School of Mathematics

Interior Point Methods for Convex Quadratic and Convex Nonlinear Programming

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Outline

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  - quadratic forms
  - duality in QP
  - first order optimality conditions
  - primal-dual framework
- **Part 2: IPM for NLP**
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  - Lagrangian
  - first order optimality conditions
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- Algebraic Modelling Languages
- **Self-concordant barrier**

IPM for Convex QP

Convex Quadratic Programs

The quadratic function

$$f(x) = x^T Q\, x$$

is convex if and only if the matrix Q is positive definite.
In such case the quadratic programming problem

$$\begin{array}{ll} \min & c^T x + \frac{1}{2} x^T Q\, x \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \end{array}$$

is well defined.

If there exists a *feasible* solution to it,
then there exists an *optimal* solution.

QP Background:

Def. A matrix $Q \in \mathcal{R}^{n \times n}$ is positive semidefinite if $x^T Q x \geq 0$ for any $x \neq 0$. We write $Q \succeq 0$.

Def. A matrix $Q \in \mathcal{R}^{n \times n}$ is positive definite if $x^T Q x > 0$ for any $x \neq 0$. We write $Q \succ 0$.

Example:

Consider quadratic functions $f(x) = x^T Q\, x$ with the following matrices:

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}.$$

Q_1 is positive definite (hence f_1 is convex).
Q_2 and Q_3 are indefinite (f_2, f_3 are not convex).

QP with IPMs

Consider the *convex* quadratic programming problem.
The **primal**

$$\begin{array}{ll} \min & c^T x + \frac{1}{2} x^T Q\, x \\ \text{s.t.} & Ax = b, \\ & x \ge 0, \end{array}$$

and the **dual**

$$\begin{array}{ll} \max & b^T y - \frac{1}{2} x^T Q\, x \\ \text{s.t.} & A^T y + s - Qx = c, \\ & x, s \ge 0. \end{array}$$

Apply the *usual* procedure:

- replace inequalities with log barriers;
- form the Lagrangian;
- write the first order optimality conditions;
- apply Newton method to them.

QP with IPMs: Log Barriers

Replace the **primal** QP

$$\begin{array}{ll} \min & c^T x + \frac{1}{2} x^T Q\, x \\ \text{s.t.} & Ax \;=\; b, \\ & x \geq 0, \end{array}$$

with the **primal barrier QP**

$$\begin{array}{ll} \min & c^T x + \frac{1}{2} x^T Q\, x - \sum_{j=1}^{n} \ln x_j \\ \text{s.t.} & Ax \;= b. \end{array}$$

QP with IPMs: Log Barriers

Replace the **dual** QP

$$\begin{array}{llcl} \max & b^T y \; - & \frac{1}{2} x^T Q\, x & \\ \text{s.t.} & A^T y \, + \, s \, - \, Qx & = & c, \\ & y \;\; \text{free}, & s \geq 0, & \end{array}$$

with the **dual barrier QP**

$$\begin{array}{ll} \max & b^T y - \frac{1}{2} x^T Q\, x + \sum_{j=1}^{n} \ln s_j \\ \text{s.t.} & A^T y + s - Qx \; = c. \end{array}$$

First Order Optimality Conditions

Consider the **primal barrier quadratic program**

$$\begin{array}{ll} \min & c^T x + \frac{1}{2} x^T Q\, x - \mu \sum_{j=1}^{n} \ln x_j \\ \text{s.t.} & Ax \;=\; b, \end{array}$$

where $\mu \geq 0$ is a barrier parameter.

Write out the **Lagrangian**

$$L(x, y, \mu) = c^T x + \frac{1}{2} x^T Q\, x - y^T (Ax - b) - \mu \sum_{j=1}^{n} \ln x_j ,$$

First Order Optimality Conditions (cont'd)

The conditions for a stationary point of the Lagrangian:

$$L(x, y, \mu) = c^T x + \frac{1}{2} x^T Q\, x - y^T (Ax - b) - \mu \sum_{j=1}^{n} \ln x_j,$$

are

$$\begin{array}{rcrcl} \nabla_x L(x, y, \mu) & = & c - A^T y - \mu X^{-1} e + Qx & = & 0 \\ \nabla_y L(x, y, \mu) & = & Ax - b & = & 0, \end{array}$$

where $X^{-1} = diag\{x_1^{-1}, x_2^{-1}, \cdots, x_n^{-1}\}$.
Let us denote

$$s = \mu X^{-1} e, \quad \text{i.e.} \quad XSe = \mu e.$$

The **First Order Optimality Conditions** are:

$$\begin{array}{rcl} Ax & = & b, \\ A^T y + s - Qx & = & c, \\ XSe & = & \mu e. \end{array}$$

Apply Newton Method to the FOC

The first order optimality conditions for the barrier problem form a large system of nonlinear equations

$$F(x, y, s) = 0,$$

where $F : \mathcal{R}^{2n+m} \mapsto \mathcal{R}^{2n+m}$ is an application defined as follows:

$$F(x, y, s) = \begin{bmatrix} Ax & - b \\ A^T y + s - Qx & - c \\ XSe & - \mu e \end{bmatrix}.$$

Actually, the first two terms of it are *linear*; only the last one, corresponding to the complementarity condition, is *nonlinear*.
Note that

$$\nabla F(x, y, s) = \begin{bmatrix} A & 0 & 0 \\ -Q & A^T & I \\ S & 0 & X \end{bmatrix}.$$

Newton Method for the FOC (cont'd)

Thus, for a given point (x, y, s)
we find the Newton direction $(\Delta x, \Delta y, \Delta s)$
by solving the system of linear equations:

$$\begin{bmatrix} A & 0 & 0 \\ -Q & A^T & I \\ S & 0 & X \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^T y - s + Qx \\ \mu e - XSe \end{bmatrix}.$$

Interior-Point QP Algorithm

Initialize

$$k = 0, \quad (x^0, y^0, s^0) \in \mathcal{F}^0, \quad \mu_0 = \frac{1}{n} \cdot (x^0)^T s^0, \quad \alpha_0 = 0.9995$$

Repeat until optimality

$k = k + 1$

$\mu_k = \sigma \mu_{k-1}$, where $\sigma \in (0, 1)$

Δ = Newton direction towards μ-center

Ratio test:

$$\begin{aligned} \alpha_P &:= \max \{\alpha > 0 : \ x + \alpha \Delta x \geq 0\}, \\ \alpha_D &:= \max \{\alpha > 0 : \ s + \alpha \Delta s \geq 0\}. \end{aligned}$$

Make step:

$$\begin{aligned} x^{k+1} &= x^k + \alpha_0 \alpha_P \Delta x, \\ y^{k+1} &= y^k + \alpha_0 \alpha_D \Delta y, \\ s^{k+1} &= s^k + \alpha_0 \alpha_D \Delta s. \end{aligned}$$

From LP to QP

QP problem

$$\begin{array}{ll} \min & c^T x + \frac{1}{2} x^T Q\, x \\ \text{s.t.} & Ax = b, \\ & x \geq 0. \end{array}$$

First order conditions (for barrier problem)

$$\begin{array}{rcl} Ax & = & b, \\ A^T y + s - Qx & = & c, \\ XSe & = & \mu e. \end{array}$$

IPMs for Convex NLP

Convex Nonlinear Optimization

Consider the nonlinear optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g(x) \leq 0, \end{array}$$

where $x \in \mathcal{R}^n$, and $f : \mathcal{R}^n \mapsto \mathcal{R}$ and $g : \mathcal{R}^n \mapsto \mathcal{R}^m$ are convex, twice differentiable.

Assumptions:
f and g are convex
$\Rightarrow$ If there exists a **local** minimum then it is a **global** one.
f and g are twice differentiable
$\Rightarrow$ We can use the **second order Taylor approximations**.

Some additional (technical) conditions
$\Rightarrow$ We need them to prove that the point which satisfies the first order optimality conditions is the optimum. *We won't use them in this course.*

Taylor Expansion of $f : \mathcal{R} \mapsto \mathcal{R}$

Let $f : \mathcal{R} \mapsto \mathcal{R}$.
If all derivatives of f are continuously differentiable at x_0, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

where $f^{(k)}(x_0)$ is the k-th derivative of f at x_0.

The *first order approximation* of the function:

$$f(x) = f(x_0) + f^{'}(x_0)(x - x_0) + r_2(x - x_0),$$

where the remainder satisfies:

$$\lim_{x \to x_0} \frac{r_2(x - x_0)}{x - x_0} = 0.$$

Taylor Expansion (cont'd)

The *second order approximation*:

$$f(x) = f(x_0) + f^{'}(x_0)(x - x_0) \\ + \frac{1}{2} f^{''}(x_0)(x - x_0)^2 + r_3(x - x_0),$$

where the remainder satisfies:

$$\lim_{x \to x_0} \frac{r_3(x - x_0)}{(x - x_0)^2} = 0.$$

Derivatives of $f : \mathcal{R}^n \mapsto \mathcal{R}$

The vector

$$(\nabla f(x))^T = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right)$$

is called the **gradient** of f at x.

The matrix

$$\nabla^2 f(x) = \left[\begin{array}{cccc} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) \\ \dots & \dots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) & \dots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{array} \right]$$

is called the **Hessian** of f at x.

Taylor Expansion of $f : \mathcal{R}^n \mapsto \mathcal{R}$

Let $f : \mathcal{R}^n \mapsto \mathcal{R}$.
If all derivatives of f are continuously differentiable at x_0, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,$$

where $f^{(k)}(x_0)$ is the k-th derivative of f at x_0.

The *first order approximation* of the function:

$$f(x) = f(x_0) + \nabla f(x_0)^T (x - x_0) + r_2(x - x_0),$$

where the remainder satisfies:

$$\lim_{x \to x_0} \frac{r_2(x - x_0)}{\|x - x_0\|} = 0.$$

Taylor Expansion (cont'd)

The *second order approximation*: of the function:

$$f(x) = f(x_0) + \nabla f(x_0)^T (x - x_0) + \frac{1}{2}(x - x_0)^T \nabla^2 f(x_0)(x - x_0) + r_3(x - x_0),$$

where the remainder satisfies:

$$\lim_{x \to x_0} \frac{r_3(x - x_0)}{\|x - x_0\|^2} = 0.$$

Convexity: Reminder

Property 1. For any collection $\{C_i \mid i \in I\}$ of convex sets, the intersection $\bigcap_{i \in I} C_i$ is convex.

Property 4. If C is a convex set and $f : C \mapsto \mathcal{R}$ is convex function, the level sets $\{x \in C \mid f(x) \leq \alpha\}$ and $\{x \in C \mid f(x) < \alpha\}$ are convex for all scalars α.

Lemma 1: If $g : \mathcal{R}^n \mapsto \mathcal{R}^m$ is a convex function, then the set $\{x \in \mathcal{R}^n \,|\, g(x) \leq 0\}$ is convex.

Proof: Since every function $g_i : \mathcal{R}^n \mapsto \mathcal{R}, i = 1, 2, ..., m$ is convex, from Property 4, we conclude that every set $X_i = \{x \in \mathcal{R}^n \,|\, g_i(x) \leq 0\}$ is convex. Next from Property 1, we deduce that the intersection

$$X = \bigcap_{i=1}^{m} X_i = \{x \in \mathcal{R}^n \,|\, g(x) \leq 0\}$$

is convex, which completes the proof.

Differentiable Convex Functions

Property 8. Let $C \in \mathcal{R}^n$ be a convex set and $f : C \mapsto \mathcal{R}$ be twice continuously differentiable over C.
(a) If $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$, then f is convex.
(b) If $\nabla^2 f(x)$ is positive definite for all $x \in C$, then f is strictly convex.
(c) If f is convex, then $\nabla^2 f(x)$ is positive semidefinite for all $x \in C$.
Let the second order approximation of the function be given:

$$f(x) \approx f(x_0) + c^T(x-x_0) + \frac{1}{2}(x-x_0)^T Q(x-x_0),$$

where $c = \nabla f(x_0)$ and $Q = \nabla^2 f(x_0)$.
From Property 8, it follows that when f is convex and twice differentiable, then Q exists and is a positive semidefinite matrix.

Conclusion:

If f is convex and twice differentiable, then optimization of $f(x)$ can (locally) be replaced with the minimization of its quadratic model.

Nonlinear Optimization with IPMs

Nonlinear Optimization via QPs:
Sequential Quadratic Programming (SQP).
Repeat until optimality:

- approximate NLP (locally) with a QP;
- solve (approximately) the QP.

Nonlinear Optimization with IPMs:
works similarly to SQP scheme.

However, the (local) QP approximations are not solved to optimality. Instead, only one step in the Newton direction corresponding to a given QP approximation is made and the new QP approximation is computed.

Nonlinear Optimization with IPMs

Derive an IPM for NLP:

- replace inequalities with log barriers;
- form the Lagrangian;
- write the first order optimality conditions;
- apply Newton method to them.

NLP Notation

Consider the nonlinear optimization problem

$$\min \ f(x) \quad \text{s.t.} \quad g(x) \leq 0,$$

where $x \in \mathcal{R}^n$, and $f : \mathcal{R}^n \mapsto \mathcal{R}$ and $g : \mathcal{R}^n \mapsto \mathcal{R}^m$ are convex, twice differentiable.

The vector-valued function $g : \mathcal{R}^n \mapsto \mathcal{R}^m$ has a derivative

$$A(x) = \nabla g(x) = \left[\frac{\partial g_i}{\partial x_j}\right]_{i=1..m,\, j=1..n} \in \mathcal{R}^{m \times n}$$

which is called the **Jacobian** of g.

NLP Notation (cont'd)

The Lagrangian associated with the NLP is:

$$\mathcal{L}(x,y) = f(x) + y^T g(x),$$

where $y \in \mathcal{R}^m$, $y \geq 0$ are Lagrange multipliers (dual variables).

The first derivatives of the Lagrangian:

$$\begin{aligned} \nabla_x \mathcal{L}(x,y) &= \nabla f(x) + \nabla g(x)^T y \\ \nabla_y \mathcal{L}(x,y) &= g(x). \end{aligned}$$

The **Hessian** of the Lagrangian, $Q(x,y) \in \mathcal{R}^{n \times n}$:

$$Q(x,y) = \nabla^2_{xx} \mathcal{L}(x,y) = \nabla^2 f(x) + \sum_{i=1}^{m} y_i \nabla^2 g_i(x).$$

Convexity in NLP

Lemma 2: If $f : \mathcal{R}^n \mapsto \mathcal{R}$ and $g : \mathcal{R}^n \mapsto \mathcal{R}^m$ are convex, twice differentiable, then the **Hessian** of the Lagrangian

$$Q(x,y) = \nabla^2 f(x) + \sum_{i=1}^{m} y_i \nabla^2 g_i(x)$$

is positive semidefinite for any x and any $y \geq 0$. If f is strictly convex, then $Q(x,y)$ is positive definite for any x and any $y \geq 0$.

Proof: Using Property 8, the convexity of f implies that $\nabla^2 f(x)$ is positive semidefinite for any x. Similarly, the convexity of g implies that for all $i = 1,2,...,m$, $\nabla^2 g_i(x)$ is positive semidefinite for any x. Since $y_i \geq 0$ for all $i = 1,2,...,m$ and $Q(x,y)$ is the sum of positive semidefinite matrices, we conclude that $Q(x,y)$ is positive semidefinite.
If f is strictly convex, then $\nabla^2 f(x)$ is positive definite and so is $Q(x,y)$.

IPM for NLP

Add slack variables to nonlinear inequalities:

$$\begin{array}{lll} \min & f(x) & \\ \text{s.t.} & g(x) + z & = 0 \\ & z & \geq 0, \end{array}$$

where $z \in \mathcal{R}^m$. Replace inequality $z \geq 0$ with the logarithmic barrier:

$$\begin{array}{lll} \min & f(x) - \mu \sum_{i=1}^{m} \ln z_i & \\ \text{s.t.} & g(x) + z & = 0. \end{array}$$

Write out the **Lagrangian**

$$L(x, y, z, \mu) = f(x) + y^T (g(x) + z) - \mu \sum_{i=1}^{m} \ln z_i,$$

IPM for NLP

For the **Lagrangian**

$$L(x, y, z, \mu) = f(x) + y^T(g(x) + z) - \mu \sum_{i=1}^{m} \ln z_i,$$

write the conditions for a stationary point

$$\begin{aligned} \nabla_x L(x, y, z, \mu) &= \nabla f(x) + \nabla g(x)^T y = 0 \\ \nabla_y L(x, y, z, \mu) &= g(x) + z = 0 \\ \nabla_z L(x, y, z, \mu) &= y - \mu Z^{-1} e = 0, \end{aligned}$$

where $Z^{-1} = diag\{z_1^{-1}, z_2^{-1}, \cdots, z_m^{-1}\}$.

The **First Order Optimality Conditions** are:

$$\begin{aligned} \nabla f(x) + \nabla g(x)^T y &= 0, \\ g(x) + z &= 0, \\ YZe &= \mu e. \end{aligned}$$

Newton Method for the FOC

The first order optimality conditions for the barrier problem form a large system of nonlinear equations

$$F(x, y, z) = 0,$$

where $F : \mathcal{R}^{n+2m} \mapsto \mathcal{R}^{n+2m}$ is an application defined as follows:

$$F(x, y, z) = \begin{bmatrix} \nabla f(x) + \nabla g(x)^T y \\ g(x) + z \\ YZe - \mu e \end{bmatrix}.$$

Note that all three terms of it are *nonlinear*.
(In LP and QP the first two terms were *linear*.)

Newton Method for the FOC

Observe that

$$\nabla F(x,y,z) = \begin{bmatrix} Q(x,y) & A(x)^T & 0 \\ A(x) & 0 & I \\ 0 & Z & Y \end{bmatrix},$$

where $A(x)$ is the **Jacobian** of g
and $Q(x,y)$ is the **Hessian** of $\mathcal{L}$.

They are defined as follows:

$$\begin{aligned} A(x) &= \nabla g(x) && \in \mathcal{R}^{m \times n} \\ Q(x,y) &= \nabla^2 f(x) + \sum_{i=1}^{m} y_i \nabla^2 g_i(x) && \in \mathcal{R}^{n \times n} \end{aligned}$$

Newton Method (cont'd)

For a given point (x, y, z) we find the Newton direction $(\Delta x, \Delta y, \Delta z)$ by solving the system of linear equations:

$$\begin{bmatrix} Q(x,y) & A(x)^T & 0 \\ A(x) & 0 & I \\ 0 & Z & Y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} -\nabla f(x) - A(x)^T y \\ -g(x) - z \\ \mu e - YZe \end{bmatrix}.$$

Using the third equation we eliminate

$$\Delta z = \mu Y^{-1} e - Ze - ZY^{-1} \Delta y,$$

from the second equation and get

$$\begin{bmatrix} Q(x,y) & A(x)^T \\ A(x) & -ZY^{-1} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -\nabla f(x) - A(x)^T y \\ -g(x) - \mu Y^{-1} e \end{bmatrix}.$$

Interior-Point NLP Algorithm

Initialize

$k = 0$

(x^0, y^0, z^0) such that $y^0 > 0$ and $z^0 > 0$, $\quad \mu_0 = \frac{1}{m} \cdot (y^0)^T z^0$

Repeat until optimality

$k = k + 1$

$\mu_k = \sigma \mu_{k-1}$, where $\sigma \in (0, 1)$

Compute $A(x)$ and $Q(x, y)$

$\Delta =$ Newton direction towards μ-center

Ratio test:

$$\begin{aligned} \alpha_1 &:= \max \{\alpha > 0 : \ y + \alpha \Delta y \geq 0\}, \\ \alpha_2 &:= \max \{\alpha > 0 : \ z + \alpha \Delta z \geq 0\}. \end{aligned}$$

Choose the step: (use trust region or line search) $\alpha \leq \min \{\alpha_1, \alpha_2\}$.

Make step:

$$\begin{aligned} x^{k+1} &= x^k + \alpha \Delta x, \\ y^{k+1} &= y^k + \alpha \Delta y, \\ z^{k+1} &= z^k + \alpha \Delta z. \end{aligned}$$

From QP to NLP

Newton direction for **QP**

$$\begin{bmatrix} -Q & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} \xi_d \\ \xi_p \\ \xi_\mu \end{bmatrix}.$$

Augmented system for QP

$$\begin{bmatrix} -Q - SX^{-1} & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \xi_d - X^{-1}\xi_\mu \\ \xi_p \end{bmatrix}.$$

From QP to NLP

Newton direction for **NLP**

$$\begin{bmatrix} Q(x,y) & A(x)^T & 0 \\ A(x) & 0 & I \\ 0 & Z & Y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} -\nabla f(x) - A(x)^T y \\ -g(x) - z \\ \mu e - YZe \end{bmatrix}.$$

Augmented system for NLP

$$\begin{bmatrix} Q(x,y) & A(x)^T \\ A(x) & -ZY^{-1} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -\nabla f(x) - A(x)^T y \\ -g(x) - \mu Y^{-1} e \end{bmatrix}.$$

Conclusion:
NLP is a natural extension of QP.

Linear Algebra in IPM for NLP

Newton direction for **NLP**

$$\begin{bmatrix} Q(x,y) & A(x)^T & 0 \\ A(x) & 0 & I \\ 0 & Z & Y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} -\nabla f(x) - A(x)^T y \\ -g(x) - z \\ \mu e - YZe \end{bmatrix}.$$

The corresponding augmented system

$$\begin{bmatrix} Q(x,y) & A(x)^T \\ A(x) & -ZY^{-1} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -\nabla f(x) - A(x)^T y \\ -g(x) - \mu Y^{-1} e \end{bmatrix}.$$

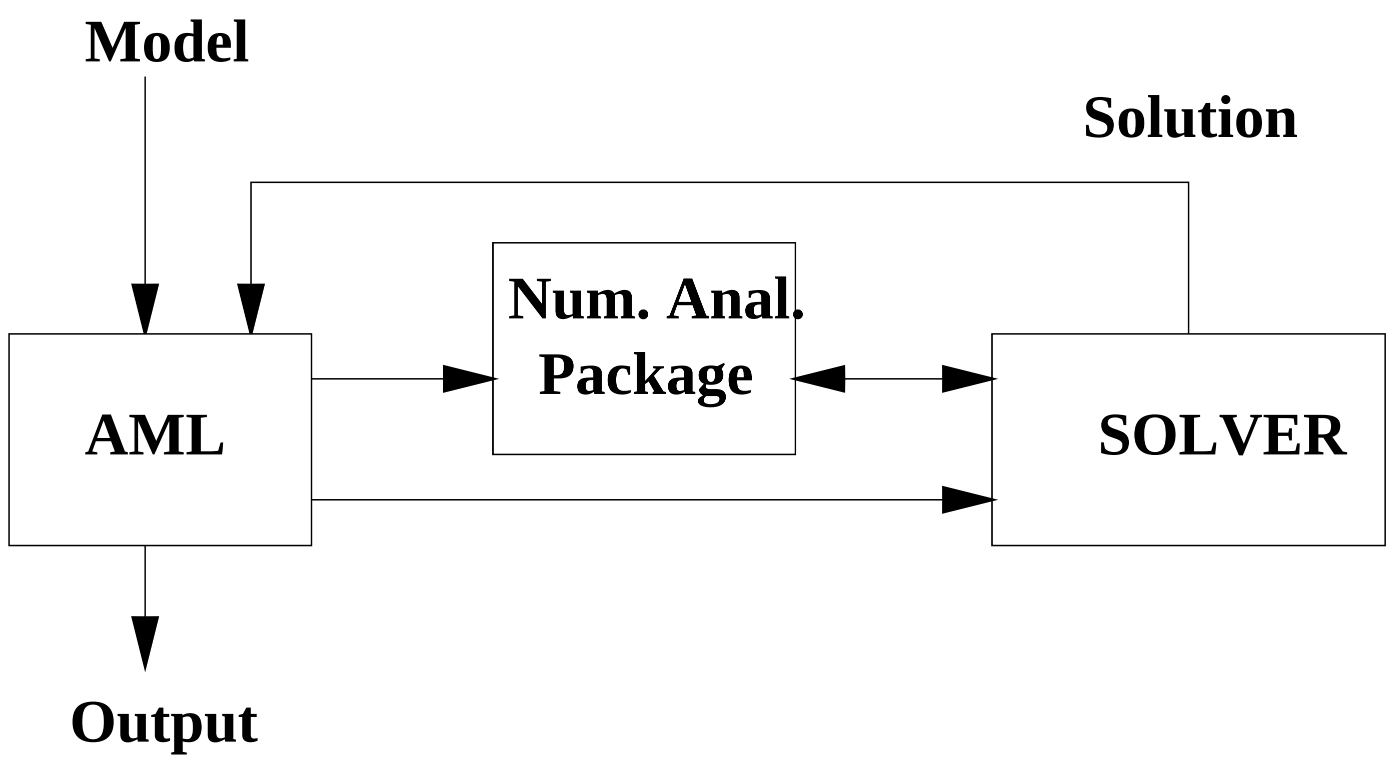
where $A(x) \in \mathcal{R}^{m \times n}$ is the **Jacobian** of g
and $Q(x,y) \in \mathcal{R}^{n \times n}$ is the **Hessian** of $\mathcal{L}$

$$A(x) = \nabla g(x)$$

$$Q(x,y) = \nabla^2 f(x) + \sum_{i=1}^{m} y_i \nabla^2 g_i(x)$$

Linear Algebra in IPM for NLP (cont'd)

Automatic differentiation is very useful ...
get $Q(x, y)$ and $A(x)$ from **Algebraic Modeling Language**.



Automatic Differentiation

AD in the Internet:

- ADIFOR (FORTRAN code for AD):
  `http://www-unix.mcs.anl.gov/autodiff/ADIFOR/`
- ADOL-C (C/C++ code for AD):
  `http://www-unix.mcs.anl.gov/autodiff/AD_Tools/adolc.anl/adolc.html`
- AD page at Cornell:
  `http://www.tc.cornell.edu/~averma/AD/`

IPMs: Remarks

- Interior Point Methods provide the unified framework for convex optimization.
- IPMs provide polynomial algorithms for LP, QP and NLP.
- The linear algebra in LP, QP and NLP is very similar.
- Use IPMs to solve very large problems.

Further Extensions:

- **Nonconvex** optimization.

IPMs in the Internet:

- LP FAQ (Frequently Asked Questions):
  `http://www-unix.mcs.anl.gov/otc/Guide/faq/`
- Interior Point Methods On-Line:
  `http://www-unix.mcs.anl.gov/otc/InteriorPoint/`
- NEOS (Network Enabled Optimization Services):
  `http://www-neos.mcs.anl.gov/`

Newton Method and Self-concordant Barriers

Another View of Newton M. for Optimization

Newton Method for Optimization

Let $f : \mathcal{R}^n \mapsto \mathcal{R}$ be a twice continuously differentiable function. Suppose we build a quadratic model $\tilde{f}$ of f around a given point x^k, i.e., we define $\Delta x = x - x^k$ and write:

$$\tilde{f}(x) = f(x^k) + \nabla f(x^k)^T \Delta x + \frac{1}{2} \Delta x^T \nabla^2 f(x^k) \Delta x$$

Now we **optimize the model** $\tilde{f}$ instead of **optimizing** f.
A minimum (or, more generally, a stationary point) of the quadratic model satisfies:

$$\nabla \tilde{f}(x) = \nabla f(x^k) + \nabla^2 f(x^k) \Delta x = 0,$$

i.e.

$$\Delta x = x - x^k = -(\nabla^2 f(x^k))^{-1} \nabla f(x^k),$$

which reduces to the usual equation:

$$x^{k+1} = x^k - (\nabla^2 f(x^k))^{-1} \nabla f(x^k).$$

$-\log x$ Barrier Function

Consider the **primal barrier linear program**

$$\min\ c^T x - \mu \sum_{j=1}^{n} \ln x_j \qquad \text{s.t.} \qquad Ax = b,$$

where $\mu \geq 0$ is a barrier parameter.
Write out the **Lagrangian**

$$L(x, y, \mu) = c^T x - y^T (Ax - b) - \mu \sum_{j=1}^{n} \ln x_j,$$

and the conditions for a stationary point

$$\begin{aligned} \nabla_x L(x, y, \mu) &= c - A^T y - \mu X^{-1} e = 0 \\ \nabla_y L(x, y, \mu) &= \qquad\qquad\quad Ax - b = 0, \end{aligned}$$

where $X^{-1} = diag\{x_1^{-1}, x_2^{-1}, \cdots, x_n^{-1}\}$.

$-\log x$ Barrier Function (cont'd)

Let us denote

$$s = \mu X^{-1} e, \quad \text{i.e.} \quad XSe = \mu e.$$

The **First Order Optimality Conditions** are:

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ XSe &= \mu e. \end{aligned}$$

- $\log x$ bf: Newton Method

The first order optimality conditions for the barrier problem form a large system of nonlinear equations

$$F(x,y,s) = 0,$$

where $F : \mathcal{R}^{2n+m} \mapsto \mathcal{R}^{2n+m}$ is an application defined as follows:

$$F(x,y,s) = \begin{bmatrix} Ax - b \\ A^T y + s - c \\ XSe - \mu e \end{bmatrix}.$$

Actually, the first two terms of it are *linear*; only the last one, corresponding to the complementarity condition, is *nonlinear*.

Note that

$$\nabla F(x,y,s) = \begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix}.$$

- $\log x$ bf: Newton Method (cont'd)

Thus, for a given point (x, y, s) we find the Newton direction $(\Delta x, \Delta y, \Delta s)$ by solving the system of linear equations:

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^T y - s \\ \mu e - XSe \end{bmatrix}.$$

$1/x^\alpha, \ \alpha > 0$ Barrier Function

Consider the **primal barrier linear program**

$$\min \ c^T x - \mu \sum_{j=1}^{n} \frac{1}{x_j^\alpha} \qquad \text{s.t.} \qquad Ax = b,$$

where $\mu \geq 0$ is a barrier parameter and $\alpha > 0$.

Write out the **Lagrangian**

$$L(x, y, \mu) = c^T x - y^T (Ax - b) + \mu \sum_{j=1}^{n} \frac{1}{x_j^\alpha},$$

and the conditions for a stationary point

$$\begin{aligned} \nabla_x L(x, y, \mu) &= c - A^T y - \mu \alpha X^{-\alpha-1} e = 0 \\ \nabla_y L(x, y, \mu) &= Ax - b = 0, \end{aligned}$$

where $X^{-\alpha-1} = diag\{x_1^{-\alpha-1}, x_2^{-\alpha-1}, \cdots, x_n^{-\alpha-1}\}$.

$1/x^\alpha, \ \alpha > 0$ **Barrier Function (cont'd)**

Let us denote

$$s = \mu \alpha X^{-\alpha-1} e, \quad \text{i.e.} \quad X^{\alpha+1} S e = \mu \alpha e.$$

The **First Order Optimality Conditions** are:

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ X^{\alpha+1} S e &= \mu \alpha e. \end{aligned}$$

$1/x^\alpha,\ \alpha > 0$ bf: Newton Method

The first order optimality conditions for the barrier problem are

$$F(x,y,s) = 0,$$

where $F : \mathcal{R}^{2n+m} \mapsto \mathcal{R}^{2n+m}$ is an application defined as follows:

$$F(x,y,s) = \begin{bmatrix} Ax - b \\ A^T y + s - c \\ X^{\alpha+1} S e - \mu \alpha e \end{bmatrix}.$$

As before, only the last term, corresponding to the complementarity condition, is *nonlinear*.

Note that

$$\nabla F(x,y,s) = \begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ (\alpha+1) X^\alpha S & 0 & X^{\alpha+1} \end{bmatrix}.$$

$1/x^\alpha,\ \alpha>0$ bf: Newton Method (cont'd)

Thus, for a given point (x, y, s) we find the Newton direction $(\Delta x, \Delta y, \Delta s)$ by solving the system of linear equations:

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ (\alpha+1)X^\alpha S & 0 & X^{\alpha+1} \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^T y - s \\ \mu\alpha e - X^{\alpha+1} S e \end{bmatrix}.$$

$e^{1/x}$ Barrier Function

Consider the **primal barrier linear program**

$$\min\ c^T x - \mu \sum_{j=1}^{n} e^{1/x_j} \qquad \text{s.t.} \qquad Ax \ = b,$$

where $\mu \geq 0$ is a barrier parameter.

Write out the **Lagrangian**

$$L(x, y, \mu) = c^T x - y^T (Ax - b) + \mu \sum_{j=1}^{n} e^{1/x_j},$$

and the conditions for a stationary point

$$\begin{array}{rcrcl} \nabla_x L(x, y, \mu) & = & c - A^T y - \mu X^{-2} \mathrm{exp}(X^{-1}) e & = & 0 \\ \nabla_y L(x, y, \mu) & = & Ax - b & = & 0, \end{array}$$

where $\mathrm{exp}(X^{-1}) = diag\{e^{1/x_1}, e^{1/x_2}, \cdots, e^{1/x_n}\}$.

$e^{1/x}$ Barrier Function

Let us denote

$$s = \mu X^{-2}\text{exp}(X^{-1})e, \quad \text{i.e.} \quad X^2\text{exp}(-X^{-1})Se = \mu e.$$

The **First Order Optimality Conditions** are:

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ X^2\text{exp}(-X^{-1})Se &= \mu e. \end{aligned}$$

$e^{1/x}$ bf: Newton Method

The first order optimality conditions are

$$F(x,y,s) = 0,$$

where $F : \mathcal{R}^{2n+m} \mapsto \mathcal{R}^{2n+m}$ is defined as follows:

$$F(x,y,s) = \begin{bmatrix} Ax & - & b \\ A^T y + s & - & c \\ X^2 \mathrm{exp}(-X^{-1}) Se & - & \mu e \end{bmatrix}.$$

As before, only the last term, corresponding to the complementarity condition, is *nonlinear*.

Note that

$$\nabla F(x,y,s) = \begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ (2X+I)\mathrm{exp}(-X^{-1}) & 0 & X^2 \mathrm{exp}(-X^{-1}) \end{bmatrix}.$$

$e^{1/x}$ bf: Newton Method (cont'd)

Newton direction $(\Delta x, \Delta y, \Delta s)$ solves the following system of linear equations:

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ (2X+I)\exp(-X^{-1})S & 0 & X^2\exp(-X^{-1}) \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^T y - s \\ \mu e - X^2 \exp(-X^{-1}) S e \end{bmatrix}.$$

Why Log Barrier is the Best?

The First Order Optimality Conditions:

$$\begin{array}{ll} -\log x: & XSe = \mu e, \\ 1/x^{\alpha}: & X^{\alpha+1}Se = \mu \alpha e, \\ e^{1/x}: & X^2 \mathrm{exp}(-X^{-1})Se = \mu e. \end{array}$$

Log Barrier ensures
the **symmetry** between the primal and the dual.
3rd row in the Newton Equation System:

$$\begin{array}{ll} -\log x: & \nabla F_3 = [S, 0, X], \\ 1/x^{\alpha}: & \nabla F_3 = [(\alpha+1)X^{\alpha}S, 0, X^{\alpha+1}] \\ e^{1/x}: & \nabla F_3 = [(2X+I)\mathrm{exp}(-X^{-1})S, 0, X^2\mathrm{exp}(-X^{-1})] \end{array}$$

Log Barrier produces
'the weakest nonlinearity'.

Self-concordant Functions

There is a nice property of the function that is responsible for a good behaviour of the Newton method.

Def Let $C \in \mathcal{R}^n$ be an open nonempty convex set.
Let $f : C \mapsto \mathcal{R}$ be a three times continuously differentiable convex function.
A function f is called **self-concordant** if there exists a constant $p > 0$ such that

$$|\nabla^3 f(x)[h,h,h]| \leq 2p^{-1/2}(\nabla^2 f(x)[h,h])^{3/2},$$

$\forall x \in C$, $\forall h : \; x + h \in C$.
(We then say that f is p-self-concordant).

Note that a self-concordant function is always well approximated by the quadratic model because the error of such an approximation can be bounded by the 3/2 power of $\nabla^2 f(x)[h,h]$.

Self-concordant Barriers

Lemma
The barrier function $-\log x$ is self-concordant on $\mathcal{R}_+$.

Proof Consider $f(x) = -\log x$.
Compute
$f^{'}(x) = -x^{-1}$, $f^{''}(x) = x^{-2}$ and $f^{'''}(x) = -2x^{-3}$
and check that the self-concordance condition is satisfied for $p = 1$.

Lemma
The barrier function $1/x^{\alpha}$, with $\alpha \in (0, \infty)$ is not self-concordant on $\mathcal{R}_+$.

Lemma
The barrier function $e^{1/x}$ is not self-concordant on $\mathcal{R}_+$.

Use self-concordant barriers in optimization