*

More on mathematical practices and texts in context

- Other aspects that change throughout history: what is an acceptable result? mathematical object? which proof is acceptable? which methods are valid? which answer is considered a good answer?
- To study the writing and reading of mathematical texts "in context", ie by taking into account the configurations in which actors and texts are, allows us to understand the influence it has on the meaning and interpretation of these texts.
- See also C. Goldstein, *Un théorème de Fermat et ses lecteurs* (1995).
- There is no universal and unequivocal dimension for mathematical meaning, it largely depends on the reader and which questions and answers are acceptable in a given place and time.

A case study: Galois' group and (some of) its reinterpretations

Ehrhardt, Caroline. 2012. *Itinéraire d'un texte mathématique: les réélaborations d'un mémoire d'Evariste Galois au dix-neuvième siècle*, Hermann, Paris. See also "Un concept mathématique, trois notions : les groupes au XIXe siècle chez Galois, Cayley, Dedekind", *Images des mathématiques*, 2010.

- Algebra until the mid-19th century: resolution of polynomial equations, in particular finding explicit general methods to find solutions to polynomial equations. Resolution must be "by radicals", that is, using $+, -\times, \div$ and roots.
- In 1826, Niels Henrik Abel (1802-1829) showed that the algebraic resolution of equations is not possible *in general*.
- Évariste Galois (1811-1832), *Mémoire sur les conditions de résolubilité des équations par radicaux* (1831) gives an answer to the question: "is an equation resoluble?" but does not give an explicit, effective method (no formula) for the resolution.

Galois' group

- Consider the permutations of the roots an equation and define the group of the equation.:

Permutations:

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{pmatrix} \quad \begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & x_3 & x_2 \end{pmatrix} \quad \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_1 & x_3 \end{pmatrix} \quad \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \end{pmatrix} \quad \begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_2 & x_1 \end{pmatrix} \quad \begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \end{pmatrix}$$

Galois group:

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{pmatrix} \quad \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \end{pmatrix} \quad \begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \end{pmatrix}$$

- If $\mathbb{Q}$, not possible solve all polynomial equations, for example $x^2 - 2 = 0$ cannot be solved. By adjoining $\sqrt{2}$ to $\mathbb{Q}$ we get a new set in which the equation is solvable.
- With adjunction, reduce the equation and "split" the group into groups containing less elements. The equation is solvable if the last group you can get has one element.
- "*La signification que l'on pouvait attribuer au « groupe » dont parlait Galois était construite à partir de l'utilisation qui en était faite, autrement dit : elle était liée exclusivement à la théorie des équations.*" (Ehrhardt 2010)

Galois and his time

- Galois' memoir was rejected by the Académie des Sciences
- The referees Lacroix and Poisson judged that the reasoning where "neither clear nor developed enough" and that it was preferable to "wait for the author to have published his work in its entirety to form a definitive opinion."
- Galois' work was at odds with the norms of mathematics at this time, and in particular with what was considered the "right way" to ask and answer to questions in algebra.
- Joseph Liouville (1809-1882) published Galois' works 15 years later in the *Journal de mathématiques pures et appliquées*.

Cayley's group

- English mathematician Arthur Cayley (1821-1895) was the first mathematician to develop a theoretical viewpoint on Galois' group concept.
- On the Theory of Groups as Depending on the Symbolical Equation $\theta^n = 1$, *Philosophical Magazin*, 4th ser., vol. 7, 1854, p. 40-47 and p. 408-409 ; *Philosophical Magazine*, 4th ser., vol. 18, 1859, p. 34-37 [reed. in *The Collected Mathematical Papers*, t. 2, p. 123-132 and t. 4, p. 88-91].
- In Cambridge, a school of algebraists was developed, since 1810, in which algebra was not about solving equations but about symbolic operations between mathematical objects. It was the study of the rules of operations in computations, independently of the nature of the objects concerned.

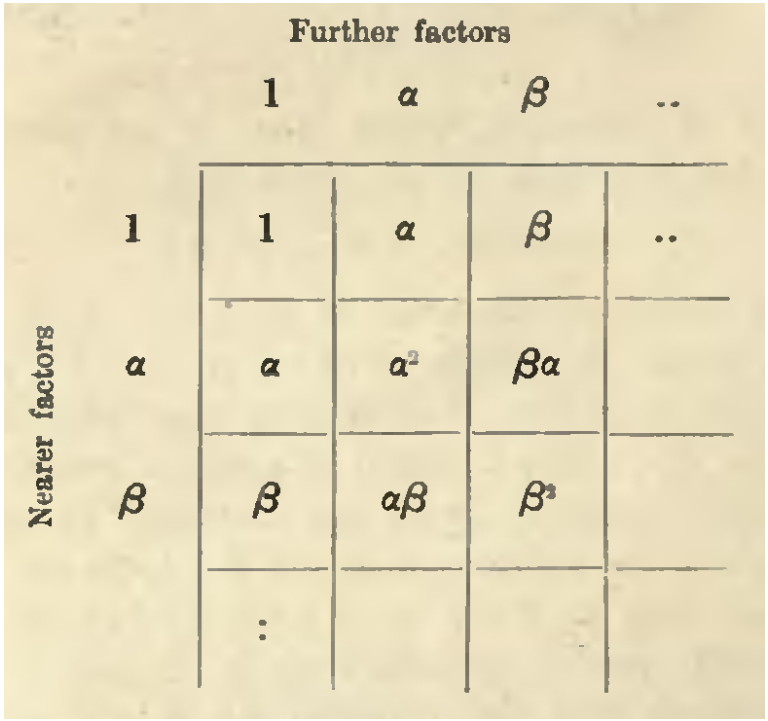
Cayley's group

- While Cayley mentions Galois' work, his own papers are clearly embedded in the Cambridge tradition.
- Cayley completely reinterprets Galois' ideas into the Cambridge tradition of algebraic algebra, with no interest for the resolution of equations, a larger perspective, a different definition for the "group" and different methods.
- *A set of symbols,*

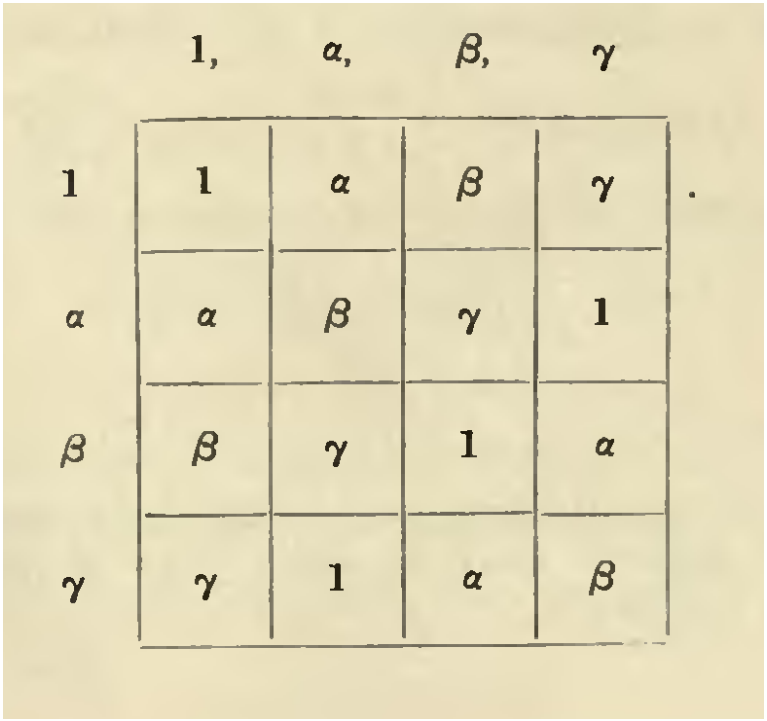
$$1, \alpha, \beta...$$

all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself, belongs to the set, is said to be a group. (Cayley 1854, p. 124)

Cayley's group



Nearer factors \ Further factors	1	α	β	..
1	1	α	β	..
α	α	α^2	βα	
β	β	αβ	β^2	
	:			



	1,	α,	β,	γ
1	1	α	β	γ
α	α	β	γ	1
β	β	γ	1	α
γ	γ	1	α	β

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Cayley's group

- The nature of the elements does not play any role in the group concept. Only the operation is a significant property of the concept.
- With both notations, Cayley puts forward organisations which are independent from the nature of the elements (which can be numbers, functions...).
- Cayley's goal is not to study abstract properties that can be deduced from the definition of a group, but to describe all groups with a given number of elements.
- Cayley's group concept is generic and organisational – but quite different from today's group concept.

Dedekind's group

- Richard Dedekind (1831-1916) was the first to teach Galois theory in Germany, when he was *Privatdozent* in Göttingen in 1856-58. Mentions Galois' and Cayley's papers.
- He imports from Cayley's paper the idea that the nature of the elements of the group does not matter. The closure property of the composition law is what matters.
- *"A complex G of g different substitutions is called a group of degree g, when every arbitrary product of substitutions all contained in G, is again contained in G.*
- Dedekind concentrates on the relations between groups of permutations and on (what we call) automorphisms between subfields of the field of complex numbers.
- His perspective is "more general" than Cayley's.
- He computes *with* the group and not *in* the group.

Dedekind's group in context

- Dedekind integrates Galois theory to number theory. He considers that adjunction is central to it, and relates it to questions related to the nature of numbers used to solve equations, and used in number theory – questions which were important in Germany since Gauss' works.
- Dedekind imports Galois' ideas into a specific epistemological framework, the so-called "conceptual" approach in mathematics that developed in Göttingen with Gauss, Lejeune-Dirichlet, Riemann and Dedekind himself: mathematics should be "guided by the thought" and rely on a minimum (if at all) of "blind computations".
- A concept is best defined by general characteristic properties (eg. conditions of closure) rather than by a specific representation (eg. a table of elements).
- But did conceptual mathematicians really not do any computations?

A case study: research practices and publication practices

- Many computations in Dedekind's drafts.
- Manuscripts (ca. 1875-1897) containing the research leading to the concept of *Dualgruppe*, equivalent to lattices.
- Starting point: "*A system* $\mathfrak{a}$ *of real or complex numbers* α*, whose sums and differences themselves belong to* $\mathfrak{a}$*, will be called a module.*"

- A module $\mathfrak{a}$ is divisible by a module $\mathfrak{b}$ iff $\mathfrak{a} \subset \mathfrak{b}$. Symbol for the divisibility between $\mathfrak{a}$ and $\mathfrak{b}$: $\mathfrak{a} < \mathfrak{b}$ (or $\mathfrak{b} > \mathfrak{a}$)
- LCM$(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} \cap \mathfrak{b}$. Symbol for LCM of $\mathfrak{a}$ and $\mathfrak{b}$: $\mathfrak{a} - \mathfrak{b}$
- The GCD of $\mathfrak{a}$ and $\mathfrak{b}$ is the module composed by all the numbers $\alpha + \beta$ with α and β respectively running through all numbers of $\mathfrak{a}$ and $\mathfrak{b}$. Symbol for GCD of $\mathfrak{a}$ and $\mathfrak{b}$: $\mathfrak{a} + \mathfrak{b}$

- Dedekind's initial interest in the operations between modules led him to explore the possibilities offered by these operations,
- Series of examples, tables, diagrams and computations often repeated in slight variations and attempted generalizations.
- Gradual generalization of computations through which Dedekind identifies the general properties and the fundamental ones.

The laws for modules

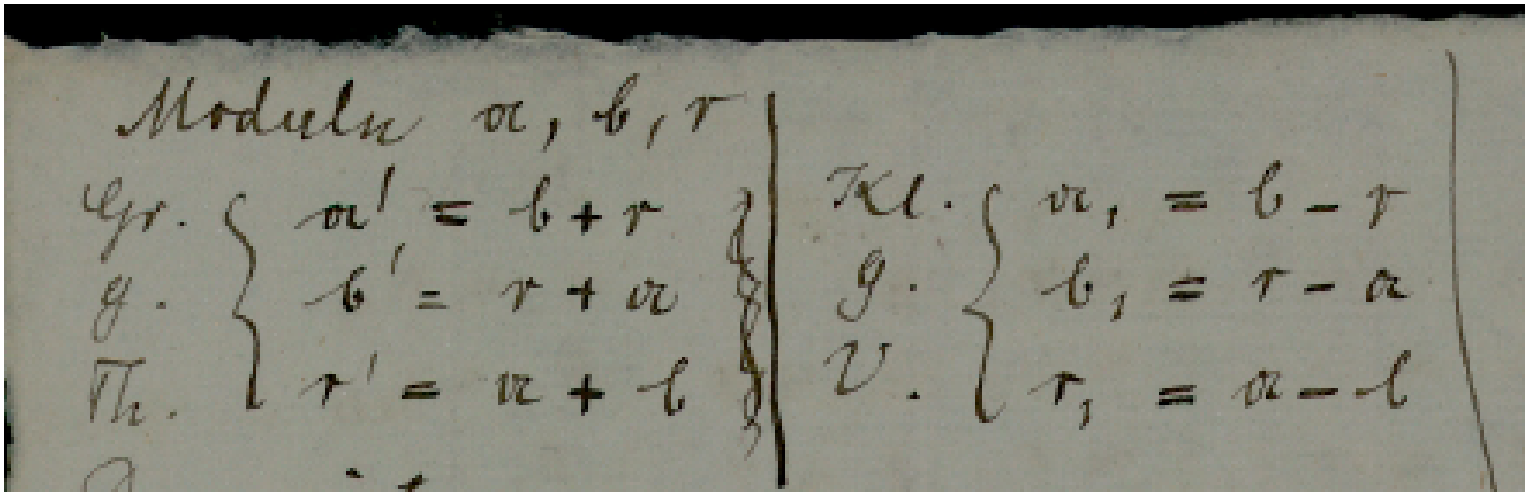
For $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ modules, we have

$$\begin{array}{c|c} \mathfrak{a}+\mathfrak{b}=\mathfrak{b}+\mathfrak{a} & \mathfrak{a}-\mathfrak{b}=\mathfrak{b}-\mathfrak{a} \\ \mathfrak{a}+(\mathfrak{b}+\mathfrak{c})=(\mathfrak{a}+\mathfrak{b})+\mathfrak{c} & \mathfrak{a}-(\mathfrak{b}-\mathfrak{c})=(\mathfrak{a}-\mathfrak{b})-\mathfrak{c} \\ \mathfrak{a}+\mathfrak{a}=\mathfrak{a} & \mathfrak{a}-\mathfrak{a}=\mathfrak{a} \\ \mathfrak{a}+(\mathfrak{a}-\mathfrak{b})=\mathfrak{a} & \mathfrak{a}-(\mathfrak{a}+\mathfrak{b})=\mathfrak{a} \end{array}$$

A certain "dualism between the notions of [LCM] and [GCD]", i.e., a valid result such as $\mathfrak{a}+(\mathfrak{b}-\mathfrak{c})=\mathfrak{a}-(\mathfrak{b}+\mathfrak{c})$ can be changed into another valid result by switching the operations: $\mathfrak{a}-(\mathfrak{b}+\mathfrak{c})=\mathfrak{a}+(\mathfrak{b}-\mathfrak{c})$.
For $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ modules with $\mathfrak{a}<\mathfrak{b}$,

$$\begin{aligned}(\mathfrak{a}+\mathfrak{b})-(\mathfrak{a}+\mathfrak{c})&=\mathfrak{a}+(\mathfrak{b}-(\mathfrak{a}+\mathfrak{c}))\\(\mathfrak{a}-\mathfrak{b})+(\mathfrak{a}-\mathfrak{c})&=\mathfrak{a}-(\mathfrak{b}+(\mathfrak{a}-\mathfrak{c}))\end{aligned}$$

Dedekind's notations



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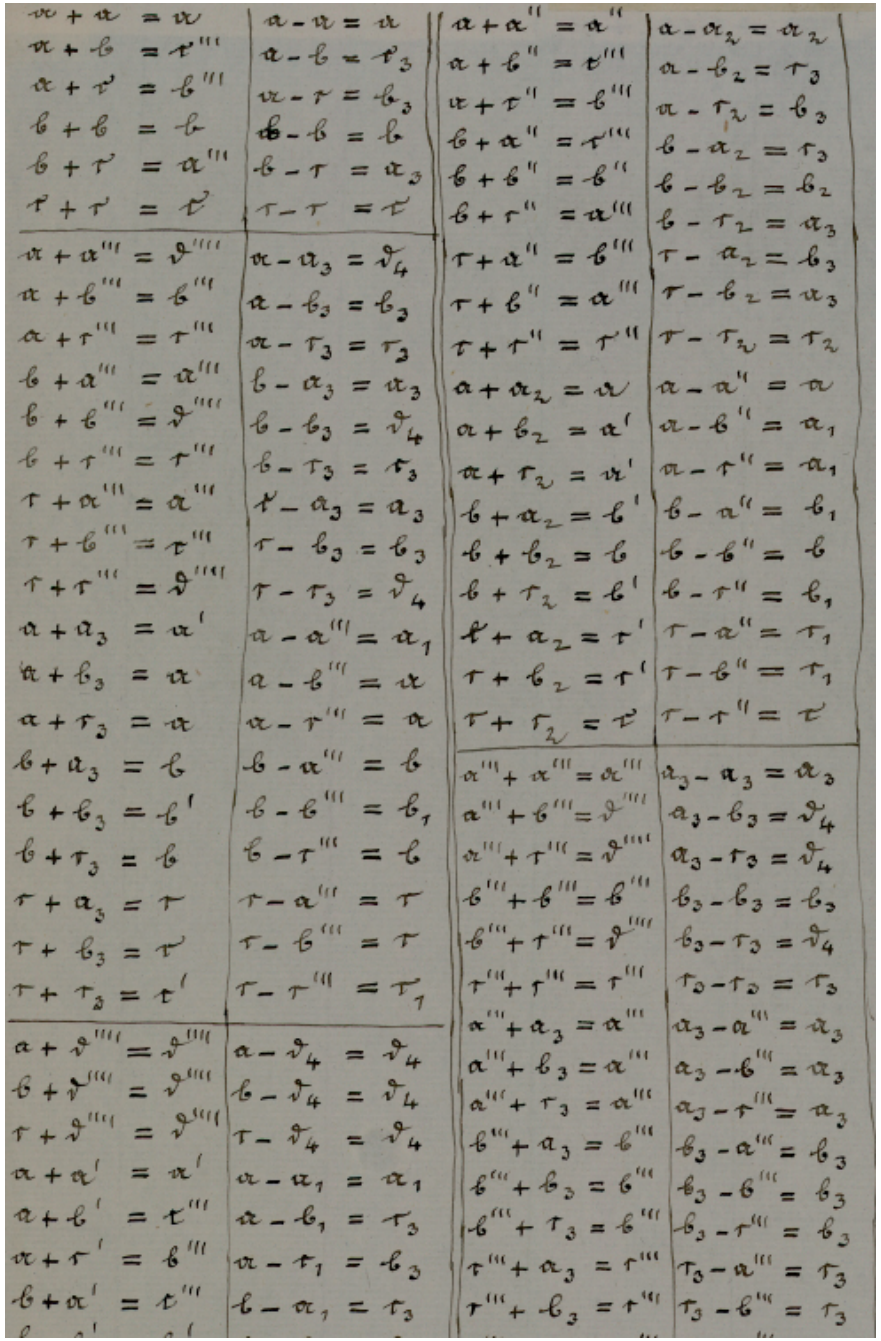

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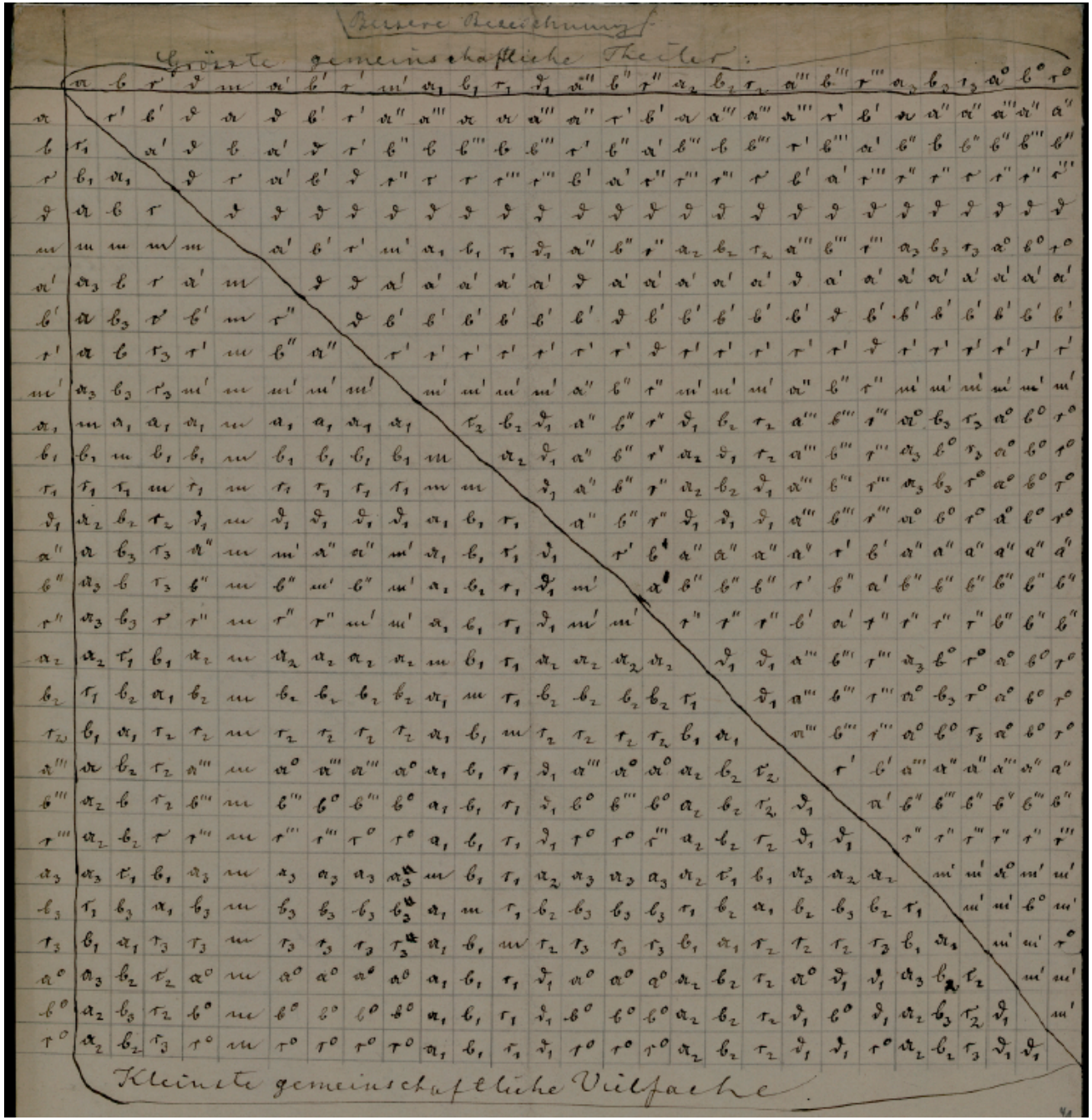

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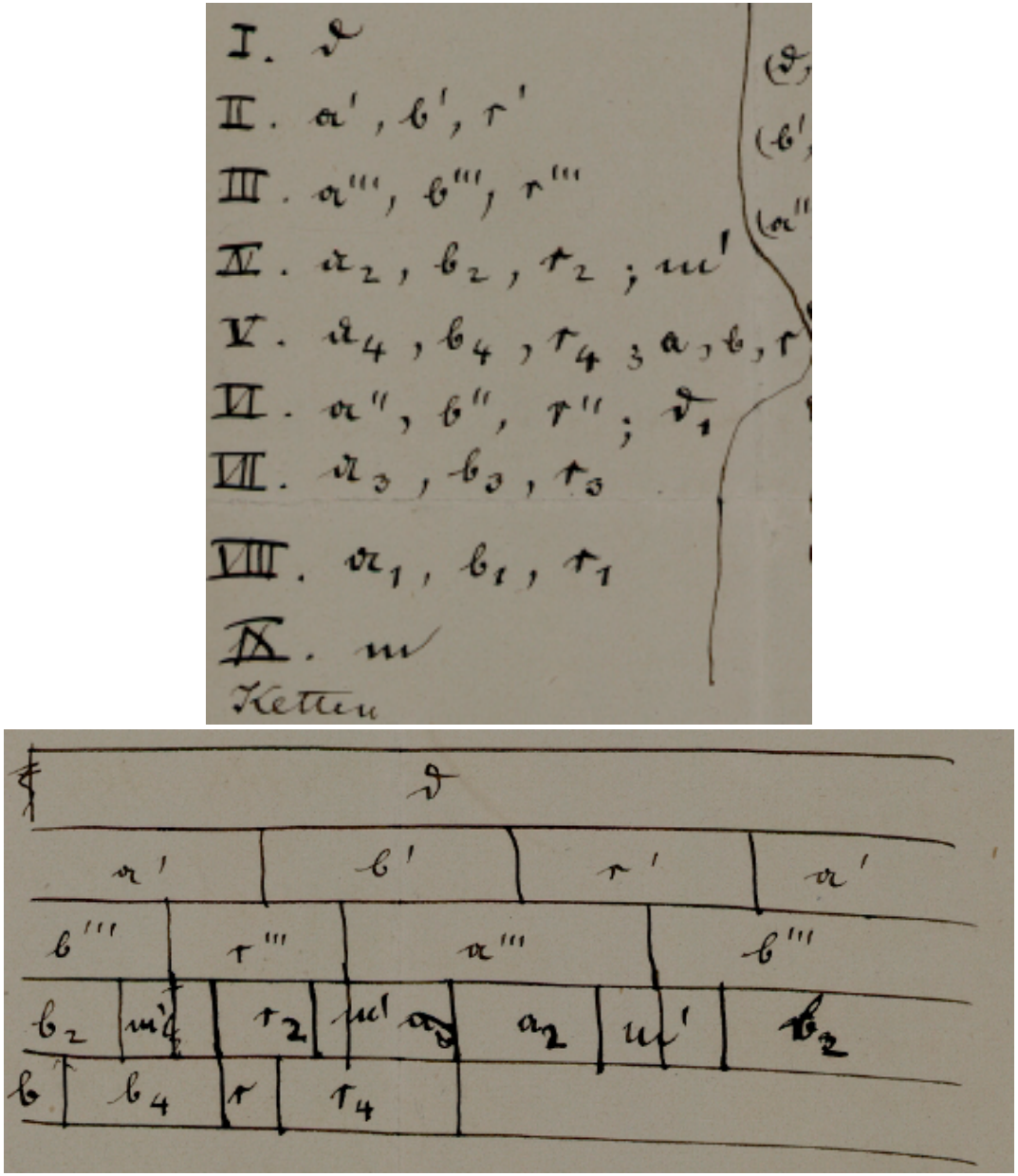

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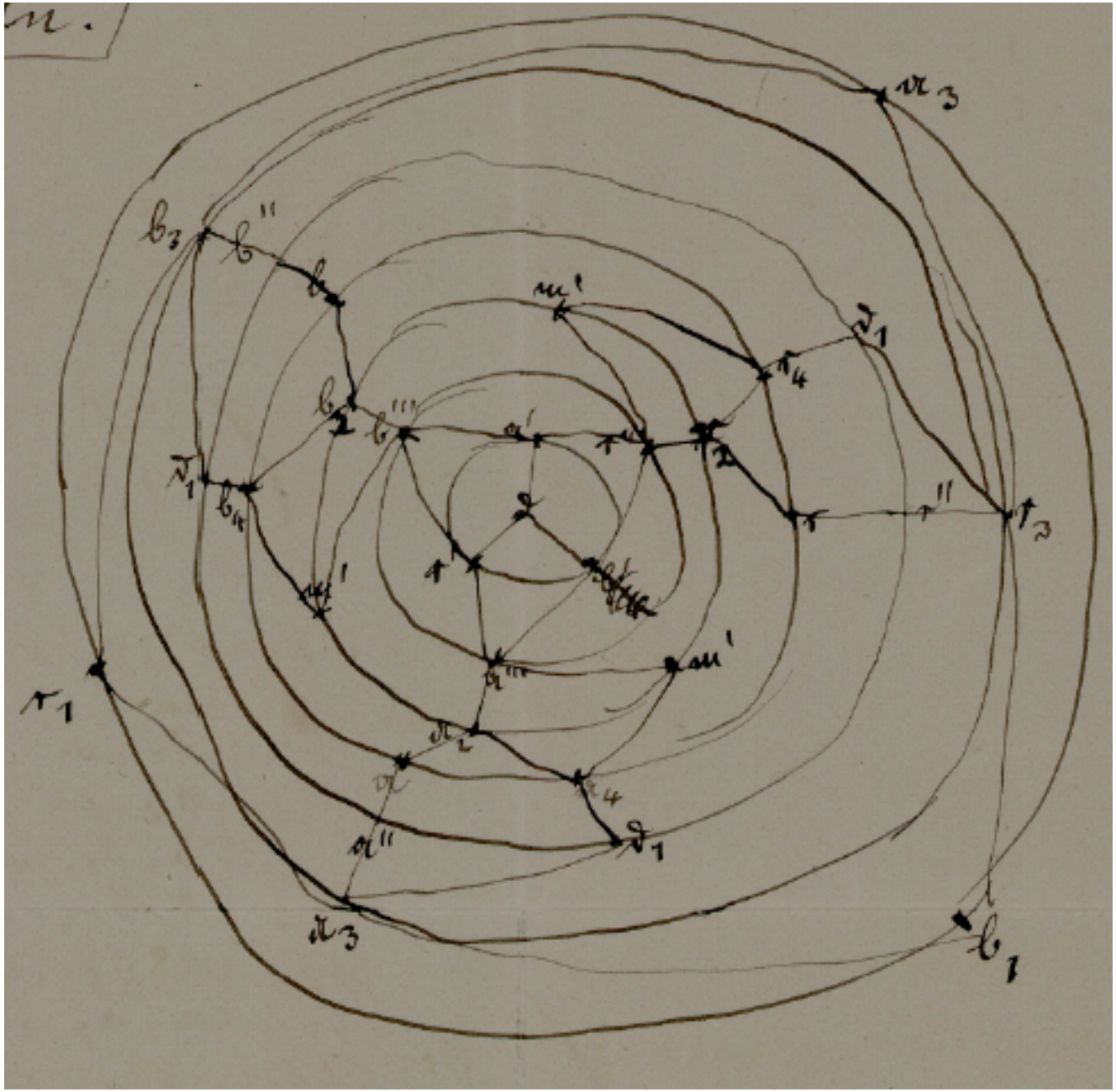

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Hidden computations?

- No computations, no diagrams, very little tables in Dedekind's publications, which are abstract and axiomatic.
- Several levels of writing in mathematics, from the first research to a publishable text.
- Drafts are a domain of private writing, freed from certain norms of writing, in which we can observe practices that are erased by the constitution of a structured, polished text suitable for communication to the scientific community.
- The principles of "conceptual" approach are largely about exposing results – rather than finding them?

Conclusion

- *"We have to construe mathematics as both a body of knowledge and a field of social practice at the same time."*
- The objectivity and purported universality of mathematics can seem to go against the idea that mathematics is a human activity, deeply anchored in historical, socio-cultural contexts.
- Contrast but not contradiction between how mathematical results, once validated by the community, are seen as definitive, and the idea that the making of these results is done locally, in a precisely situated historical context and that the validation criteria are changing.