A glimpse of Mean Field Games theory

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A very humble introduction based on the work by J. M. Lasry and P. L. Lions.
Slides strongly based on the lectures notes by P. Cardaliaguet

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Plan

Introductory Example: Swimmers on a beach

We have swimmers that want to

- Be near the sea.
- Not far from the parking.
- Not near each other

How can we model an optimal repartition of the swimmers?

We will suppose that the swimmers are identical, i.e. they have the same preferences.

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A symmetric static game with N players

We have

- $N =$ number of players (or agents).
- Set of actions Q (the same for all the players).
- If player i choose the action $x_i \in Q$ and the other choose $(x_j)_{j \neq i}$, then the cost for the player i is $F(x_i, (x_j)_{j \neq i})$.
- For every x the function $F(x, \cdot)$ is symmetric.

In our example:

- $N=$ number of swimmers.
- Q = the beach.
- We can take as cost:

$$F(x_i,(x_j)_{j\neq i}) = \alpha \text{dist}(x_i, \text{Sea}) + \beta \text{dist}(x_i, \text{Parking}) - \gamma \frac{1}{N-1} \sum_{j\neq i} |x_j - x_i|,$$

where α, β, $\gamma > 0$.

Nash equilibria

In this talk we will suppose that $Q \subseteq \mathbb{R}^d$ is compact.

Definition

We say that $(\bar{x}_1, \dots, \bar{x}_N) \in Q^N$ is a Nash equilibrium for a game with pure strategies if

$$F(\bar{x}_i, (\bar{x}_j)_{j \neq i}) \leq F(y, (\bar{x}_j)_{j \neq i}) \quad \text{for all } y \in Q.$$

The meaning is that *no player can improve its utility by unilaterally changing its action*.

Drawbacks:

- Under no-convexity assumptions we don't have existence in general.
- Multiplicity of Nash equilibria and difficulty for selecting a *desirable one*.

Definition

We say that $(\overline{\pi_1}, \ldots, \overline{\pi_N}) \in \mathcal{P}(Q)^N$ is a Nash equilibrium for a game with mixed strategies if for all $\pi \in \mathcal{P}(Q)$,

$$\int_{Q^N} F(x_i, (x_j)_{j \neq i}) \mathrm{d}\overline{\pi_i}(x_i) \ldots \mathrm{d}\overline{\pi_N}(x_N) \leq \int_{Q^N} F(x_i, (x_j)_{j \neq i}) \mathrm{d}\pi(x_i) \ldots \mathrm{d}\overline{\pi_N}(x_N).$$

It is the same idea than before but *the player choose their strategy randomly and the cost is the mean*.

- Under reasonable assumptions now we have existence of Nash equilibria (Nash 1950). The argument is based on an application of the Ky-Fan fixed point theorem for set-valued mappings (We can use as a topology in $\mathcal{P}(Q)^N$ the Kantorovic-Rubistein distance).
- Moreover under our specific structure we have a symmetric equilibrium of the form $(\bar{\pi}, \ldots, \bar{\pi})$.

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What happens when $N \uparrow \infty$?

A heuristic idea: Consider the space $Q^N/\mathcal{S}^N$ (the quotient space between Q^N and the permutation relation). Every point in $(x_1, \dots, x_N) \in Q^N/\mathcal{S}^N$ (or an equivalence class to be more precise) can be identified with

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}.$$

Since every $m \in \mathcal{P}(Q)$ can be obtained as a limit of empirical measures (Law of the Large numbers), we have formally that "$Q^N/\mathcal{S}^N \to \mathcal{P}(Q)$". Therefore, it is natural to think that symmetric functions $u_N : Q^N \to \mathbb{R}$, or equivalently functions defined on $Q^N/\mathcal{S}^N$ "converge to a function $U : \mathcal{P}(Q) \to \mathbb{R}$". We will see that the above argument can be formalized under some assumptions.

- Preliminary: Symmetric functions of N variables and their limits when $N \to \infty$.

Let $u_N : Q^N \to \mathbb{R}$ be differentiable and symmetric, i.e.

$$u_N(x_1, \ldots, x_N) = u_N(x_{\sigma(1)}, \ldots, x_{\sigma(N)}), \quad \text{for every permutation } \sigma \text{ of } \{1, \ldots, N\}.$$

We suppose that

- There exists $C > 0$ (independent of N) such that

$$\|u_N\|_\infty \leq C.$$

  and

$$\|D_{x_i} u_N\|_\infty \leq \frac{C}{N} \qquad \forall\, i \in \{1, \ldots, N\}.$$

Given $x = (x_1, \dots, x_N) \in Q^N$ we define $m_N^x \in \mathcal{P}(Q)$ as

$$m_N^x := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}.$$

Theorem

There exists a continuous function $U : \mathcal{P}_1(Q) \to \mathbb{R}$, such that as $N \uparrow \infty$, up to some subsequence, we have that

$$\lim_{N \uparrow \infty} \sup_{x \in Q^N} |u_N(x) - U(m_N^x)| = 0.$$

The proof is an application of Ascoli theorem.

We now comeback to our game: In what follows we explicit the dependence on N of the game by denoting $F_N : Q \times Q^{N-1} \to \mathbb{R}$ for the cost function. We assume that F_N is continuous and differentiable with respect to $(x_j)_{j \neq i}$ and that

- $$\|F_N\|_\infty \leq C$$
  and
  $$|D_{x_j} F_N(x_i, (x_j)_{j \neq i})| \leq \frac{C}{N} \quad \forall\, j \neq i.$$

This means that the cost for the player i depends weakly on the decisions of the other players.

Adapting our previous arguments, we see that, up to some subsequence, F_N is uniformly near a function $F : Q \times \mathcal{P}_1(Q) \to \mathbb{R}$. More precisely,

$$F_N(y, (x_j)_{j \neq i}) = F\left(y, \frac{1}{N-1}\sum_{j \neq i} \delta_{x_j}\right) + o(1).$$

The latter function F "defines the cost function for the continuous game" where a *generic* player wants to minimize a cost $F(x, m)$ that depends on its own decision x and the strategies m of the rest of the players.

Now that we have "defined" our game, what is the correct notion of equilibrium? To find it, we will suppose a *good case* for the finite games, i.e. that the Nash equilibria exist and we will try to obtain an information at the limit.

Theorem

Suppose that for each N, $\bar{x}^N := (\bar{x}_1^N, \ldots, \bar{x}_N^N)$ is a Nash equilibrium for the game with N players. Then, up to some subsequence, their "discrete" distribution $m_N^{\bar{x}^N}$ converge to a "continuous distribution" $\bar{m} \in \mathcal{P}_1(Q)$ satisfying

$$\int_Q F(x, \bar{m}) \mathrm{d}\bar{m}(x) = \inf_{m \in \mathcal{P}(Q)} \int_Q F(x, \bar{m}) \mathrm{d}m(x). \qquad (*)$$

Equivalently,

$$supp(\bar{m}) \subseteq \operatorname{argmin} F(\cdot, \bar{m}).$$

[Sketch of the proof] By compactness, up to some subsequence, there exists $\bar{m} \in \mathcal{P}_1(Q)$ such that $m_N^{\bar{x}^N} \to \bar{m}$. Let us suppose the existence of $\bar{x} \in \text{supp}(\bar{m})$ and $\bar{y} \in Q$ such that

$$F(\bar{y}, \bar{m}) < F(\bar{x}, \bar{m}).$$

By continuity the same inequality holds on a ball B around $\bar{x}$. Since $m_N^{\bar{x}^N}(B) \to \bar{m}(B)$ and $\bar{m}(B) \neq 0$ we have the existence of $x_i^N \in B$ such that

$$F(\bar{y}, \bar{m}) < F(\bar{x}_i^N, \bar{m}).$$

Thus by the convergence of $F_N(\cdot, m_N^{\bar{x}^N}) \to F(\cdot, \bar{m})$ we get a contradiction.

The above result motivates the following definition

Definition

We say that $\bar{m}$ is an equilibrium of the mean field game (F, Q), where $F : Q \times \mathcal{P}_1(Q) \to \mathbb{R}$ if

$$\int_Q F(x, \bar{m}) \mathrm{d}\bar{m}(x) = \inf_{m \in \mathcal{P}(Q)} \int_Q F(x, \bar{m}) \mathrm{d}m(x). \qquad (MFG)$$

or equivalently

$$\operatorname{supp}(\bar{m}) \subseteq \operatorname{argmin}\ F(\cdot, \bar{m}).$$

In the beach example

$$F(x,m) = \alpha \text{dist}(x, \text{Sea}) + \beta \text{dist}(x, \text{Parking}) - \gamma \int_Q |y - x| \mathrm{d}m(y).$$

Even if the finite games have no Nash equilibria it can be proved that the continuous games have them. Two approaches:

- Fixed point argument.
- Limit of Nash equilibria of games with mixed strategies.

Theorem

Let $(\bar{\pi}^N, \dots, \bar{\pi}^N)$ be a sequence of Nash equilibria with mixed strategies for the finite game. Then $\bar{\pi}^N$ converges to a solution $\bar{m}$ of (MFG).

The proof is based on a deep result in probability theory known as the Hewitt-Savage theorem.

Existence of an ε-Nash equilibria for the finite game with mixed strategies

Theorem

Let $\bar{m}$ be a continuous Nash equilibrium. For all $\varepsilon > 0$ there exists $N_\varepsilon > 0$ such that for all $N \geq N_\varepsilon$, we have $(\bar{m}, \dots, \bar{m})$ is an ε- Nash equilibrium for the game with N players, i.e.

$$F_N(\bar{m}, (\bar{m}, \dots, \bar{m})) \leq F_N(m, (\bar{m}, \dots, \bar{m})) + \varepsilon. \quad \forall\, m \in \mathcal{P}(Q).$$

Uniqueness

We have

Theorem

Assume that F satisfies the following monotonicity assumption

$$\int_Q [F(y, m_1) - F(y, m_2)] \, \mathrm{d}(m_1 - m_2)(y) > 0 \quad \forall \; m_1 \neq m_2.$$

Then the solution of (MFG) is unique.

Proof:
Suppose that we have two different solutions m_1 and m_2 of (MFG). Then

$$\int_Q F(y, m_1)\mathrm{d}m_1 \leq \int_Q F(y, m_1)\mathrm{d}m_2,$$

$$\int_Q F(y, m_2)\mathrm{d}m_2 \leq \int_Q F(y, m_2)\mathrm{d}m_1.$$

Adding both inequalities yields a contradiction with the monotonicity assumption.

Remark: Even if the continuous problem has a unique solution, it can happen that uniqueness does not hold for the finite problems.

How can we calculate an equilibrium?

Let us suppose that we have local interactions i.e. $F : Q \times [0, \infty) \to \mathbb{R}$. In this case, the (MFG) is

$$\int_Q F(y, \bar{m})\bar{m}(y)\mathrm{d}y = \inf_{m \in \mathcal{P}_{ac}(Q)} \int_Q F(y, \bar{m}(y))m(y)\mathrm{d}y.$$

Define

$$\Phi(x, m) = \int_0^m F(x, r)\mathrm{d}r.$$

We have

Proposition

Assume that $\bar{m}$ minimize

$$m \rightarrow \int_Q \Phi(x, m(x))\mathrm{d}x.$$

Then $\bar{m}$ is a Nash equilibrium of the continuous game.

The above result correspond to a coordination or efficiency principle for the continuous game.

Example: Take *formally* $Q = \mathbb{R}^2$

$$F(x, m) := \frac{1}{2}|x|^2 + \log m(x).$$

In this case (MFG) can be written as

$$F(x, \bar{m}(x)) = \inf_y \; F(y, \bar{m}(y)) \quad \text{in } \bar{m} > 0.$$

Setting $\bar{\lambda} := \inf_y \; F(y, \bar{m}(y))$, we get

$$\bar{m}(x) = e^{\bar{\lambda}} e^{-\frac{|x|^2}{2}},$$

and so $\bar{m}$ is Gaussian.

Plan

An insight on the dynamic case

PDE system associated to a Nash equilibrium in a stochastic differential game N-players (A. Bensoussan and J. Freshe, 1984)

$$-\partial_t u_i^N(X,t) - \frac{\sigma^2}{2}\Delta_X u_i^N(X,t) + H\left(x_i, D_{x_i} u_i^N(X), (x_j)_{j\neq i}\right)$$
$$+\sum_{j\neq i} D_p H\left(x_j, t, D_{x_j} u_j^N(X,t)\right)\cdot D_{x_j} u_i^N(X,t) = F\left(x_i, (x_j)_{j\neq i}\right),$$
$$u_i^N(X,T) = G\left(x_i, (x_j)_{j\neq i}\right).$$

where $X = (x_1, \dots, x_N) \in (\mathbb{R}^d)^N$ and $i = 1, \dots, N$ and $\sigma > 0$.

- Under symmetry assumptions, the solutions $u^N(x_i; (x_j)_{j \neq i}, t)$ converge to some u and the discrete distribution of the players converge to $m(\cdot)$ (the continuous distribution). Moreover, u and m solve

$$
\begin{array}{rcl}
-\partial_t u - \frac{\sigma^2}{2}\Delta u + H(x, Du) & = & F(x, m(t)), \\
\partial_t m - \frac{\sigma^2}{2}\Delta m - \operatorname{div}\left(m \frac{\partial H}{\partial p}(x, Du)\right) & = & 0, \\
u(x, T) = G(x, m(T)) \quad \text{for } x \in \mathbb{R}^d & , & m(0) = m_0 \in \mathcal{P}_1.
\end{array}
$$

- Numerical schemes for the above problem have been studied by Achdou and Capuzzo-Dolcetta ('10), Lachapelle, Salomon and Turinici ('10), Achdou, Camilli, Capuzzo-Dolcetta ('11), Guéant ('12), Achdou, Camilli, Capuzzo-Dolcetta ('12), Carlini-S. ('13, work in progress)

A model first order problem

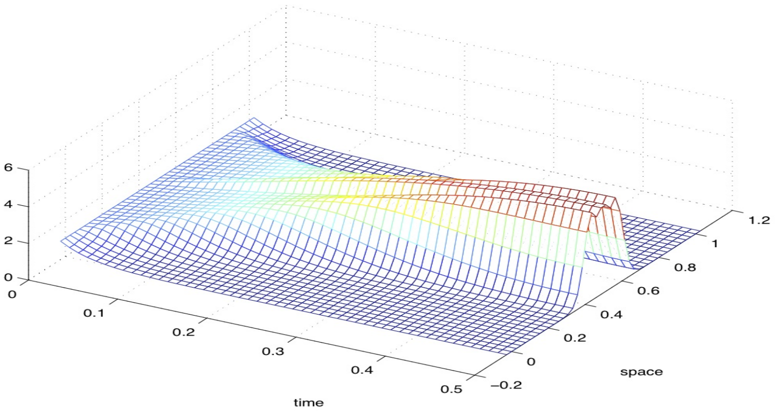
We consider the first order case $\sigma = 0$ and the particular case of a quadratic Hamiltonian:

$$\left.\begin{array}{rcl} -\partial_t u + \frac{1}{2}|Du(t,x)|^2 & = & F(x, m(t)), \\ \partial_t m - \operatorname{div}\big(mDu\big) & = & 0, \\ u(x,T) = G(x, m(T)) \quad \text{for } x \in \mathbb{R}^d & , & m(0) = m_0 \in \mathcal{P}_1. \end{array}\right\} \text{(MFG)}$$

• With E. Carlini ('12) we provided a fully discrete approximation such that:

- It is well posed.
- When the discretization parameter tends to 0, we have the convergence to (u, m).

Exemple: We take $F(x, m) = c(x - 0.5)^2 + \rho_\sigma * [\rho_\sigma * m(t)]\,(x)$ and $G \equiv 0$



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A mean field game model for electrical vehicles in the smart grids, by R. Couillet, S. Perlaza, H. Tembine and M. Debbah

- Electrical vehicles have batteries which can be charged and discharged.
- They are energy consuming devices and mobile energy sources (store and transport)
- The approach is that reliability is improved if the vehicles can buy and sell energy to the smart-grid.
- The price depend on the existing demand on the grid.
- What are the optimal charge and discharge policies?

- The model with N vehicles. Let us set for a vehicle k
  - $x^{(k)}(t) \in [0, 1]$ for the energy stored.
  - $g^{(k)}(t)$ for the energy consumption rate.
  - $\alpha^{(k)}(t)$ for the energy provisioning rate (buy or sell).
  - $p(\alpha(t), t)$ for prize of the energy, determined by the strategies of all the owners of the vehicles.

  Thus
$$\frac{d}{dt}x^{(k)}(t) = \alpha^{(k)}(t) - g^{(k)}(t).$$

and each owner wants to minimize

$$J^{(k)}(\alpha^{(k)}, \alpha^{(-k)}) = \int_0^T \left\{ \alpha^{(k)} p(\alpha, t) + h^{(k)}(\alpha^{(k)}) + f^{(k)}(x^{(k)}) \right\} \mathrm{d}t + \kappa^{(k)}(x^{(k)}(T))$$

- The mean field game model

Using the symmetry assumption we can drop the super indexes (k) and model the dynamics of a "generic" player by the SDE

$$\mathrm{d}x(t) = [\alpha(t) - g(t)]\mathrm{d}t - \sigma g(t)\mathrm{d}W(t) + \mathrm{d}N(t),$$

where $\sigma \neq 0$ and N is a reflecting process that ensures that $x \in [0, 1]$ a.s. The cost is modeled as

$$\mathbb{E}\left(\int_0^T \{\alpha(t)p(m(t), t) + h(\alpha(t), t) + f(x(t), t)\}\,\mathrm{d}t + \kappa(x(T))\right),$$

where

$$p(m(t), t) = D(\cdot, t)^{-1}\left[g(t) + \frac{d}{dt}\int_0^1 x\mathrm{d}m(t)(x)\right],$$

is the price when the expected mean consumption $g(t) + \frac{d}{dt}\int_0^1 x\mathrm{d}m(t)(x)$ and the total energy demand $D(\cdot, t)$ are given at time t.

Setting $h(\alpha) = \frac{1}{2}|\alpha|^2$, the above yields the following MFG system (omitting the function arguments)

$$\begin{aligned} -\partial_t u - \frac{1}{2}\sigma^2 g^2 \Delta u + \frac{1}{2}|Du + p(m)|^2 + gDu = & \quad f, \\ \partial m - \frac{1}{2}\sigma^2 g^2 \Delta m - \operatorname{div}\left[(Du + p(m) + g)m\right] = & \quad 0, \\ m(0) = m_0, \quad u(T, \cdot) = \kappa(\cdot). & \end{aligned}$$

- Questions:
  - It is this system well posed?
  - Numerical resolution?

Plan

References

On the theory of MFG problems:

- P. Cardaliaguet. *Notes on Mean Field Games*.
- J.-M. Lasry and P.-L. Lions. *Mean field games*.
- P.-L. Lions. *Cours au Collège de France*.

Numerical methods for MFG problems

- Y. Achdou and I. Capuzzo Dolcetta. *Mean field games: Numerical methods.*
- M. Bardi and I. Capuzzo Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations.*
- O. Guéant. *Mean field game equations with quadratic Hamiltonian: a specific approach*
- A. Lachapelle, Salomon and G. Turinici. *Computation of mean field equilibria in economics*.

- *A semi-discrete approximation for a first order mean field game problem*, with F. Camilli.
- *A fully-discrete Semi-Lagrangian approximation for a first order mean field game problem*, with E. Carlini.

Engineering literature

- Caines, Huang, Malhamé. *Large population stochastic wireless power control problem*
- Couillet, Perlaza, Tembine, Debbah. *Electrical Vehicles in the Smart Grid: A Mean Field Game Analysis.*