Irregular Particle Systems with Absorption

CONTENTS

This topic addresses Task 3 of the project "Particle Systems Meet Systemic Risk", coordinated by Gaoyue Guo and supported by an ANR JCJC grant of €233,810. While this funding represents a significant contribution, it does not fully cover the total estimated project budget of €310,990. As a result, the originally planned two-year postdoctoral position dedicated to this topic had to be cancelled, limiting our capacity to achieve the project's objectives in a timely manner. The requested complementary funding would allow us to restore this position, ensuring that the proposed research can be conducted effectively and that Task 3 can be fully.

1 DESCRIPTION AND FORMULATION

1.1 MOTIVATIONS AND PERSPECTIVES

The concept of singularity, which concerns conditions under which a system may undergo catastrophic breakdown, is a central and fascinating topic in mathematical physics. This project focuses on irregular particle systems, whose singular behavior can arise from *singular coefficients* (cross-interactions between particles) and *hitting times* (absorption at certain barriers). To model such singular phenomena in large particle systems and to analyze *systemic risk*, we adopt mean-field strategies. These strategies study the behavior of high-dimensional random models by approximating them with simpler models that average over many degrees of freedom.

Provided a particle system[1], we are particularly interested in the conditions under which events at the individual level could trigger a collapse of the entire system. More specifically, the systemic risk refers to the danger of opposite traits as below:

- Extinction of particles: the system may collapse if all particles are absorbed. A key question is whether the absorption of a small fraction of particles can induce severe instability in the system, manifesting as domino effects.
- Over-concentration or collision of particles: the system may fail due to excessive clustering. We aim to determine whether the system can maintain long-term stability under such conditions.

Complexity of particle systems makes it a challenge to define adequate indicators of systemic risk, and we aim to develop quantitative methodologies for analyzing the systemic risk.

1.2 MODEL DESCRIPTION

To study relevant questions, a mathematical framework is needed. Consider a particle system composed of N particles, labeled by $[N] := \{1, \dots, N\}$. The state of particle i is described by a stochastic process X^i taking values in $\mathbb{R}^d$ with $d \in \{1,2,3\}$. Particle i is said to be *absorbed* if X^i reaches some barrier $\partial\Omega$, where $\Omega \subset \mathbb{R}^d$ is fixed. In the absence of absorption, the particles $X^1, \dots, X^N$ evolve according to stochastic differential equations (SDEs).

When a particle X^j hits $\partial\Omega$ at time t, every surviving particle i experiences an instantaneous shock:

$$X^i_{t-} \implies X^i_{t-} - L(X^j_t),$$

[1] "Particle" has various interpretations under different models, and "particle" may refer to "individual", "agent", "cell", etc.

where $L : \mathbb{R}^d \to \mathbb{R}^d$ is a given function. where $L : \mathbb{R}^d \to \mathbb{R}^d$ is a given function. This shock can in turn trigger the immediate absorption of other particles, leading to *cascades of absorption*. After such an event, the remaining particles continue their dynamics until another particle reaches the barrier, and the process repeats. Rephrasing mathematically, one has for $t \in [0, \tau_i]$,

$$X_t^i \;=\; X_0^i + \int_0^t b(X_s^i, \mu_s^N)\mathrm{d}s + \sigma_N W_t^i - \underbrace{\sum_{j\neq i} \mathbf{1}_{\{\tau_j \le t\}} L(X_{\tau_j}^j)}_{\text{instantaneous impact due to absorption}}, \tag{1}$$

where

$$\underbrace{\tau_i \;:=\; \inf\left\{t \ge 0 :\; X_t^i \in \partial\Omega\right\}}_{\text{absorption time of } i} \quad \text{and} \quad \mu_t^N \;:=\; \sum_{j=1}^N \underbrace{\mathbf{1}_{\{\tau_j > t\}}\delta_{X_t^j}}_{\text{surviving particle up to } t}.$$

Here b (drift) is a measurable function, $\sigma_N > 0$ (volatility) and $W^1, \dots, W^N$ are independent $\mathbb{R}^d-$Brownian motions. Letting $N \to \infty$, we recover formally the mean-field limit: for $t \in [0, \tau]$,

$$X_t \;=\; X_0 + \int_0^t b(X_s, \mu_s)\mathrm{d}s + \sigma_\infty W_t - \mathbb{E}\big[\mathbf{1}_{\{\tau \le t\}} L(X_\tau)\big], \tag{2}$$

where $\sigma_\infty \ge 0$, $\tau := \inf\{t \ge 0 :\; X_t \in \partial\Omega\}$ and $\mu_t := \mathbb{P}[X_t \in \cdot,\, \tau > t]$ denotes the sub-probability measure. This formulation provides a tractable framework for analyzing the systemic risk in high-dimensional particle systems.

1.3 Research questions

We now specify the context and list the main questions under consideration. The singularity in the drift $b(x, m)$ is given by

$$\underbrace{b(x, m) \;:=\; \int_{\mathbb{R}^d} w(x - y) m(\mathrm{d}y),}_{\text{molecular interaction}} \quad \text{where } w : \mathbb{R}^d \setminus \{0\} \longrightarrow \mathbb{R}^d.$$

Roughly speaking, b stands for electromagnetic type forces of attraction or repulsion between particles.

- For $d = 1$, $\Omega = \mathbb{R}_+$ and $L \equiv \beta$, the models (1) and (2) were first introduced by Delarue et al. [5, 6] in a probabilistic framework to study large networks of interacting spiking neurons, and later developed by Hambly et al. [9] and Nadtochiy and Shkolnikov [10] (see also [2, 3, 4] for PDE formulations). Due to the singular, non-Markovian indicators $\mathbf{1}_{\{\tau_j > t\}}$ and $\mathbf{1}_{\{\tau > t\}}$, the well-posedness of (1) and (2) is not straightforward within standard McKean-Vlasov theory, and uniqueness may fail. The literature introduces two notions of solution—*physical solution* and *minimal solution*—to establish well-posedness and propagation of chaos. In our work, we consider the Dyson model with absorption, i.e., $w(x) = 1/x$ and $\sigma_N = 2/\sqrt{N}$. By adapting the arguments in [12], we aim to prove global well-posedness of (1) and (2), establish the absence of collisions for both $\beta = 0$ and $\beta > 0$, and, following [1], prove convergence of the finite-N system (1) to its mean-field limit (2).

- For $d \in \{2, 3\}$ and $L \equiv 0$, we consider the Keller-Segel model (stochastic counterpart) with absorption, i.e., $w(x) = -\chi x/|x|^d$, $\sigma_N = \sqrt{2}$. In the classical Keller-Segel model (without absorption), cells X^i move in $\mathbb{R}^d$ under *chemotaxis*, emitting a chemical product that attracts other cells. Denoting the empirical measure of cells by $\nu_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$, the chemical concentration λ_t^N evolves according to Poisson's equation:

$$\begin{cases} \mathrm{d}X_t^i = -\nabla\lambda_t^N(X_t^i)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t^i \\ \Delta\lambda_t^N = \chi\nu_t^N \quad \Longrightarrow \quad \lambda_t^N = V * \nu_t^N \end{cases} \Longrightarrow \; \mathrm{d}X_t^i = \int_{\mathbb{R}^d} \mathbf{1}_{\{y \neq X_t^i\}} w(X_t^i - y)\nu_t^N(\mathrm{d}y)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t^i, \tag{3}$$

where $V : \mathbb{R}^d \to \mathbb{R}$ is the potential so that $w = -\nabla V$ and $*$ denotes the convolution operator.

This interaction is singular in following sense: let $N \to \infty$ and denote by $[0, T_{\max}]$ the maximal interval in which the mean-field limit of (3) is well defined. Depending on the value of χ, either no collision occurs at any time, i.e., $T_{\max} = \infty$, or formation of a cluster of cells in finite time, i.e., $T_{\max} < \infty$. We extend (3) to the case that cells move inside Ω and are absorbed at $\partial\Omega$. To handle similar issues, the key is to choose a suitable sequence of mollifiers $(\eta_\varepsilon)_{\varepsilon>0}$ so that we may consider b_ε defined by the convolution below

$$b_\varepsilon(x, m) \ := \ \int_{\mathbb{R}^d} b(y, m)\eta_\varepsilon(x - y)\mathrm{d}y.$$

Using the tightness argument for well chosen $(\varepsilon_k)_{k\geq 1}$, existence of the solutions to (1) and (2) follows. To show wellposedness and propagation of chaos, the challenge is uniqueness of the solutions to (1) and (2), and we shall extend the arguments in [11] (without absorbing barrier) to our case.

Finally, an important question concerns the effect of the absorbing boundary on the system's lifespan. For instance, in $d = 2$, $T_{\max} = \infty$ if $\chi < 8\pi$ and $T_{\max} < \infty$ if $\chi > 8\pi$, see e.g. [8]. Introducing absorption reduces particle density, potentially preventing collisions. Following [7], we aim to rigorously analyze whether absorbed particles can lengthen the lifetime of a system that would otherwise blow up, or even render it infinite.

2 Related references

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