Variance reduction techniques for American options and applications to Power plants valuation

- Stochastic control problems are standard in energy management

Short-term: unit commitment problem minimizing production costs to satisfy a stochastic demand;

Long-term: investment decisions evaluating power plants flexibility (Gas turbines etc.), real options, when to invest ?

Increasing stochasticity due to demand, market prices (electricity, fuels, Co2), production (with the emergence of intermittent energies) ...

$$v_0(x) = \sup_{\nu} \mathbb{E}\left[\sum_{k=0}^{n} f_k(X_k^{\nu_k}, \nu_k)\right] .$$

- American option pricing is a specific stochastic control problem from financial mathematics, for which a great variety of numerical methods and variance reduction techniques have been proposed

$$u_0(x) = \sup_{\tau \in T_n} \mathbb{E}\left[\sum_{k=0}^{n} f(X_k)\mathbf{1}_{k=\tau}\right] .$$

- Control variate, Quasi Monte Carlo, Antithetic variables, Importance Sampling [7, 2, 1, 8, 6] ...

- Goal: To extend and develop variance reduction techniques for the more complex case of physical assets valuation (risk analysis, investment) and optimization.

Optimal stopping problem

- $(X_k)_{k\geq 0}$: a Markov chain taking values in $(E_k, \mathcal{E}_k)_{n\geq 0}$, with
  - initial distribution on E_0: $\mu_0 = \mathrm{Law}(X_0)$;
  - Markov transition from E_{k-1} to E_k: $M_k(x_{k-1}, dx_k)$.
    For any measurable function φ defined on E_k,
    $M_k(\varphi)$ stands for the conditional expectation function on E_{k-1}:

$$
\begin{aligned}
M_k(\varphi)(x_{k-1}) &= \int_{E_k} M_k(x_{k-1}, dx_k)\ \varphi(x_k)\ , \quad x_{k-1} \in E_{k-1} \\
&= \mathbb{E}\left(\varphi(X_k) \,|\, X_{k-1} = x_{k-1}\right)\ .
\end{aligned}
$$

- f_k: a sequence of non-negative measurable *payoff* functions on E_k.
- Goal: to compute the Snell envelope u_k given by

$$
\begin{cases}
u_n = f_n \\
u_k = f_k \vee M_{k+1}(u_{k+1})\ , \quad \text{for any } 0 \leq k < n\ .
\end{cases}
$$

- Backward operator $\mathcal{H}_k$, for $k \leq l \leq n$:

$$\begin{aligned} u_k &= \mathcal{H}_{k+1}(u_{k+1}) = f_k \vee M_{k+1}(u_{k+1}) = \mathcal{H}_{k,l}(u_l) \ , \quad \text{for any } k \leq l \leq n \\ \mathcal{H}_{k,l} &= \mathcal{H}_{k+1} \circ \mathcal{H}_{k+1,l} \ , \quad \text{with the convention} \quad \mathcal{H}_{k,k} = Id \ . \end{aligned}$$

- Lipschitz property: $\left|\mathcal{H}_{k,l}(u) - \mathcal{H}_{k,l}(v)\right| \leq M_{k,l}\left(|u-v|\right)$.
- Backward approximation operator $\widehat{\mathcal{H}}_{k+1}$: $\widehat{u}_k = \widehat{\mathcal{H}}_{k+1}(\widehat{u}_{k+1}) = f_k \vee \widehat{M}_{k+1}(\widehat{u}_{k+1})$.
  - Local error $\left|\mathcal{H}_{k+1}(u) - \widehat{\mathcal{H}}_{k+1}(u)\right| \leq |(M_{k+1} - \widehat{M}_{k+1})(u)|$.
  - Error propagation: $u_k - \widehat{u}_k = \sum_{l=k}^{n} \left[\widehat{\mathcal{H}}_{k,l}(\mathcal{H}_{l+1}(u_{l+1})) - \widehat{\mathcal{H}}_{k,l}(\widehat{\mathcal{H}}_{l+1}(u_{l+1}))\right]$,

Lemma 1: Robustness Lemma

For any $0 \leq k < n$, on the state space $\widehat{E}_k$.

$$|u_k - \widehat{u}_k| \quad \leq \quad \sum_{l=k}^{n-1} \widehat{M}_{k,l} |(M_{l+1} - \widehat{M}_{l+1})(u_{l+1})| \ .$$

This Lemma has been applied in [4] to bound the error induced by various type of approximation schemes . . .

- Cut-off type models
- Euler approximation models
- Interpolation type models
- Quantization tree models
- Monte Carlo approximation models . . .

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- Path-space Markov chain
  - X'_n Markov chain with transitions $M'_k(x_{k-1}, dx'_k)$ from E'_{k-1} into E'_k.
  - $X_n = (X'_0, \ldots, X'_n) \in E_n = (E'_0 \times \ldots \times E'_n)$ Markov chain with transition kernels

$$M_k(x_{k-1}, dy_k) = \delta_{x_{k-1}}(dy_{k-1})\, M'_k(y'_{k-1}, dy'_k)\ , \text{ for any } \begin{cases} x_{k-1} & = & (x'_0, \ldots, x'_{k-1}) \in E_{k-1} \\ y_k & = & (y'_0, \ldots, y'_k) \in E_k \ . \end{cases}$$

- Let us introduce a sequence of probability measures (η_k) defined on (E_k) st,

$$\eta_k(f) := \frac{\mathbb{E}\left(f(X_k) \prod_{p=0}^{k-1} G_p(X_p)\right)}{\mathbb{E}\left(\prod_{p=0}^{k-1} G_p(X_p)\right)}\ , \quad \text{for any measurable function } f \text{ on } E_k\ ,$$

where $(G_k)_{0 \le k < n}$ are non-negative functions defined on $(E_k)_{0 \le k < n}$ designed
  - to concentrate the computational effort in regions pointed out by the payoff e.g. $G_k(x'_{k-1}, x'_k) = \exp(-\alpha(\log f_k(x'_k) - \log f_{k-1}(x'_{k-1})))$ for $\alpha > 0$.
  - by a first estimation of the snell envelope $G_k(x'_{k-1}, x'_k) = \frac{\hat{u}_k(x'_k)}{\hat{u}_{k-1}(x'_{k-1})}$ known to approximate the zero variance measure see [6].
- The sequence of measures (η_k) satisfies the following dynamic $\eta_0 = \mu_0$ and

$$\eta_k = \Phi_k(\eta_{k-1})\ , \quad \text{where} \quad (\Phi_k(\eta))(f) = \frac{\eta(G_{k-1} M_k(f))}{\eta(G_{k-1})}\ .$$

In the same vein as Broadie-Glasserman models ([2] (2004)), the conditional expectation is replaced by a simple expectation.

- Assumption: $M'_n(x'_{n-1}, \cdot) \ll \lambda_n$ with

$$(H_0) \qquad H_n(x'_{n-1}, x_n) = \frac{dM_n(x'_{n-1}, \cdot)}{d\lambda_n}(x_n) > 0 \ , \quad \forall (x_{n-1}, x_n) \in (E_{n-1} \times E_n) \ ,$$

where H_n is supposed to be known up to a normalizing constant.

Lemma 2: Snell envelope under the new measure

Under Assumption (H_0), $\quad u_k(x_k) = f_k(x_k) \vee \eta_{k+1}\left(R^{\eta_k}_{k+1}(x_k, \cdot) u_{k+1}\right) \ ,$

where $\eta_{k+1} = \Phi_{k+1}(\eta_k)$ and for any measure η on E_k,

$$R^{\eta}_{k+1}(x_k, x_{k+1}) = \frac{\eta(G_k) H_{k+1}(x_k, x_{k+1})}{\eta(G_k H_{k+1}(\cdot, x_{k+1}))} \ , \quad \text{for any} \quad (x_k, x_{k+1}) \in E_k \times E_{k+1} \ .$$

- Next step: approximating the instrumental measure (η_k) by particle systems, with (G_k) known up to a normalizing constant.

Particle system $(\xi_k) := (\xi_k^i)_{1\leq i\leq N} \in E_k^N$ evolving according the following dynamic

$$\xi_k \in E_k^N \xrightarrow{\text{Selection}} \widehat{\xi}_k := \left(\widehat{\xi}_k^i\right)_{1\leq i\leq N} \in E_k^N \xrightarrow{\text{Mutation}} \xi_{k+1} \in E_{k+1}^N \ .$$

• Let η_k^N and $\widehat{\eta}_k^N$ be the occupation measures of the genealogical tree model after the mutation and the selection steps;

$$\eta_k^N := \frac{1}{N} \sum_{1\leq i\leq N} \delta_{\xi_k^i} \quad \text{and} \quad \widehat{\eta}_k^N := \frac{1}{N} \sum_{1\leq i\leq N} \delta_{\widehat{\xi}_k^i} \ .$$

1. Initialization: $\xi_0 = (\xi_0^i)_{0\leq i\leq N_0}$, i.i.d. random copies of X_0.
2. Selection: $\xi_k \rightsquigarrow \hat{\xi}_k$
   with probability $\propto (G_k(\xi_k^i))_{1\leq i\leq N}$ select N particles $\widehat{\xi}_k := \left(\widehat{\xi}_k^i\right)_{1\leq i\leq N}$ among the N particles $\xi_k = (\xi_k^i)_{1\leq i\leq N}$
3. Mutation: $\widehat{\xi}_k \rightsquigarrow \xi_{k+1}$
   Each $\widehat{\xi}_k^i$ evolves independently to $\xi_{k+1}^i = x$ randomly chosen with the distribution $M_{k+1}(\widehat{\xi}_k^i, x)$, with $1 \leq i \leq N$.

Convergence of the occupation measures to the underlying measure

Lemma 2: Convergence of the particle approximation of the measures

For any $p \geq 1$, we denote by p' the smallest even integer greater than p.

- For any $k \geq 0$ and any integrable function $f \in \mathbb{L}_{p'}(\eta_n)$ on E_{k+1}, we have

$$\mathbb{E}\left(\eta_{k+1}^N(f)|\mathcal{F}_k^N\right) = \Phi_{k+1}(\eta_k^N)(f)$$

$$\sqrt{N}\mathbb{E}\left(\left|(\eta_k^N - \Phi_k(\eta_{k-1}^N)(f)\right|^p |\mathcal{F}_k^N\right)^{\frac{1}{p}} \leq 2\ a(p)\ \left[\Phi_{k+1}(\eta_k^N)(|f|^{p'})\right]^{\frac{1}{p'}}$$

- If moreover for any k, $\sup_{x,y} G_k(x)/G_k(y) < \infty$ then for any bounded measurable function f, with $\|f\| \leq 1$,

$$\sqrt{N}\ \mathbb{E}\left(\left|[\eta_n^N - \eta_n](f)\right|^p\right)^{1/p} \leq c(p,n)\ ,$$

  with the collection of constants $(a(p))$ and $(c(p,n))$ depending only on p and (p,n) resp. (see [3]).

Convergence of the *Interacting stochastic Mesh* approximation scheme

$$\begin{cases} \hat{u}_n = f_n \\ \hat{u}_k(x_k) = f_k(x_k) \vee \eta^N_{k+1}\left(R^{\eta^N_k}_{k+1}(x_k,\cdot)u_{k+1}\right) \end{cases}, \quad \text{for any } 0 \leq k < n \ , \ \text{with}$$

$$R^{\eta^N_k}_{k+1}(x_k, x_{k+1}) = \frac{\eta^N_k(G_k)H_{k+1}(x_k, x_{k+1})}{\eta^N_k(G_k H_{k+1}(\cdot, x_{k+1}))} \ , \quad \text{for any} \quad (x_k, x_{k+1}) \in E_k \times E_{k+1} \ .$$

Theorem: Convergence of the Snell envelope approximation scheme

For any $0 \leq k \leq n$ and any integer $p \geq 1$ and $x \in E_k$,

$$\|(\widehat{u}_k - u_k)(x)\|_{L_p} \leq \sum_{k<l<n} \frac{2\ a(p)}{\sqrt{N}} q_{k,l} \left[M_{k,l+1}(u^{p'}_{l+1})(x)\right]^{\frac{1}{p'}} \ , \quad \text{with}$$

$$q_{k,l} := \left[\|h_{k+1}\|\|h_{l+1}\| \prod_{m=k}^{l} \|G_m\|^2\right]^{\frac{p'-1}{p'}} \quad \text{and} \quad h_k(x_k) := \sup_{x,y \in E_{k-1}} \frac{H_k(x,x_k)}{H_k(y,x_k)} \ .$$

Variance reduction techniques for American options and applications to Power plants valuation

Price dynamics and payoffs

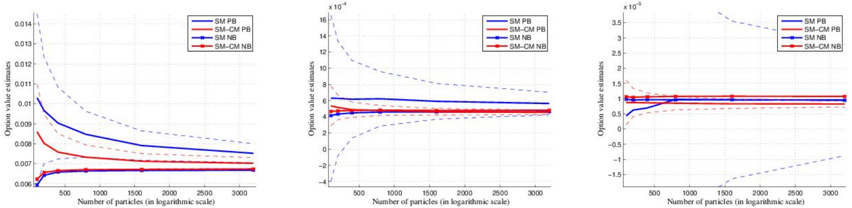
- Asset prices (X_t) follows a geometric Brownian motion under the risk-neutral measure,
$$\frac{dX_t(i)}{X_t(i)} = rdt + \sigma_i dz_t^i \ , \quad \text{for assets } i = 1, \cdots, d \ ,$$
where z^i, for $i = 1, \cdots, d$ are independent standard Brownian motions.
- Interest rate $r = 5\%$ annually, $X_{t_0}(i) = 1$, for all $i = 1, \cdots, d$, volatilities $\sigma_i = 20\%$ annually, for arithmetic puts and $r = 10\%/d$, $\sigma_i = 20\%/\sqrt{d}$ for the geometric puts.
- Maturity $T = 1$ year, $n + 1 = 11$ equally distributed exercise opportunities.
- Geometric average put option with payoff $(K - \prod_{i=1}^d X_T(i))_+$;
- Arithmetic average put option with payoff $(K - \frac{1}{d}\sum_{i=1}^d X_T(i))_+$.
- Potential functions:
$$\begin{cases} G_1(x_1) = (f_1(x_1) \vee \varepsilon)^\alpha \ , \\ G_k(x_k) = \frac{(f_k(x_k)\vee\varepsilon)^\alpha}{(f_{k-1}(x_{k-1})\vee\varepsilon)^\alpha} \ , \quad \text{for all } k = 2, \cdots, n-1 \ , \end{cases}$$
where f_k are the payoff functions and $\alpha \in (0, 1]$ and $\varepsilon > 0$ are parameters fixed in our simulations to the values $\alpha = 1/5$ and $\varepsilon = 10^{-7}$.

Variance ratio ($\frac{Var(\hat{v}_{SM})}{Var(\hat{v}_{SMCM})}$) and (within parentheses) Bias ratio ($\frac{\mathbb{E}(\hat{v}_{SM})-\mathbb{E}(\hat{v}_{SMCM})}{E(\hat{v}_{SM})}$) computed over 1000 runs for $N = 3200$ mesh points.

Payoff	K	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
Geometric Put	0.95	1 (1%)	1 (3%)	1 (6%)	1 (9%)	1 (10%)
	0.85	5 (2%)	8 (6%)	6 (11%)	4 (14%)	3 (14%)
	0.75	18 (6%)	28 (11%)	18 (17%)	16 (18%)	11 (16%)
Arithmetic Put	0.95	1 (1%)	3 (2%)	3 (7%)	4 (13%)	5 (18%)
	0.85	5 (2%)	13 (6%)	24 (19%)	56 (24%)	100 (20%)
	0.75	18 (6%)	71 (15%)	363 (14%)	866 (16%)	– (–)

(For the arithmetic put, when $d = 5$ and $K = 0.75$, the 1000 estimates provided by the standard SM algorithm were all equal to zero, hence the associated variance ratio has not been reported).

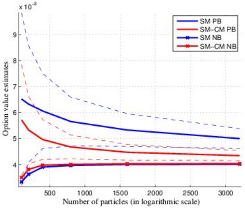
Arithmetic put with $d = 3$ assets



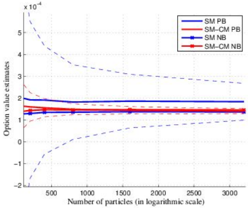
(a) $K = 0.95$ (b) $K = 0.85$ (c) $K = 0.75$

Positively-biased option values estimates (average estimates with 95% confidence interval computed over 1000 runs) and Negatively-biased option values estimates (average estimates over the 1000 runs each forward estimate being evaluated over 10000 forward Monte Carlo simulations), computed by the SM algorithm (Stochastic Mesh in blue line) and the SMCM algorithm (Stochastic Mesh with particle change of measure in red line), as a function of the number of mesh points.

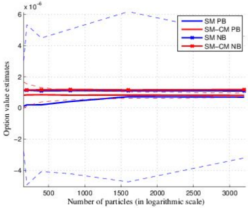
Arithmetic put with $d = 4$ assets



(d) $K = 0.95$



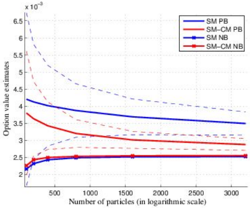
(e) $K = 0.85$



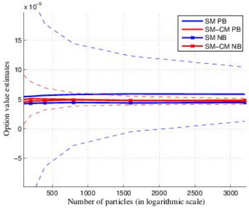
(f) $K = 0.75$

Positively-biased option values estimates (average estimates with 95% confidence interval computed over 1000 runs) and Negatively-biased option values estimates (average estimates over the 1000 runs each forward estimate being evaluated over 10000 forward Monte Carlo simulations), computed by the SM algorithm (Stochastic Mesh in blue line) and the SMCM algorithm (Stochastic Mesh with particle change of measure in red line), as a function of the number of mesh points.

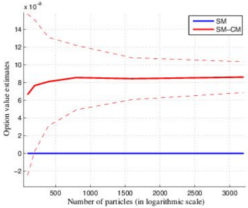
Arithmetic put with $d = 5$ assets



(g) $K = 0.95$



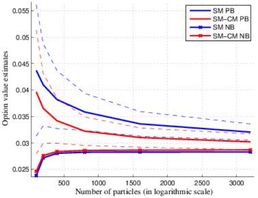
(h) $K = 0.85$



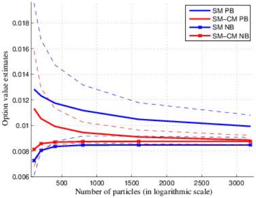
(i) $K = 0.75$

Positively-biased option values estimates (average estimates with 95% confidence interval computed over 1000 runs) and Negatively-biased option values estimates (average estimates over the 1000 runs each forward estimate being evaluated over 10000 forward Monte Carlo simulations), computed by the SM algorithm (Stochastic Mesh in blue line) and the SMCM algorithm (Stochastic Mesh with particle change of measure in red line), as a function of the number of mesh points.

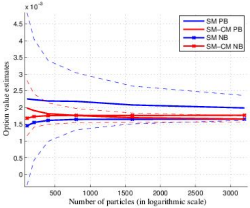
Geometric put with $d = 3$ assets



(j) $K = 0.95$



(k) $K = 0.85$



(l) $K = 0.75$

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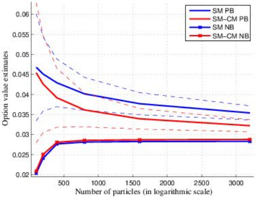
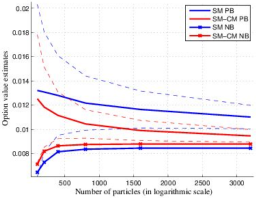
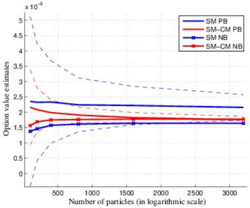
Geometric put with $d = 4$ assets



(m) $K = 0.95$



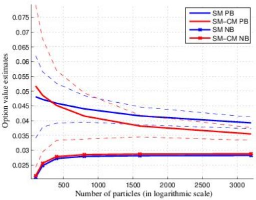
(n) $K = 0.85$



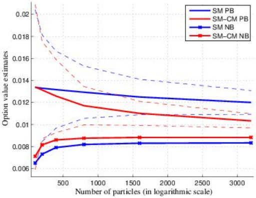
(o) $K = 0.75$

Positively-biased option values estimates (average estimates with 95% confidence interval computed over 1000 runs) and Negatively-biased option values estimates (average estimates over the 1000 runs each forward estimate being evaluated over 10000 forward Monte Carlo simulations), computed by the SM algorithm (Stochastic Mesh in blue line) and the SMCM algorithm (Stochastic Mesh with particle change of measure in red line), as a function of the number of mesh points.

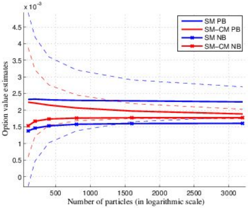
Geometric put with $d = 5$ assets



(p) $K = 0.95$



(q) $K = 0.85$



(r) $K = 0.75$

Positively-biased option values estimates (average estimates with 95% confidence interval computed over 1000 runs) and Negatively-biased option values estimates (average estimates over the 1000 runs each forward estimate being evaluated over 10000 forward Monte Carlo simulations), computed by the SM algorithm (Stochastic Mesh in blue line) and the SMCM algorithm (Stochastic Mesh with particle change of measure in red line), as a function of the number of mesh points.

Variance reduction techniques for American options and applications to Power plants valuation

Characteritics of the thermal asset

• Study period: one year discretized into $n+1$ regular instants.
• Payoff: at each time step t_k, one decides to produce electricity on the period $[t_k, t_{k+1})$, with a *power* $P_k \in \{0, P_k^{min}, P_k^{max}\}$ generating the payoff:

$$\varphi_k(P,S) = P\Big(S^1 - H_k^{rate}(P)\big(S^2 + J_k S^3 + F_k\big)\Big)\delta t \ , \qquad (1)$$

• Start-up costs: if the thermal unit is not producing at the preceding period $[t_{k-1}, t_k)$, then starting the unit at time t_k will imply an additional cost C_k^{start}.

- $S = (S^1, S^2, S^3)$ the electricity, fuel (coal) and Co2 spot prices (the stochastic process S is modelized as a function of an $\mathbb{R}^d$ valued Markov process X i.e. $S_{t_k} = f_k(X_{t_k})$);
- H_k^{rate} is the heat rate i.e. the unity of fuel (in Tc) needed to produce one MWh of electricity (for technical reasons it decreases with P);
- J_k represents the rate of released Co2 for each unity of burnt fuel (to compensate for the environmental damage one has to pay JS^3 Pounds for each Tc of burnt coal);
- F_k denotes the variable fuel cost;

• Goal: to maximize the expected return of the thermal unit over strategies $(P_k)_{k=0,\cdots n}$ adapted to the information structure $(\mathcal{F}_k)_{k=0,\cdots n}$ generated by the discrete observations of the stochastic process X i.e. $\mathcal{F}_k = \sigma(X_{t_0}, \cdots, X_{t_k})$.

$$V_0(x) = \max_{(P_k)_{k=0,\cdots n-1}} \sum_{k=1}^{n} \mathbb{E}_{0,x}[\varphi_k(P_k, S_{t_k}) - C_k^{start} \mathbf{1}_{P_{k-1}=0 \cap P_k>0}] \ , \qquad (2)$$

• Some model simplifications:

- Minimal running durations are not considered: the discretization step chosen in the present study (12 hours) being greater than the actual minimal running duration (4 hours), this constraint is then naturally inactivated.
- Forced outages are not considered: their are taken into account by multiplying the real values of P^{max} and P^{min} by an average disponibility coefficient.

These approximations are proved to be acceptable in first approximation (with no significant impact on pricing and hedging results) in our empirical studies.

Dynamic Programming Principle For each time step $k = 0, 1, \cdots n+1$, we define

- V_k^+ and V_k^- defined on $\mathbb{R}^d$

$$\begin{cases} x \mapsto V_k^+(x) & \text{max expected value} & \text{knowing it was started on } [t_{k-1}, t_k) \\ x \mapsto V_k^-(x) & & \text{knowing it was not started on } [t_{k-1}, t_k). \end{cases}$$

- $\tilde{\varphi}_k$ defined for all $x \in \mathbb{R}^d$, $P \in \{0, P^{min}, P^{max}\}$,

$$\tilde{\varphi}_k(P, x) = \varphi_k(P, f_k(x)) \,. \quad (3)$$

Recalling that $S_k = f_k(X_{t_k})$ where X is a Markov process, we obtain

Dynamic programming equation

$$\begin{cases} V_n^+(x) = \max_{P_n} \tilde{\varphi}_n(P_n, x) \\ V_n^-(x) = \max_{P_n} \{\tilde{\varphi}_n(P_n, x) - C_n^{start} \mathbf{1}_{P_n > 0}\} \\ V_k^+(x) = \max_{P_k} \left\{\tilde{\varphi}_k(P_k, x) + \mathbb{E}_{k,x}[V_{k+1}^+(X_{t_{k+1}})] \mathbf{1}_{P_k > 0} + \mathbb{E}_{k,x}[V_{k+1}^-(X_{t_{k+1}})] \mathbf{1}_{P_k = 0}\right\} \\ V_k^-(x) = \max_{P_k} \left\{\tilde{\varphi}_k(P_k, x) + (\mathbb{E}_{k,x}[V_{k+1}^+] - C_k^{start}) \mathbf{1}_{P_k > 0} + \mathbb{E}_{k,x}[V_{k+1}^-] \mathbf{1}_{P_k = 0}\right\} \,, \end{cases}$$

- Reformulation

$$\begin{cases} \Phi_k(x) = \max\left\{\tilde{\varphi}_k(P,x) \,|\, P \in \{P^{min}, P^{max}\}\right\} \\ V_k^+(x) = \max\left\{\Phi_k(x), u_k(x)\right\} + \mathbb{E}_{k,x}[V_{k+1}^+(X_{t_{k+1}})] \text{ with } V_{n+1}^+ \equiv 0 \ , \\ u_k(x) = \mathbb{E}_{k,x}\left[\min\left\{\max\left\{u_{k+1}(X_{t_{k+1}}) - \Phi_{k+1}(x), -C_{k+1}^{start}\right\}, 0\right\}\right] \text{ with } u_n \equiv 0 \ . \end{cases}$$

Let us introduce the sequence $\bar{P}_k$ such that

$$\bar{P}_k(x) = \mathrm{Arg}\max\left\{\tilde{\varphi}_k(P,x) \,|\, P \in \{P_k^{min}, P_k^{max}\}\right\} \ ,$$

Optimal strategies

$$\begin{cases} P_k^+(x) & \text{knowing the Power plant was started on } [t_{k-1}, t_k) \\ P_k^-(x) & \text{knowing the Power plant was not started on } [t_{k-1}, t_k). \end{cases}$$

P_k^+ (knowing that the central produces on $[t_{k-1}, t_k)$) and P_k^- (knowing that the central does not produce on $[t_{k-1}, t_k)$) are deduced as follows

$$\begin{cases} P_k^+(x) &= & P_k^-(x) = 0 & \text{if} \quad u_k(x) - \Phi_k(x) \geq 0 \\ P_k^+(x) &= & P_k^-(x) = \bar{P}_k & \text{if} \quad u_k(x) - \Phi_k(x) < -C_k^{start} \\ P_k^+(x) &= & \bar{P}_k(x) \text{ and } P_k^- = 0 & \text{if} \quad 0 > u_k(x) - \Phi_k(x) > -C_k^{start} \ . \end{cases}$$

The sequence of functions $(u_k)_{k=0,\cdots n}$ can be learnt in a first step with a first set of simulations.

Electricity prices are modeled by a two factors model whereas coal and Co2 prices are modeled by one long-term factor model. Let

- $B = (B^1, B^2, B^3, B^4)$ be a four dimensional standard Brownian motion
- $R = (R_{i,j})_{1 \leq i,j \leq 4}$ the correlation matrix between the electricity short term factor, the electricity long term factor, the coal and the CO2.
- $Z := R^{1/2} W$.

We define the $d = 4$-dimensional Markov process $X = (X^1, X^2, X^3, X^4)^T$

$$\left\{ \begin{array}{lcl} dX_t & = & \mu_t dt + \sigma_t dB_t \quad \text{with} \\ \mu_t & = & (-aX_t^1, 0, 0, 0)^T \\ \sigma_t & = & diag([\sigma_S, \sigma_L, \sigma_{coal}, \sigma_{Co2}])R^{1/2} \ . \end{array} \right. \tag{4}$$

and $V(t) = (V^1(t), V^2(t), V^3(t))^t$ then for $i = 1, 2, 3$,

$$S_t^i = F^i(0,t) \exp\left\{ -\frac{1}{2} V^i(t) + (AX_t)^i \right\} := f_t^i(X_t) \quad \text{with} \quad A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{5}$$

- Thermal asset parameters:
  - Delivery period from 04 January 2011 to 03 January 2012
  - time step $t_{k+1} - t_k = 12$ hours;
  - minimal capacity $P_k^{min} = 0.915 * 318 MW$;
  - Maximal capacity $P_k^{max} = 0.915 * 400 MW$;
  - Heat Rate at minimal capacity $H^{rate}(P_k^{min}) = 0.43$ Tc/MWh;
  - Heat Rate at maximal capacity $H^{rate}(P_k^{max}) = 0.37$ Tc/MWh;
  - Rate of released CO2 $J = 2.21$ TCo2/Tc;
  - Variable Fuel Cost $F = 23$ Euros/Tc;
  - Startup costs $C^{start} = 27800$ Euros.
- The price processes are modeled by a $3-d$ stochastic process $S = (S^1, S^2, S^3)$:
  - electricity prices S^1: two factor-model with $\sigma_S = 50\%$ $\sigma_L = 21\%$ and $a = 34$ in an annual basis;
  - CO2 prices S^2: one factor-model with $\sigma_{coal} = 27\%$ in an annual basis;
  - Coal prices S^3: one factor-model with $\sigma_{Co2} = 20\%$ in an annual basis;
  - the initial futures curve is flat $[60, 14 * EP, 115 * DP]$;
  - the correlation matrix R between electricity, coal and CO2 is supposed to be the identity matrix.

Tests principles:

- For both approaches, we have implemented $M = 1000$ independent tests;
- We compute the variance of each approach over the $M = 1000$ tests;

N	50	100	500	1000	5000
$\mathbb{E}[\hat{V}^{Stand}]$ $(\sigma(\hat{V}^{Stand}))$	50.981 (3.633)	51.188 (2.536)	51.029 (1.139)	51.061 (0.811)	51.023 (0.337)
$\mathbb{E}[\hat{V}^{IS}]$ $(\sigma(\hat{V}^{IS}))$	51.092 (1.161)	51.005 (0.879)	50.975 (0.399)	50.985 (0.281)	50.965 (0.126)
$\frac{Var^{stand}}{Var^{IS}}$	10	8	8	8	7

Estimates mean, standard variance (in million of Pounds) and variance ratio for the standard approach and Importance Sampling (IS) approach computed on 1000 tests.

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