

Post-doctoral Research Project

Title: Wave turbulence for dispersive PDEs

Wave turbulence theory is a kinetic theory describing the statistical physics of (weakly) nonlinear systems of random wave equations out of thermal equilibrium. It is the extension of Boltzmann's kinetic theory for dilute gases to nonlinear dispersive systems and it owes its name (*Wave turbulence*) to the description it provides of spectral energy cascades for nonlinear waves, much like what Kolmogorov's theory of hydrodynamic turbulence does. The aim of wave turbulence is to understand the effective (macroscopic) behavior and energy-dynamics of systems where many waves interact nonlinearly at the microscopic level according to some dispersive equations, through the derivation of some *kinetic wave* equation satisfied by the energy densities.

The kinetic wave equation has been formally derived in different physical contexts such as water waves, plasma models etc. (see [8] for a textbook treatment of the subject), in the limit where the number of interacting waves goes to infinity and the strength of their interaction goes to 0.

From a mathematical standpoint, the rigorous derivation of an irreversible kinetic equation from a reversible nonlinear dispersive PDE remains an open problem in many cases. The only results known at present concern the derivation of the kinetic equation for the cubic nonlinear Schrödinger equation (NLS) (see the celebrated result of Deng-Hani [3] and references therein to previous partial results) and for a stochastic KdV-type equation by Staffilani-Tran [9]. The general approach is the following: given a certain weakly nonlinear dispersive equation, set on a large d -dimensional torus of size L and where the strength of the nonlinearity is represented by some small parameter ε , one considers a periodic solution $u(t, x)$ with well-prepared random initial data (its Fourier coefficients are independent, identically distributed normalized Gaussian variables) and looks at the evolution of the expectation of the Fourier-space mass densities $\mathbb{E}|\hat{u}(t, k)|^2$, where $k \in L^{-1}\mathbb{Z}^d$. These are the quantities that completely determine the law of a family of jointly gaussian variables and hence the quantities to look at if one believes in the propagation of chaos (the initial independence and gaussianity of the Fourier coefficients survives over time in the limit $L \rightarrow \infty$). First, one has to construct the solution $u(t, x)$ to the starting problem on some time interval $[0, \delta T]$, where $T = T(L, \varepsilon) > 0$ is to be determined and $\delta > 0$ is possibly small but independent of ε, L . This is done using the following ansatz

$$u(t, x) = \sum_{n \leq N} u_n(t, x) + v(t, x)$$

where u_0 is the linear evolution, u_n are iterated self-interactions of the linear solution u_0 and v is some remainder term whose existence is to be proven. The size N of the decomposition usually has to grow logarithmically with L to ensure a good control on the error term v . Second, one has to study the limit (the order with which one takes the limit is fundamental)

$$\lim_{\substack{L \rightarrow \infty \\ \varepsilon \rightarrow 0}} \mathbb{E}|\hat{u}(tT, k)|^2, \quad t \in [0, \delta]$$

and prove that $\mathbb{E}|\hat{u}(tT, k)|^2$ is well approximated by the solution of an effective equation (heuristically obtained beforehand).

A major role in deriving the effective dynamics of the correlations associated with a dispersive PDE (or system of dispersive PDEs) is that of resonances. In the case of cubic (NLS) studied in [3], the set of exact resonances is given by

$$E = \left\{ (k_1, k_2, k_3) \in \frac{1}{L}\mathbb{Z}^d \mid \omega(k_1 - k_2 + k_3) - \omega(k_1) + \omega(k_2) - \omega(k_3) = 0 \right\}$$

and the set of quasi-resonances by

$$Q = \left\{ (k_1, k_2, k_3) \in \frac{1}{L}\mathbb{Z}^d \mid 0 \neq |\omega(k_1 - k_2 + k_3) - \omega(k_1) + \omega(k_2) - \omega(k_3)| \leq \varepsilon^2 \right\},$$

where $\omega(k) = |k|^2$ is the dispersive relation for the Schrödinger equation, $L \gg 1$ is the size of the box on which the equation is considered and $\varepsilon \ll 1$ is the strength of the nonlinear interaction. If the set of quasi-resonances dominates that of exact resonances, the effective dynamics one obtains is said to be of *kinetic* type. If instead the exact resonances prevail over quasi-resonances, the effective dynamics is *discrete* (a third regime where the “size” of the above sets is comparable is known in physics as *mesoscopic* regime). The set E always contains trivial resonances $k_1 = k_2$ or $k_2 = k_3$, but for the Schrödinger equation their contribution to the dynamics cancels out due to $\mathbb{U}(1)$ invariance. The right set to consider, after a renormalization argument, is instead

$$\tilde{E} = \left\{ (k_1, k_2, k_3) \in \frac{1}{L}\mathbb{Z}^d \mid \omega(k_1 - k_2 + k_3) - \omega(k_1) + \omega(k_2) - \omega(k_3) = 0 \wedge k_2 \neq k_1 \wedge k_2 \neq k_3 \right\}$$

and such set, under appropriate scaling laws relating the parameters L and ε , is never dominant compared to Q . The effective dynamics of the correlations is hence a kinetic equation and its rigorous derivation is obtained by Deng-Hani in [3] in the time-interval $[0, T_{\text{kin}}\delta]$ where $T_{\text{kin}} = \varepsilon^{-2}$ is the kinetic time and $\delta > 0$ is independent of L and ε .

In the case of *generic* systems of dispersive equations, there is no reason to believe that there are enough symmetries to ensure the effective dynamics of the correlations to be of kinetic type. Instead, the correlations should behave more like in the *discrete* wave turbulence or finite box effect regime (see for instance [4, 5, 6, 10]), because trivial (exact) resonances do not cancel each other and play a significant part in the effective dynamics. Such type of behavior has been showed to arise in a recent work by de Suzzoni-Stingo-Touati [2], in which they investigate a quadratic system of two Klein-Gordon equations. After a normal form reduction that “transforms” the nonlinear interaction from quadratic to cubic, the authors rigorously prove that a non-trivial effective dynamics already emerges at times of order α^{-1} , where α is the strength of the new cubic nonlinearity. The analysis uses, as common in the literature concerning these problems, a precise diagrammatic representation of the solution to the Klein-Gordon system, which in [2] is adapted to the normal form procedure and to the Klein-Gordon dispersive relation. This dispersive relation forbids, as opposed to the Schrödinger equation, the use of number theoretic results. Instead, the authors rely on a low/high frequency analysis at the diagrammatic level for which they introduce new combinatorial tools, which is a new approach compared to previous works (see the celebrated result of Lukkarinen-Spohn [7] and the work on inhomogeneous setting by Ampatzoglou-Collot-Germain [1]) where the frequency analysis is performed at the level of the equation.

Following the results of [2] and the methods developed therein, a very interesting project for a post-doctoral fellow would be that of studying other dispersive models (or rather systems of dispersive models) that are not phase invariant and to rigorously prove the emerge of a discrete effective regime as describe above.

References

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