Free energy in non-convex mean-field spin glass models

Hong-Bin Chen (IHES)
2024 Greater Paris Area Mathematics Postdocs Day
16.10.2024

Outline

- The Sherrington—Kirkpatrick model
- Convex vs non-convex
- A PDE approach

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Dean's problem

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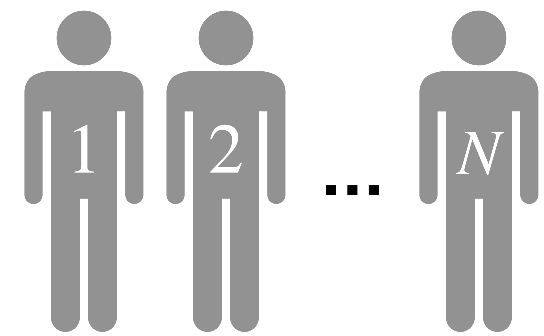
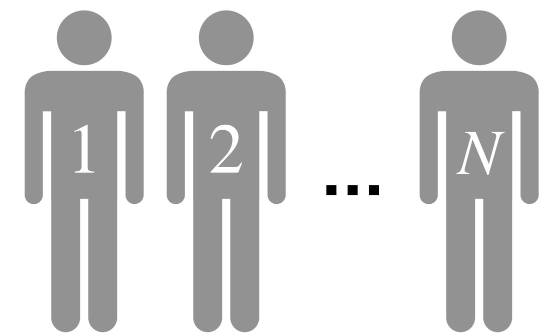
Dean's problem

$N \in \mathbb{N}$ – **number of students**



Dean's problem

$N \in \mathbb{N}$ – **number of students** 

$g_{ij} \in \mathbb{R}$ – **how much** 

Dean’s problem

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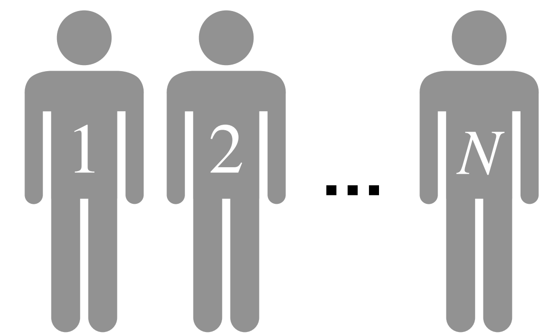

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Dean's problem

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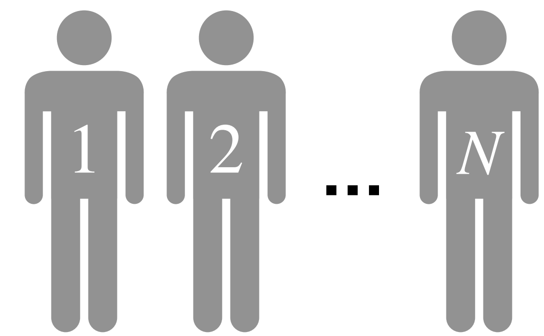

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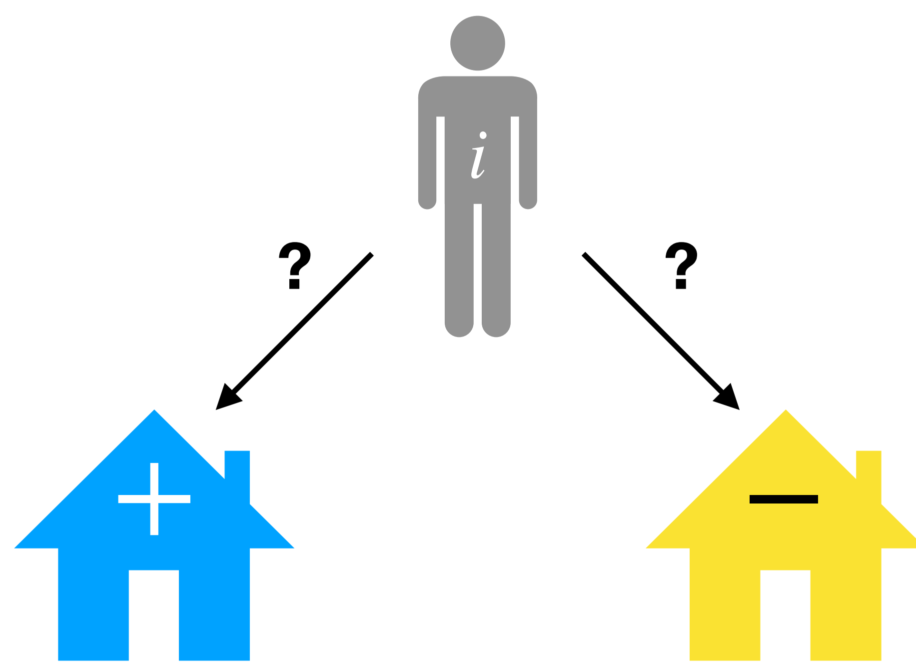

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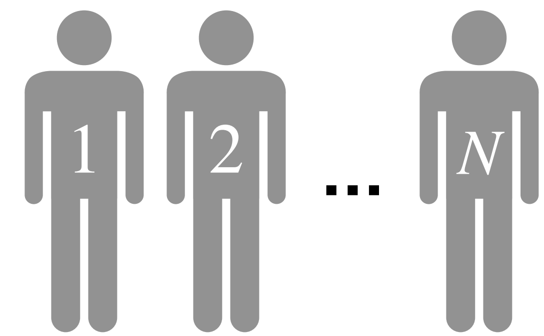

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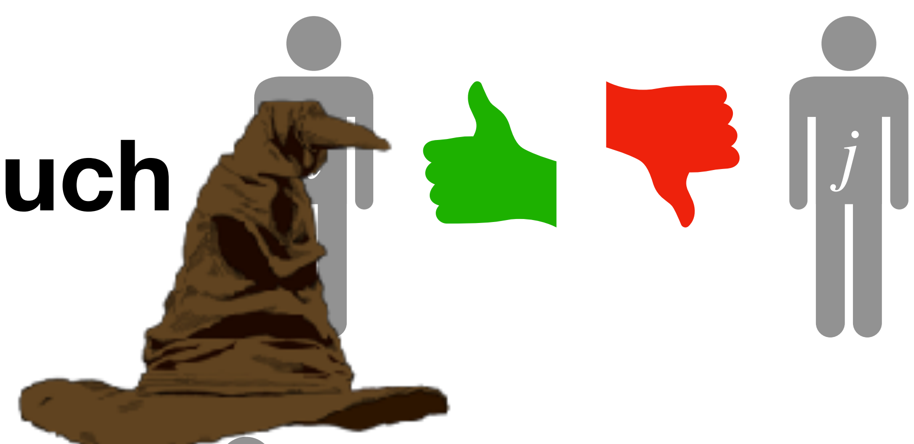

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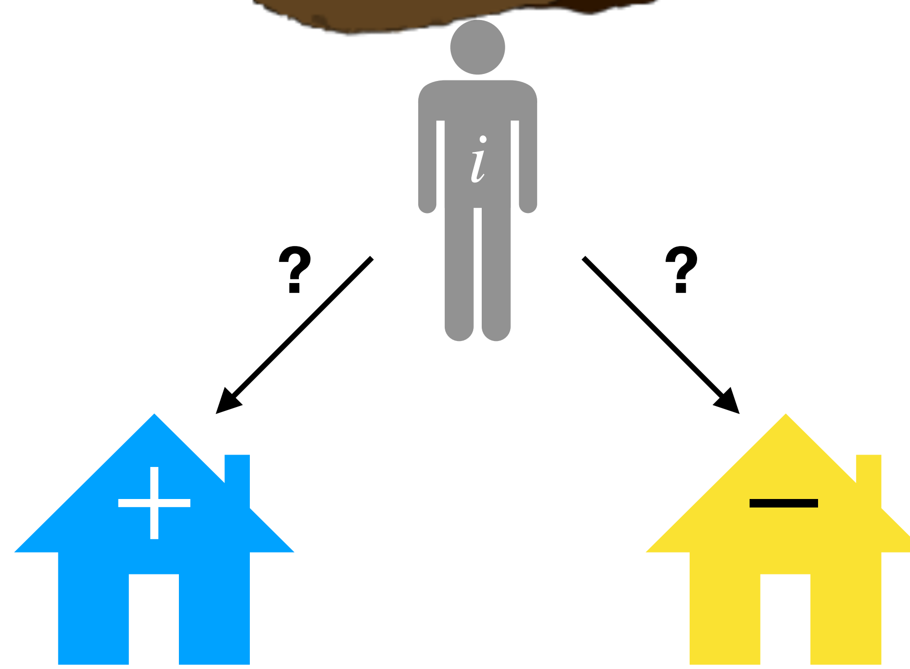
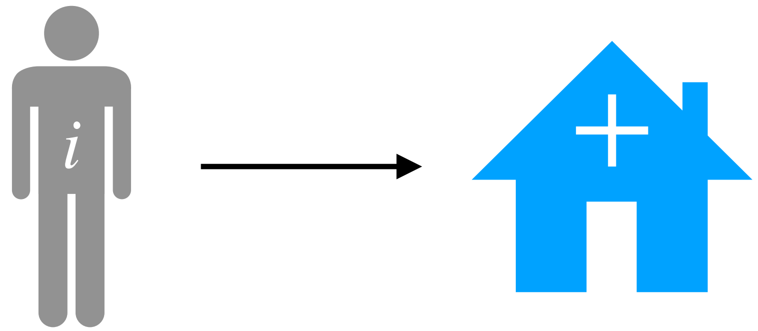


Dean’s problem

$$\sigma_i = +1$$



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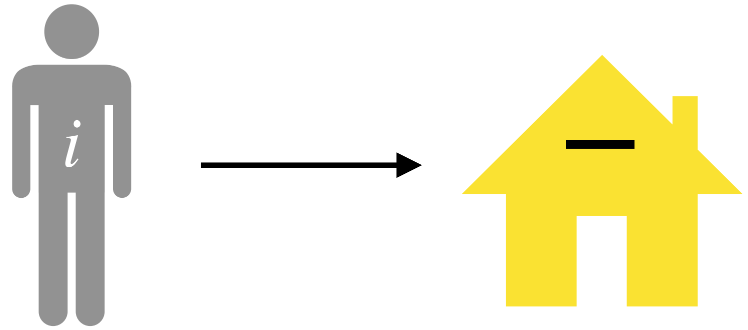

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$\sigma_i = +1$

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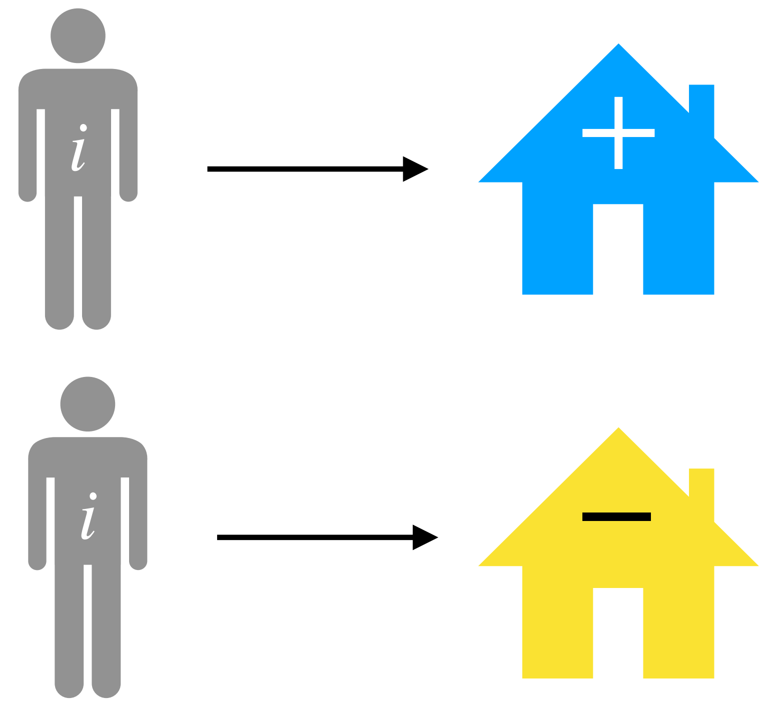

Assignment $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N) \in \{-1, +1\}^N$

Dean's problem

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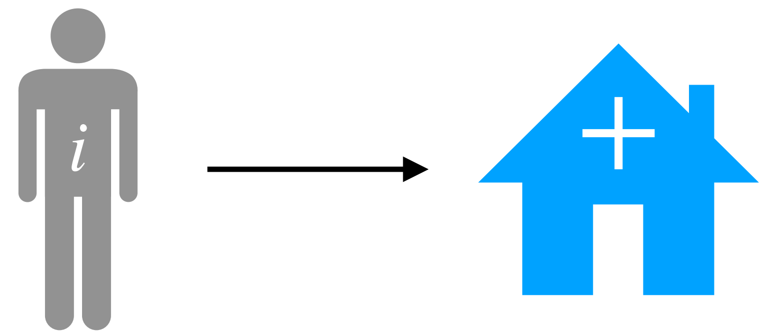

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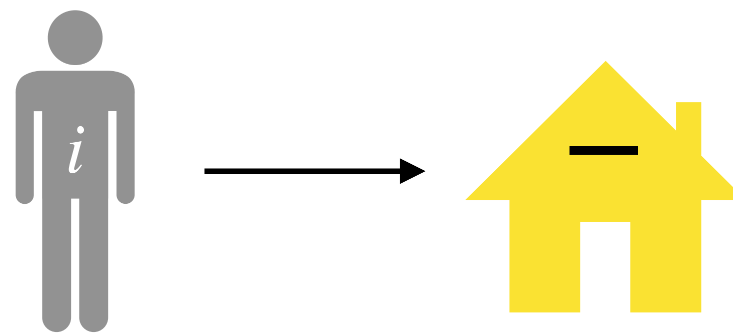

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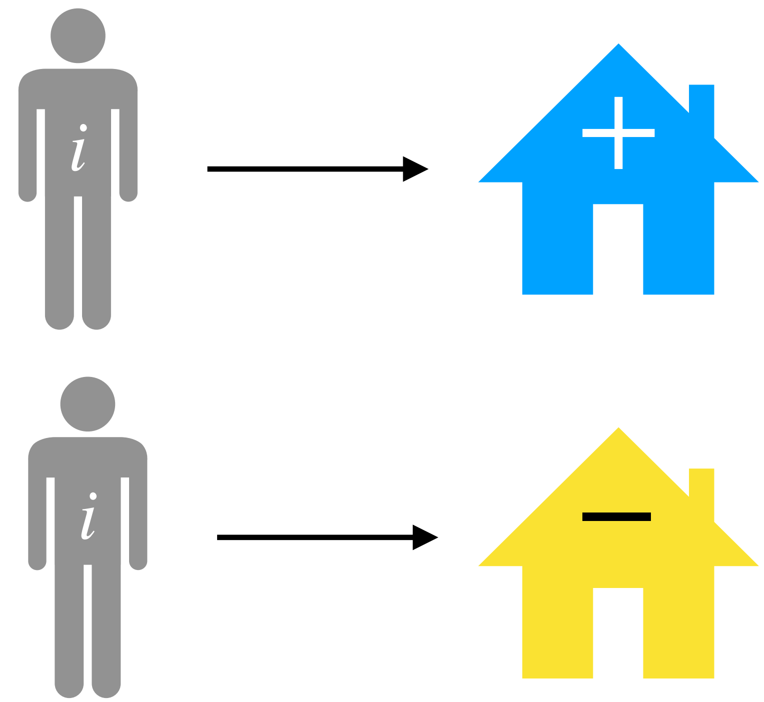

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Goal: find σ to maximise $H_N(\sigma)$

Dean's problem

$N = 2$ **To maximize** $H_2(\sigma) = \frac{1}{\sqrt{2}} g_{12}\sigma_1\sigma_2$

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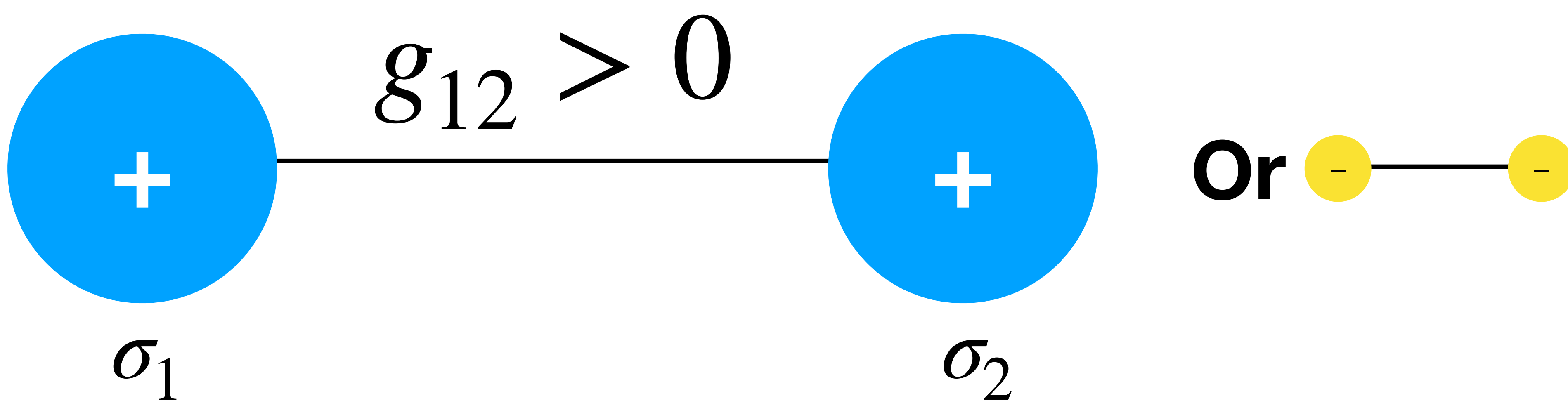

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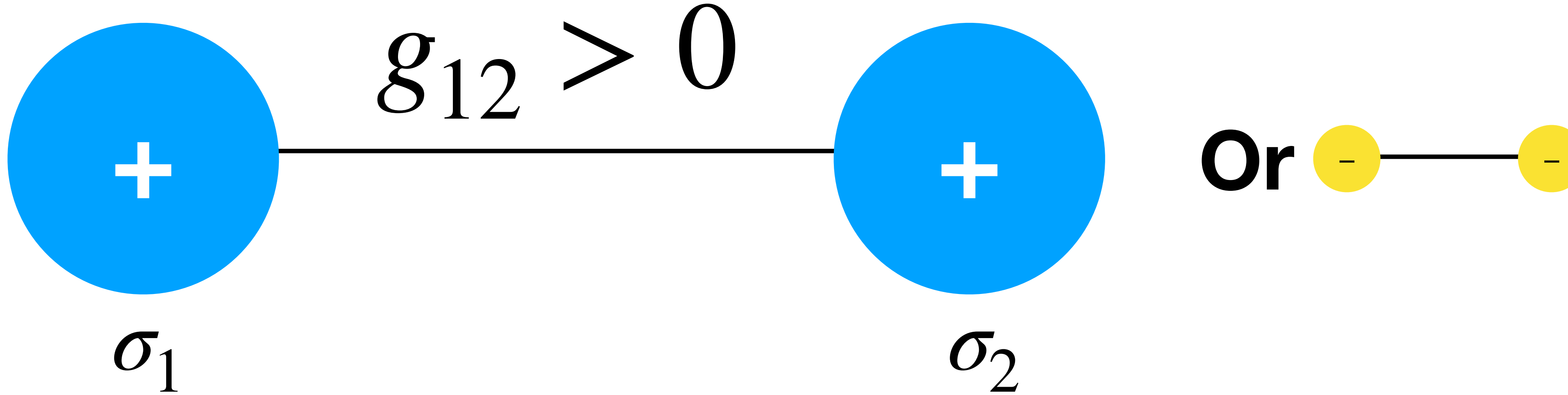

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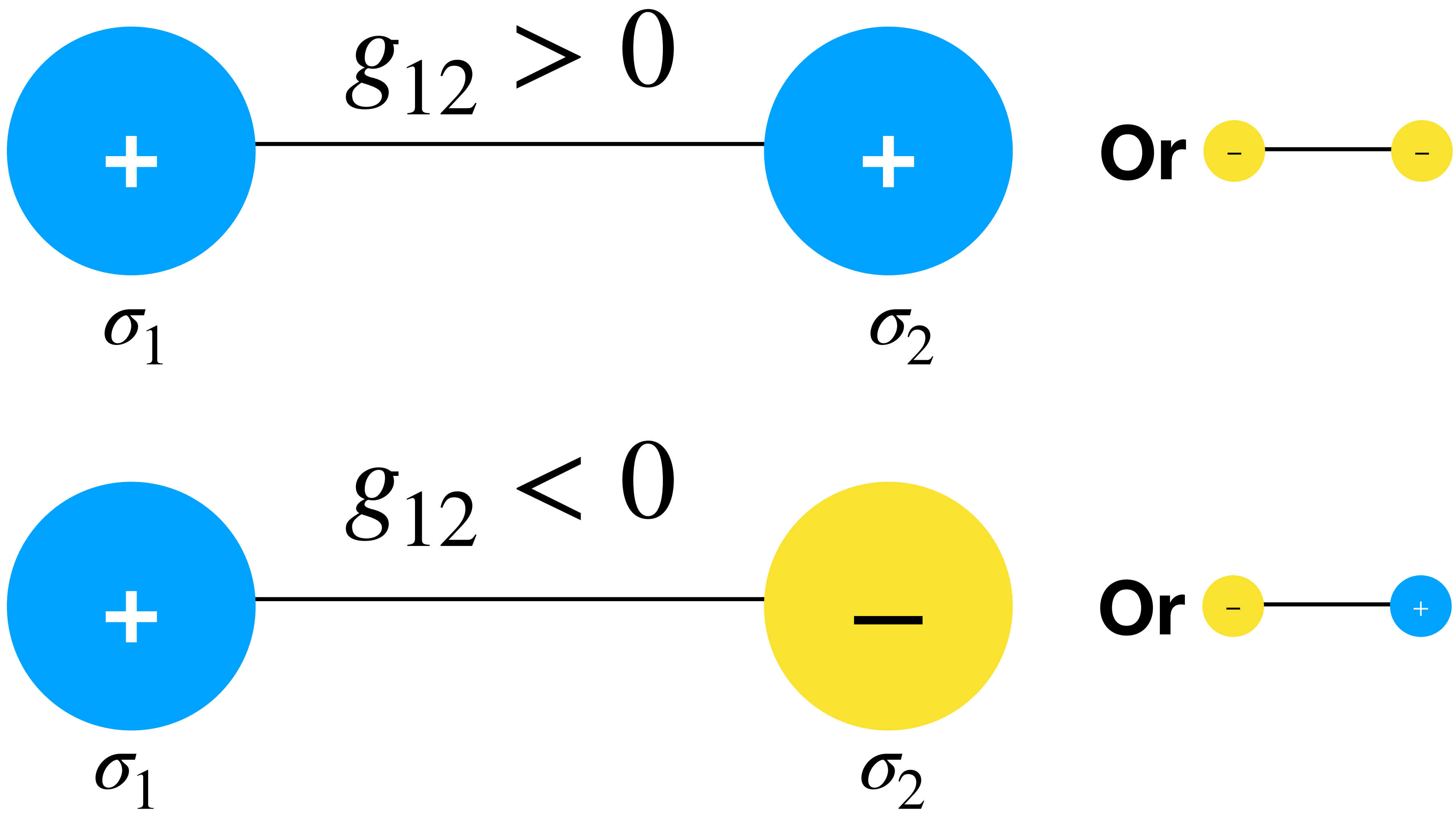

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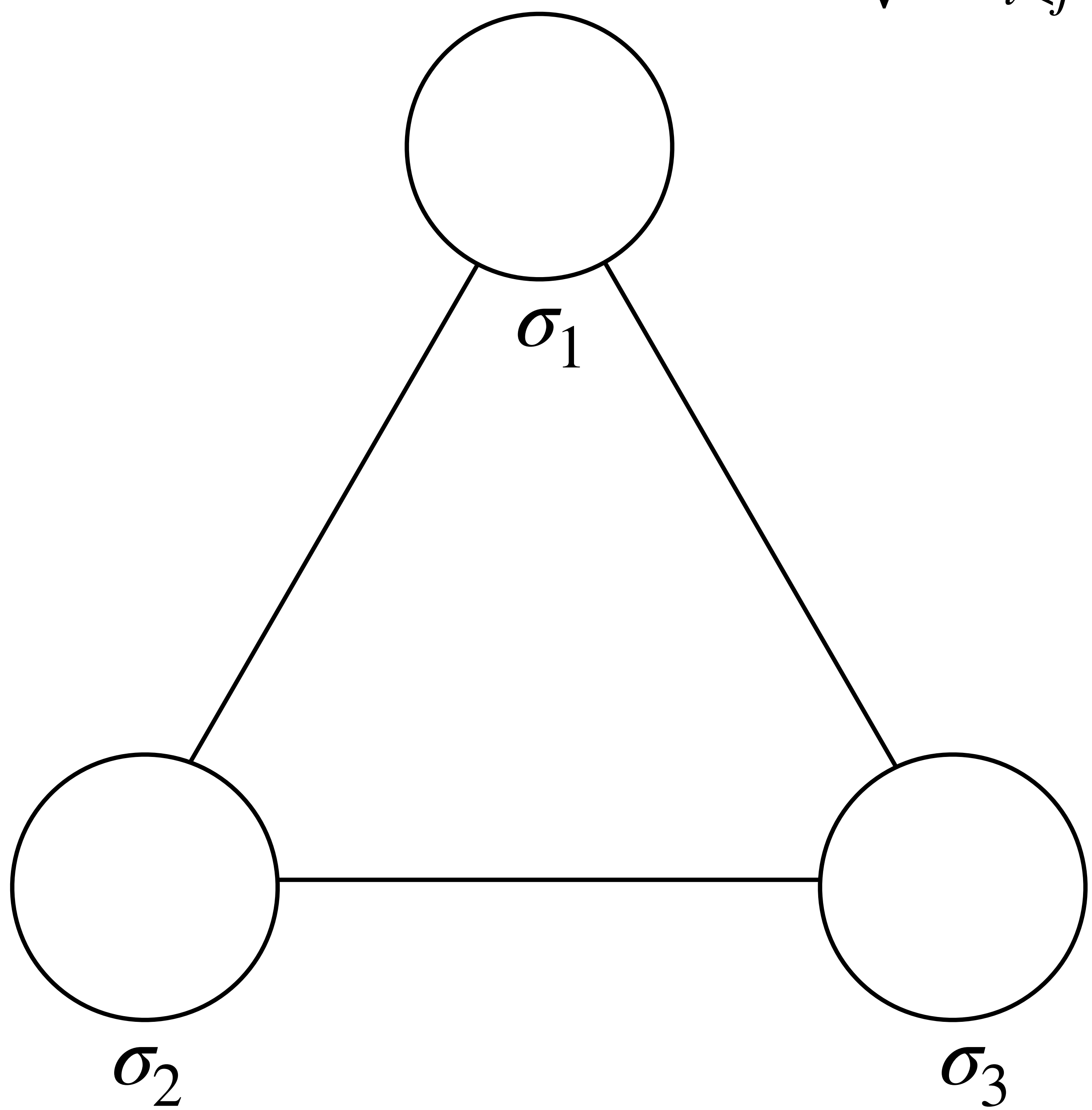

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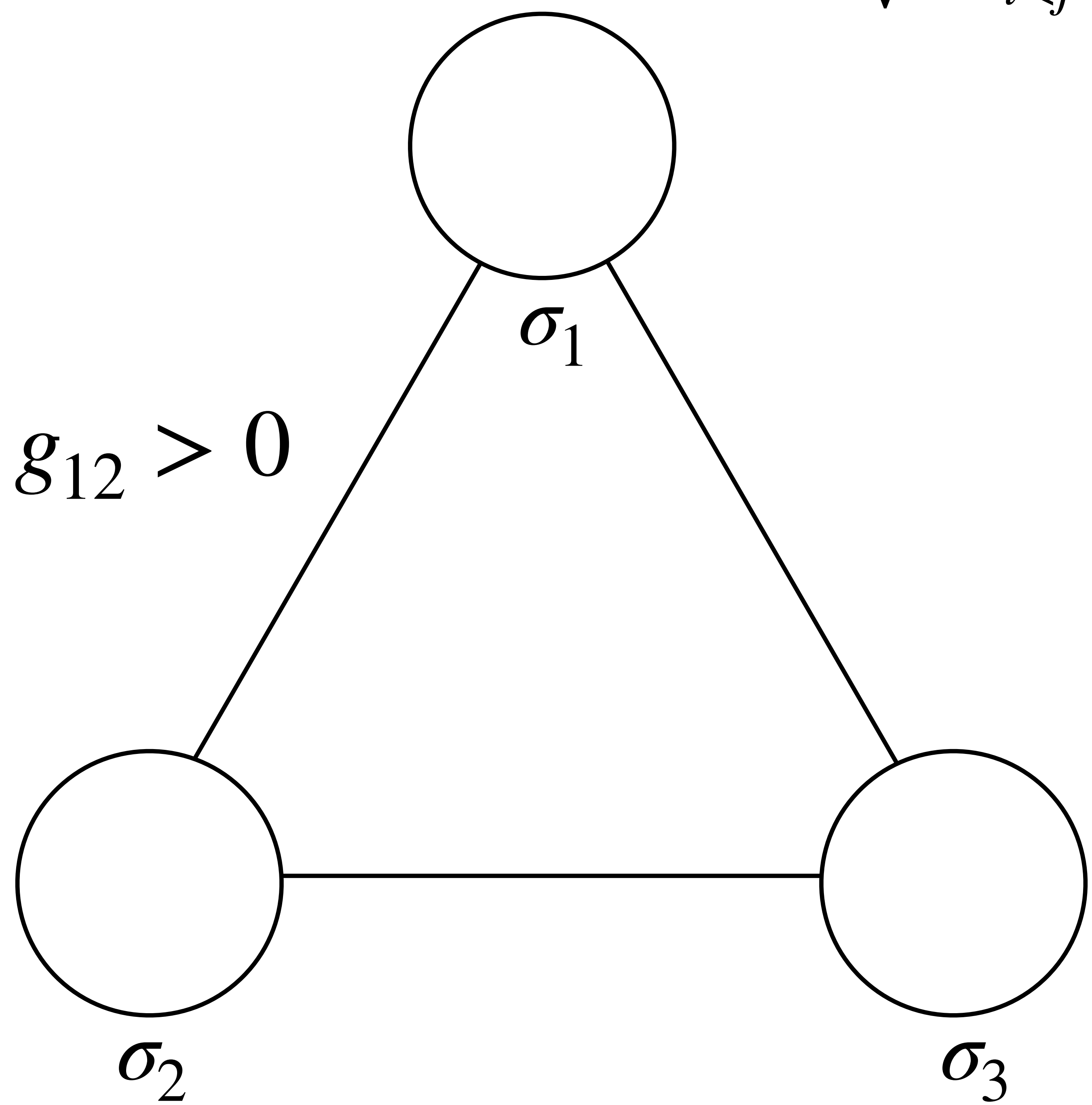

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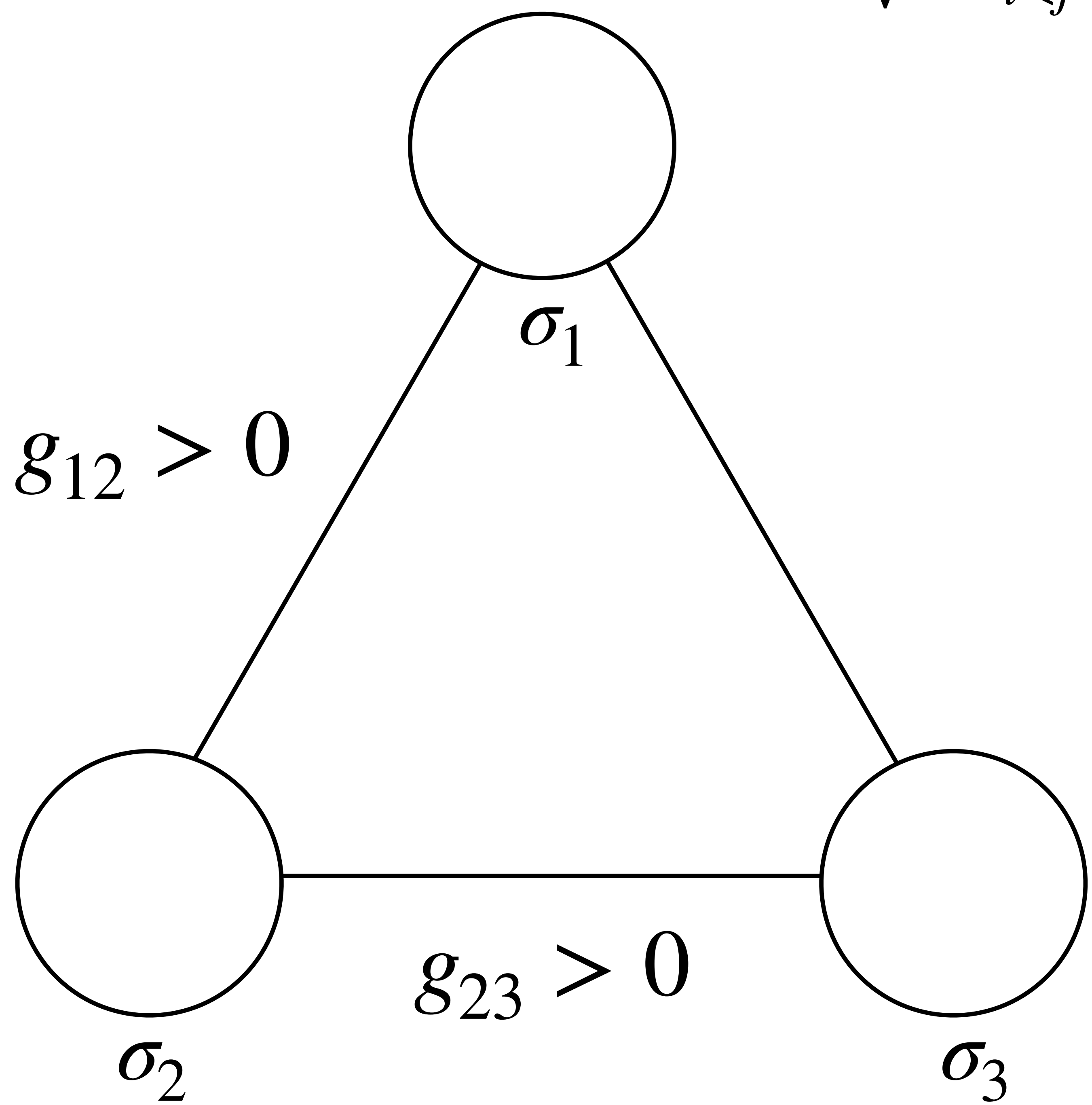

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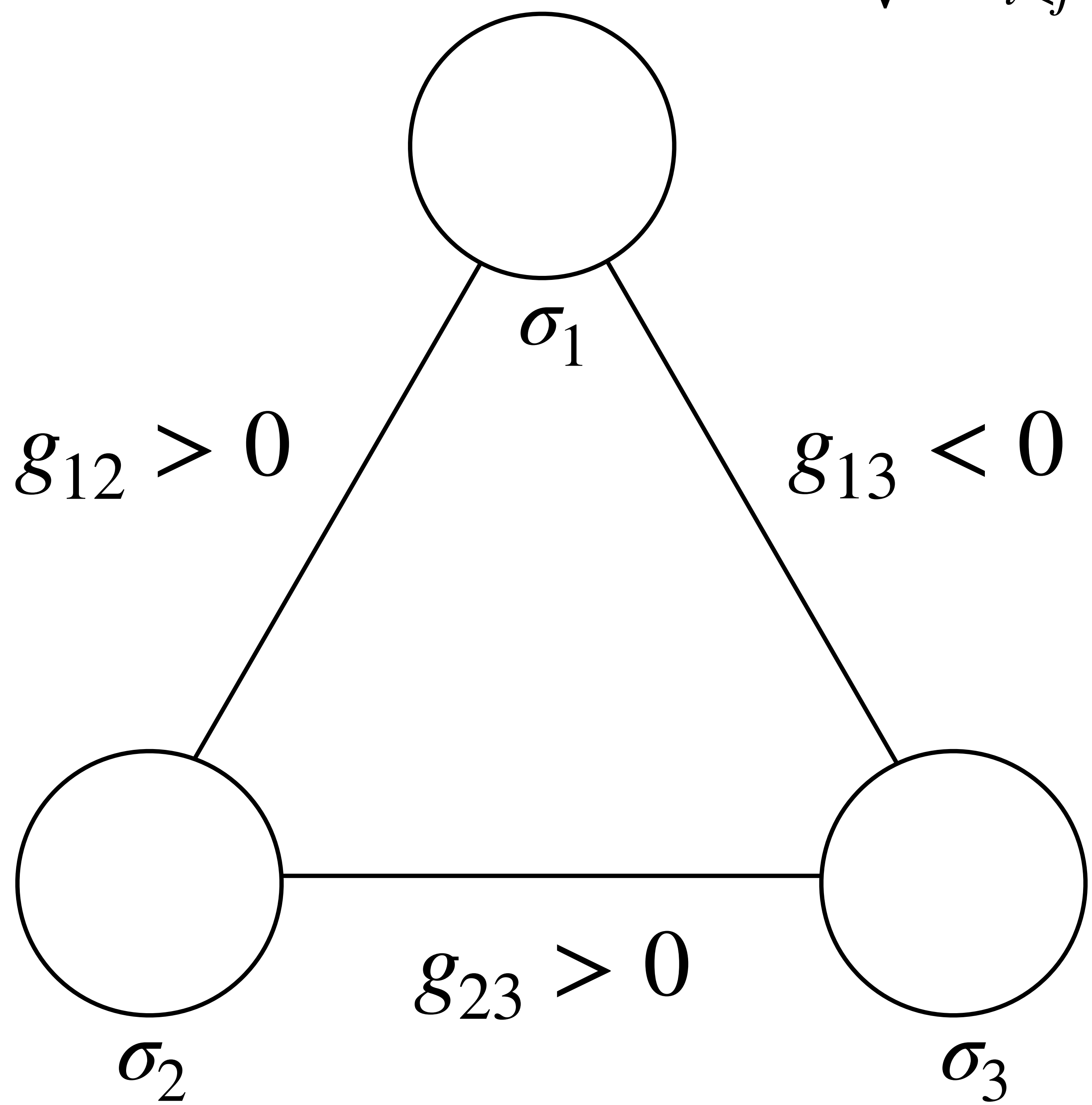

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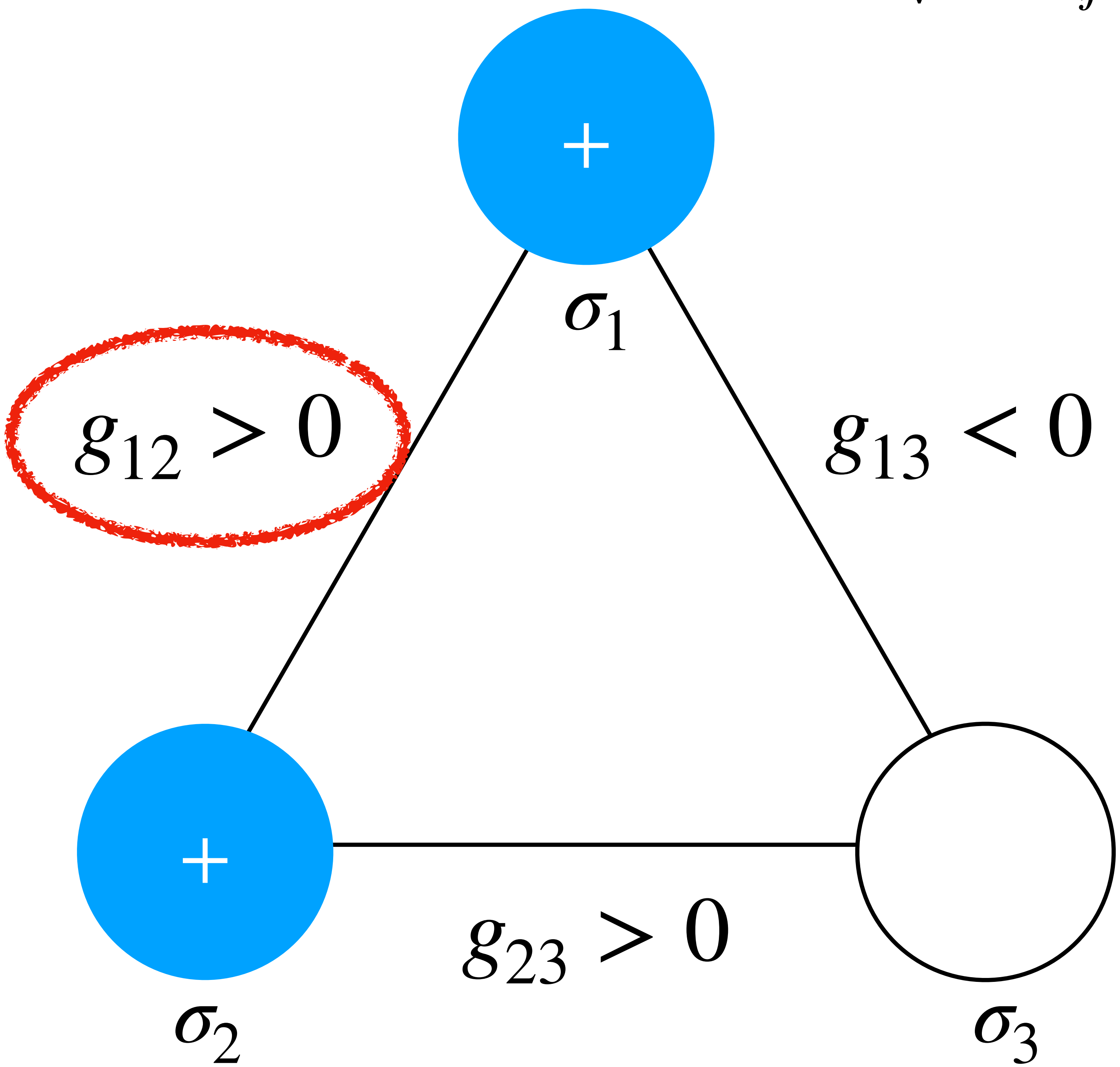

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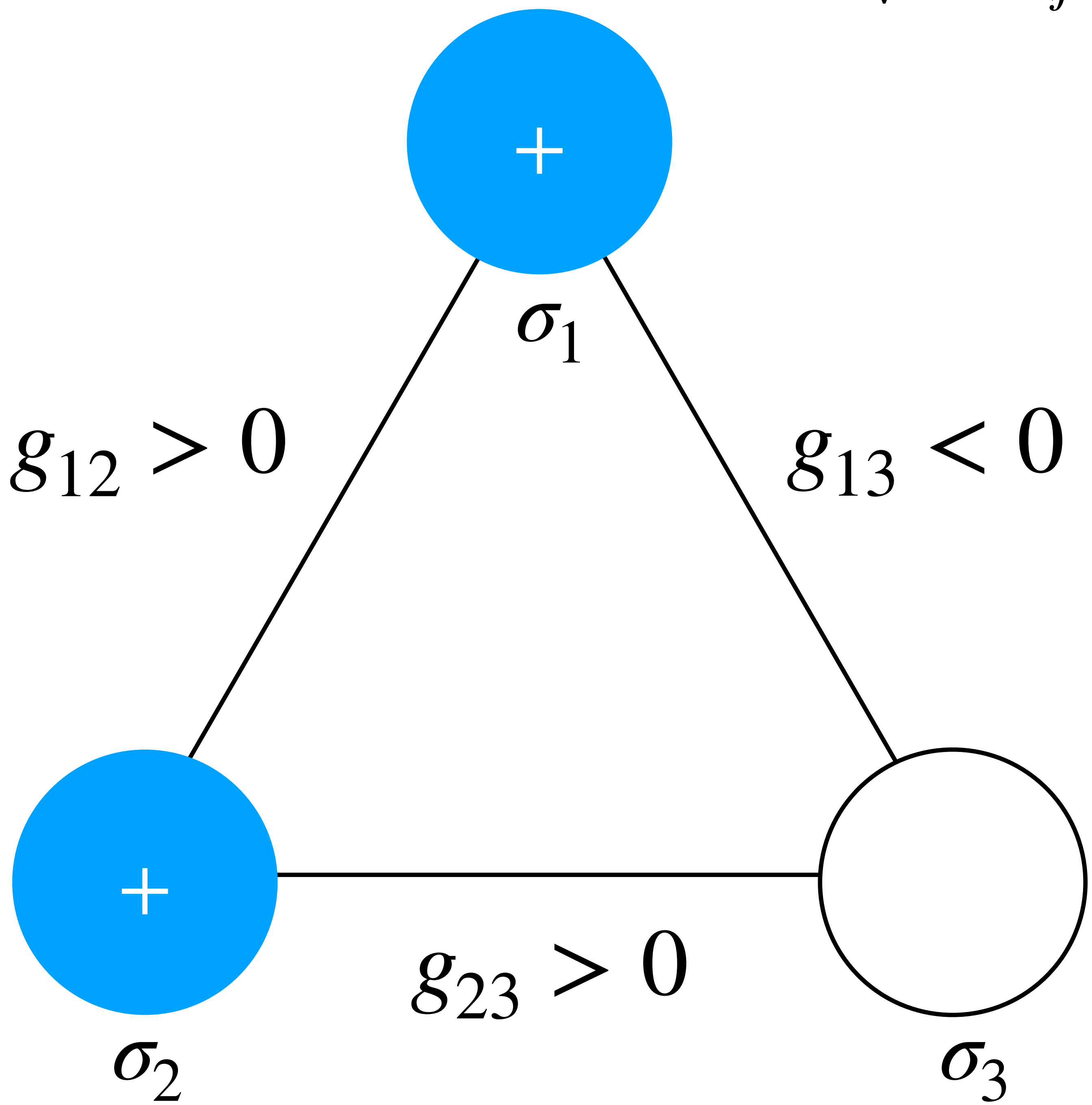

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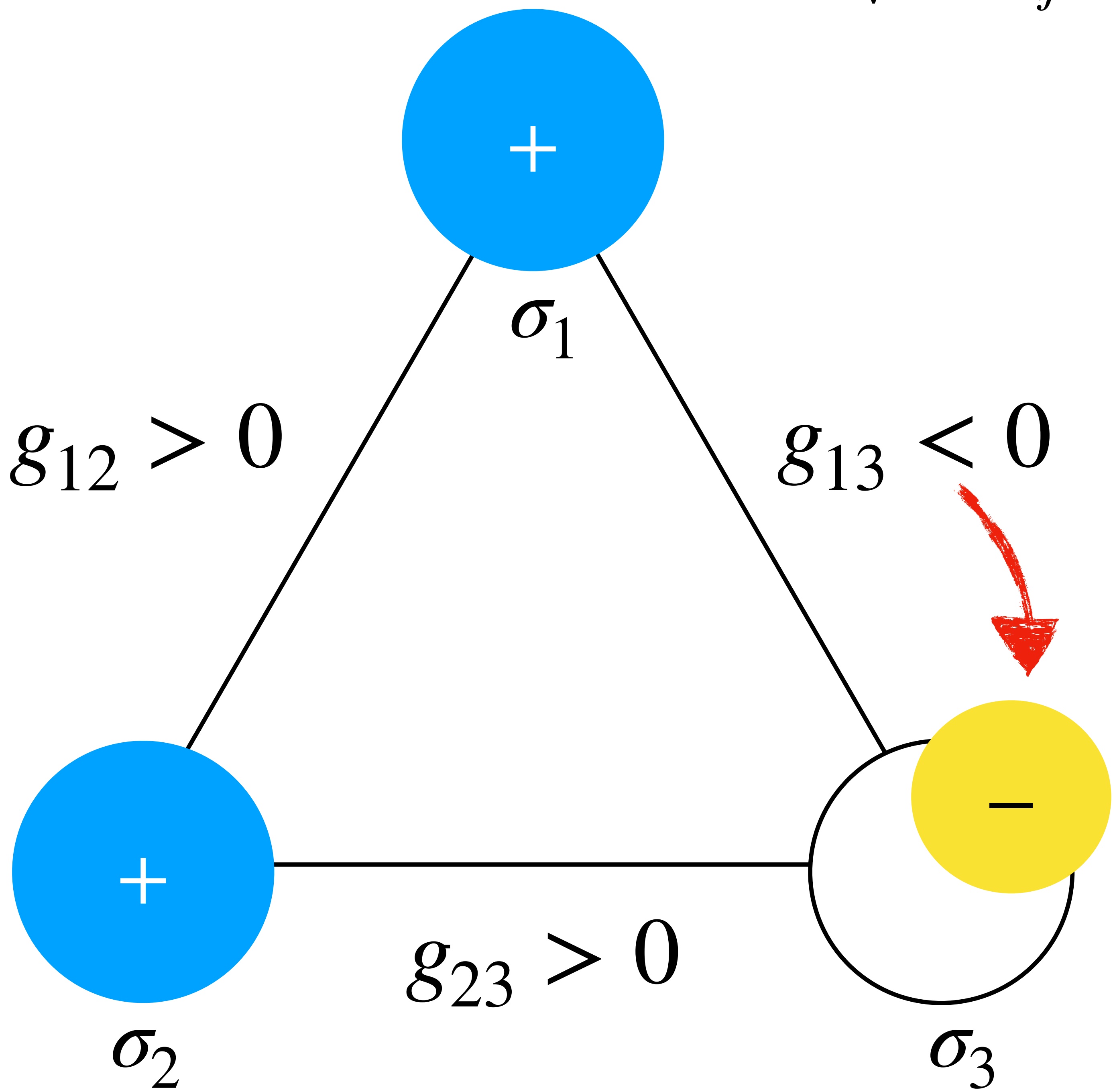

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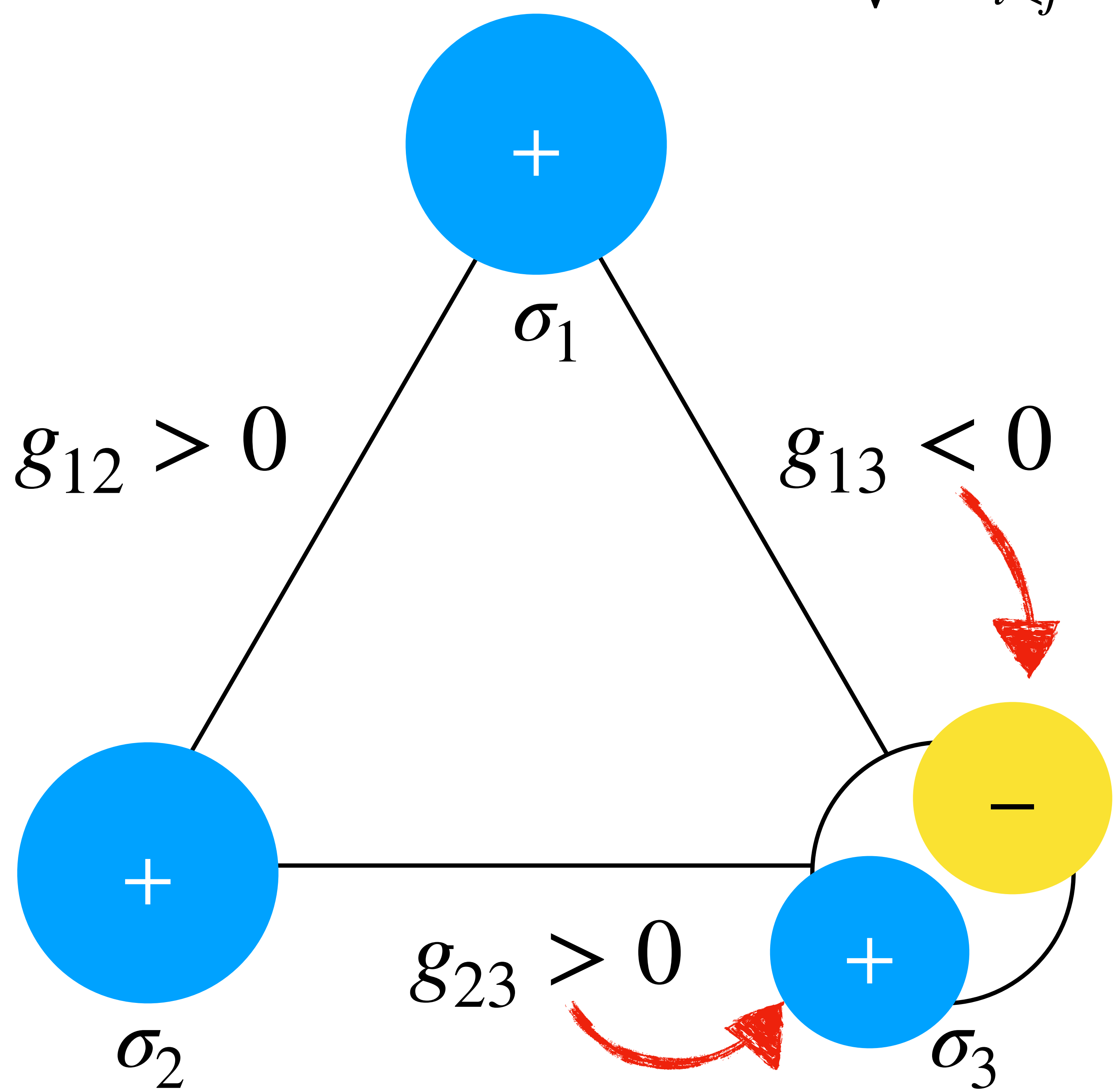

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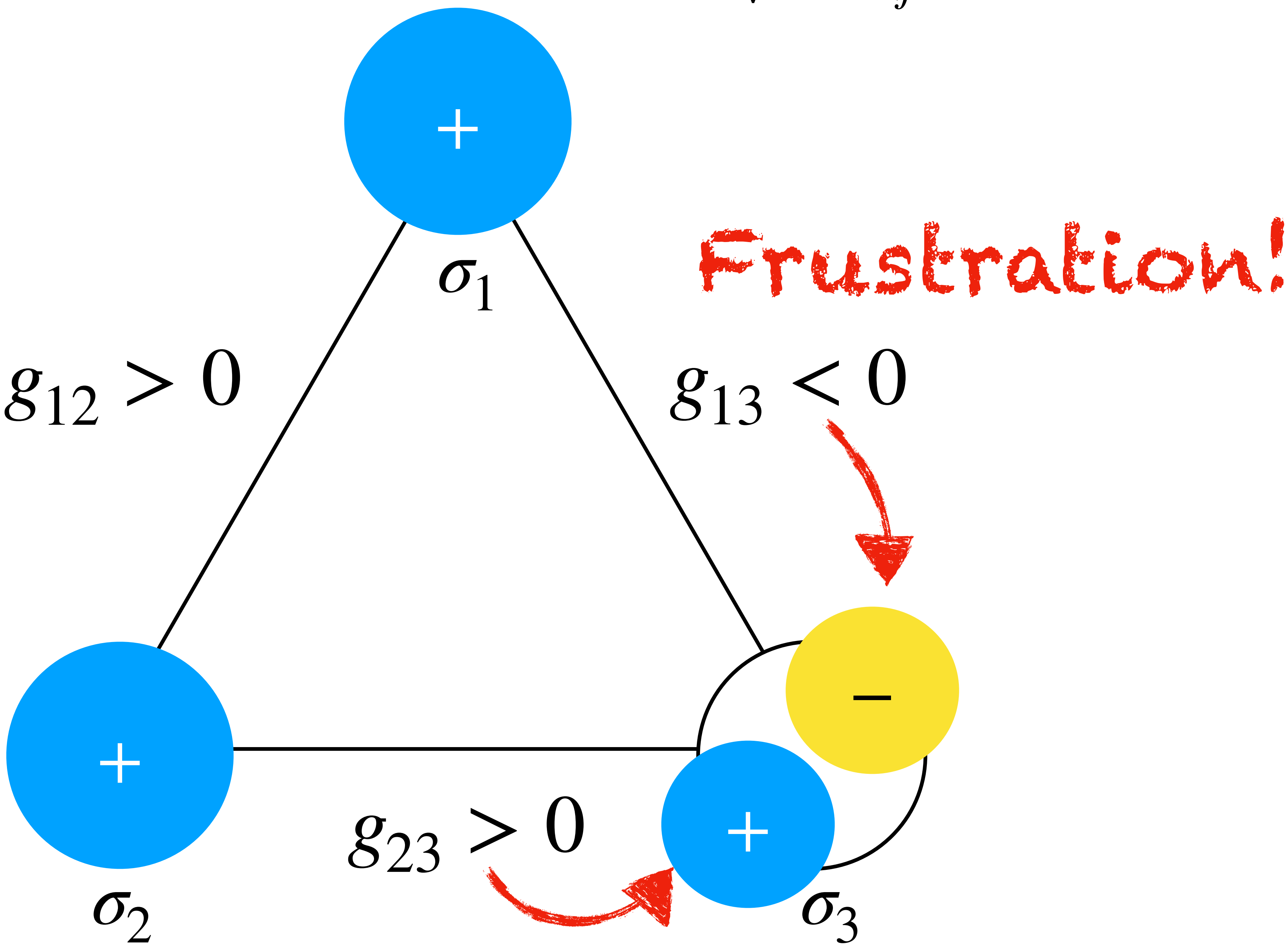

Sherrington—Kirkpatrick model ('75)

$N = 6$

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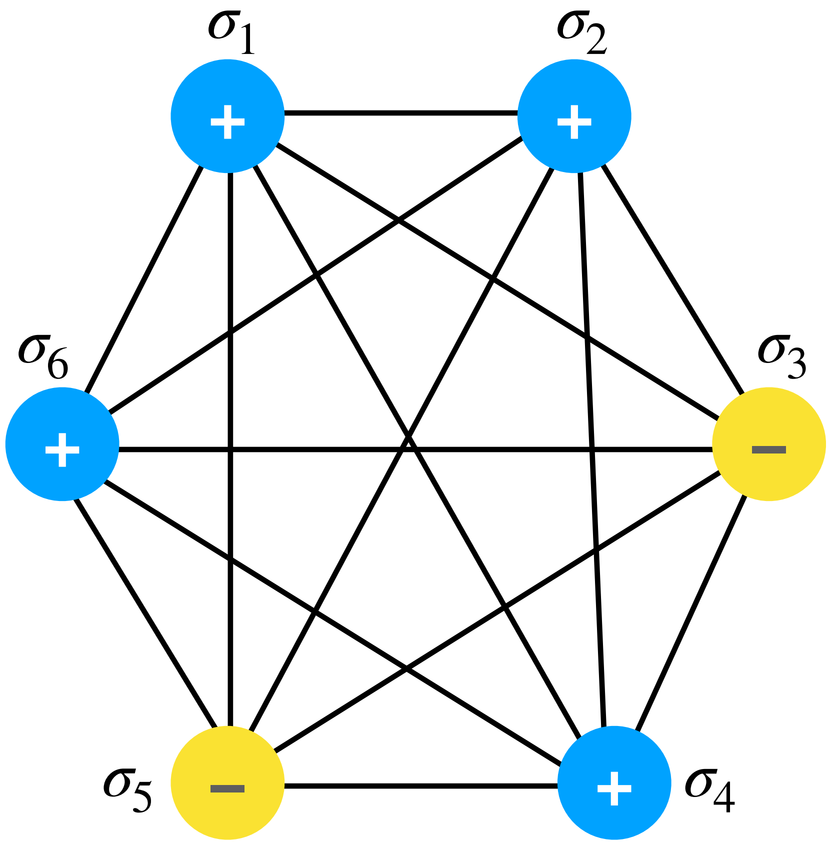

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Sherrington—Kirkpatrick model ('75)

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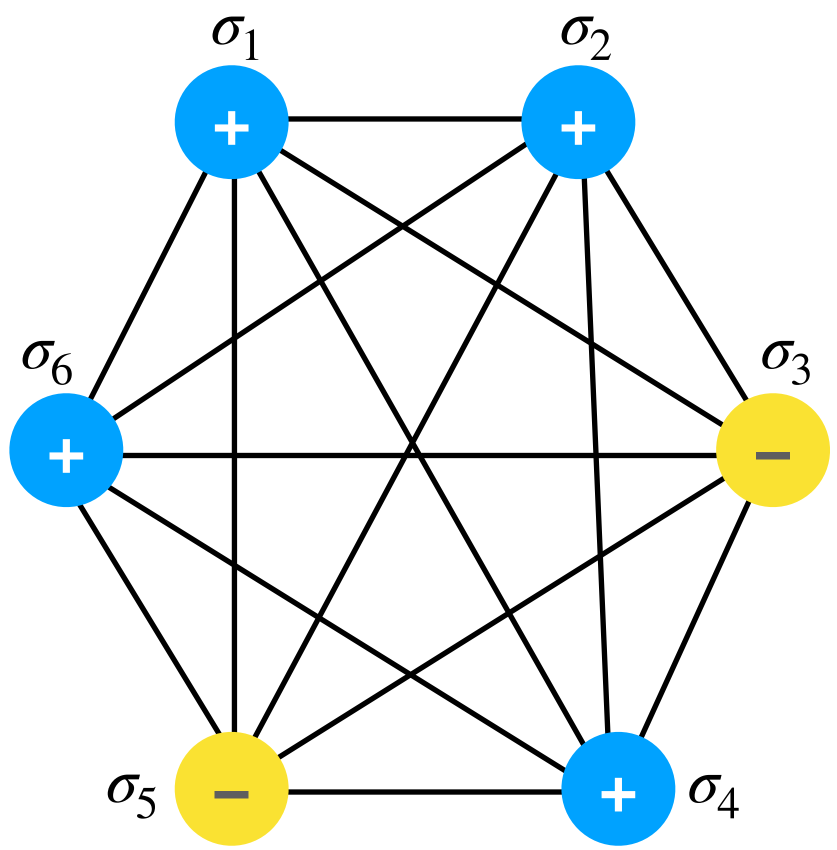

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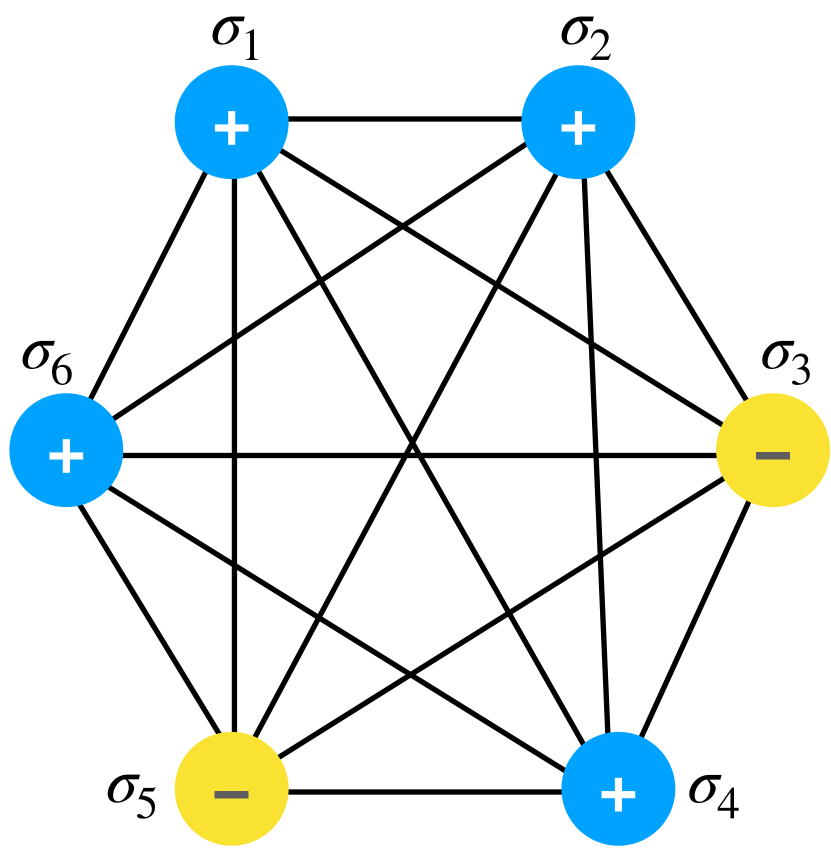

$N = 6$

$\left(g_{ij}\right)_{ij}$ **i.i.d. standard Gaussian**

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Average (g_{ij}) **i.i.d. Gaussian**

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Dean's problem

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Dean's problem

Interested in $\lim_{N \to \infty} F_N(\beta)$

Solving the SK model

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} g_{ij}\sigma_i\sigma_j \qquad F_N(\beta) = \frac{1}{N}\mathbb{E}\log\sum_{\sigma} e^{\beta H_N(\sigma)}$$

Solving the SK model

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Parisi ('80) $\lim_{N\to\infty} F_N(\beta) = \inf_{\mu} \mathscr{P}_\beta(\mu)$ **Nobel 2021**

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Probability measures $[0,1]$

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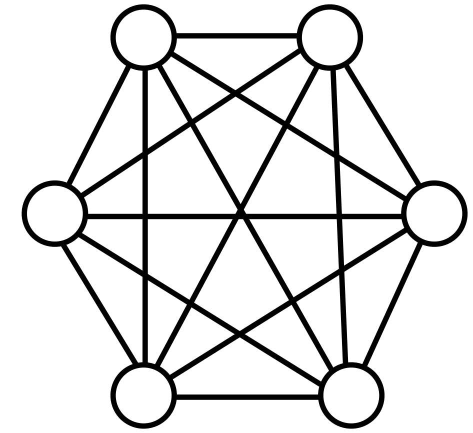
Math:

- **Guerra ('03)** $\lim F_N(\beta) \le \inf \mathscr{P}_\beta(\mu)$
- **Talagrand ('06)** $\lim F_N(\beta) \ge \inf \mathscr{P}_\beta(\mu)$ **Abel 2024**
- **Panchenko ('13+)** gave a more insightful proof

SK is convex



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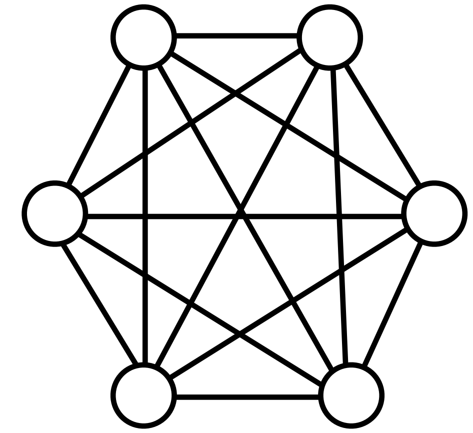

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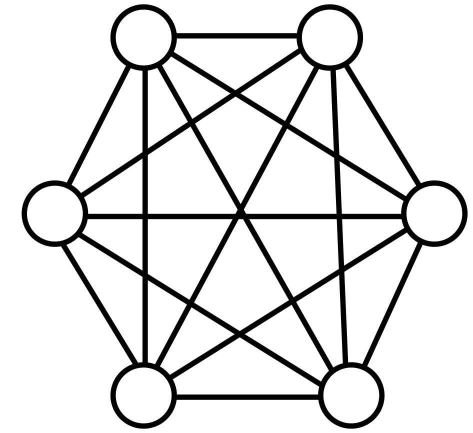

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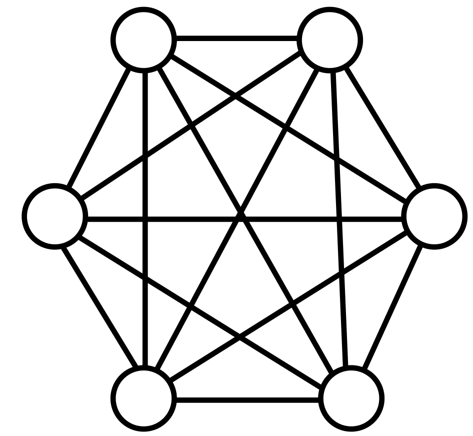
$$= N\left(\frac{\sigma \cdot \sigma'}{N}\right)^2$$

SK is convex



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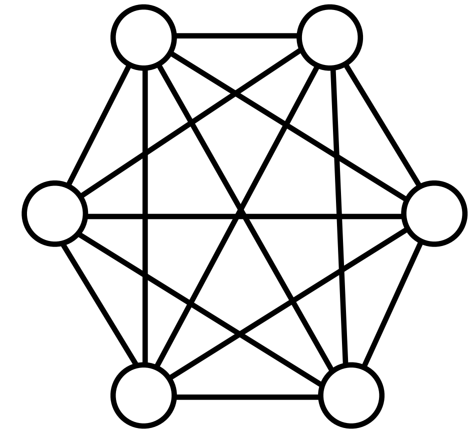

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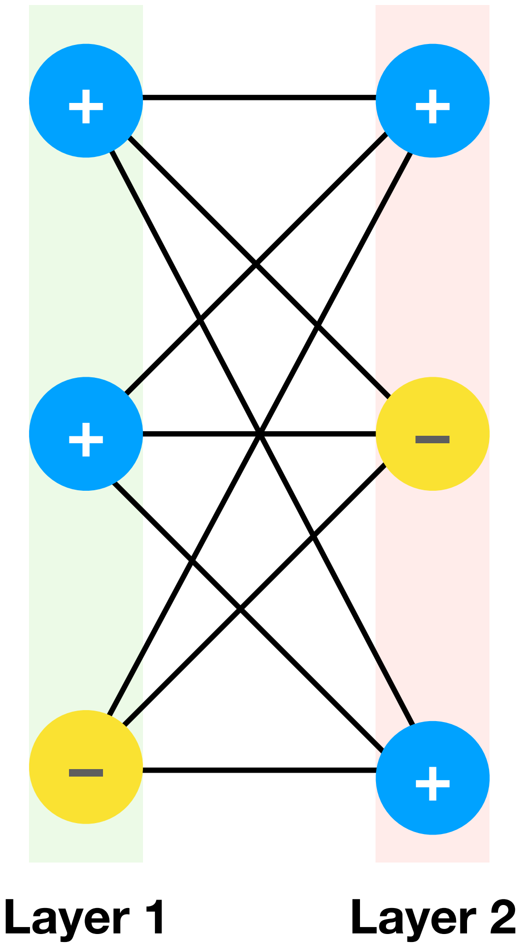
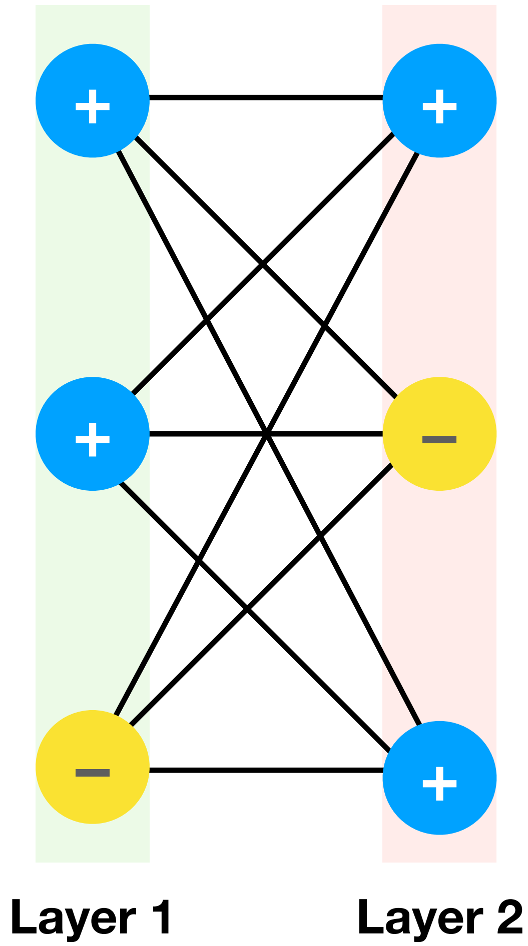
Overlap $\sim$ **# spins with the same sign**

Nonconvex: bipartite model



$N = 3$

Nonconvex: bipartite model



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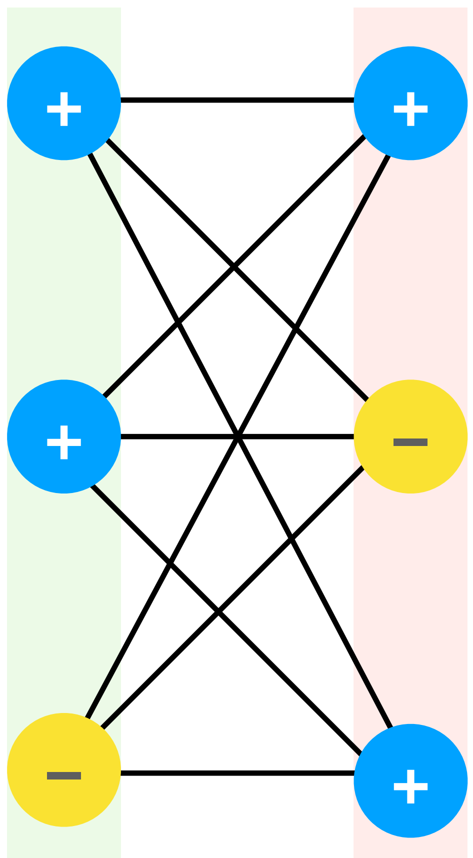
Nonconvex: bipartite model



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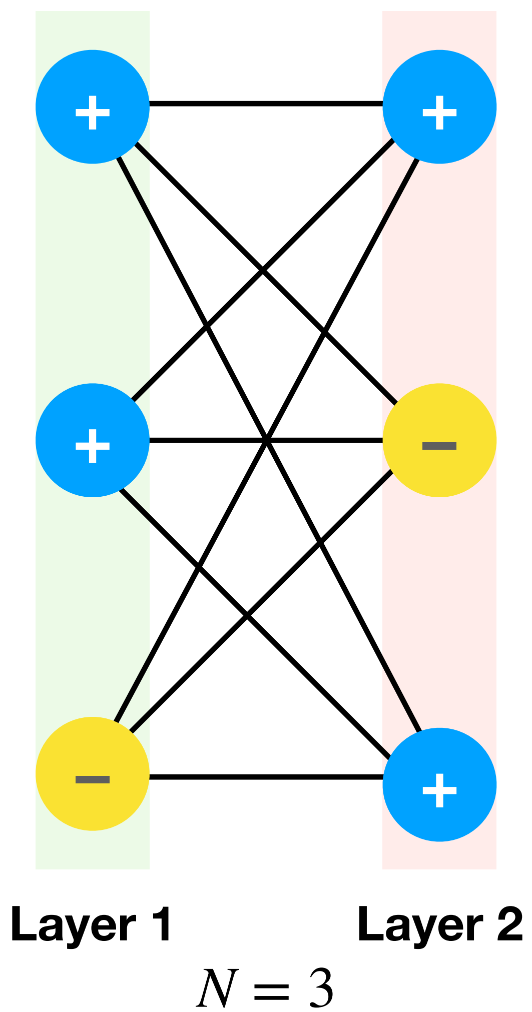

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Problem: theory based on Parisi formula is not available!

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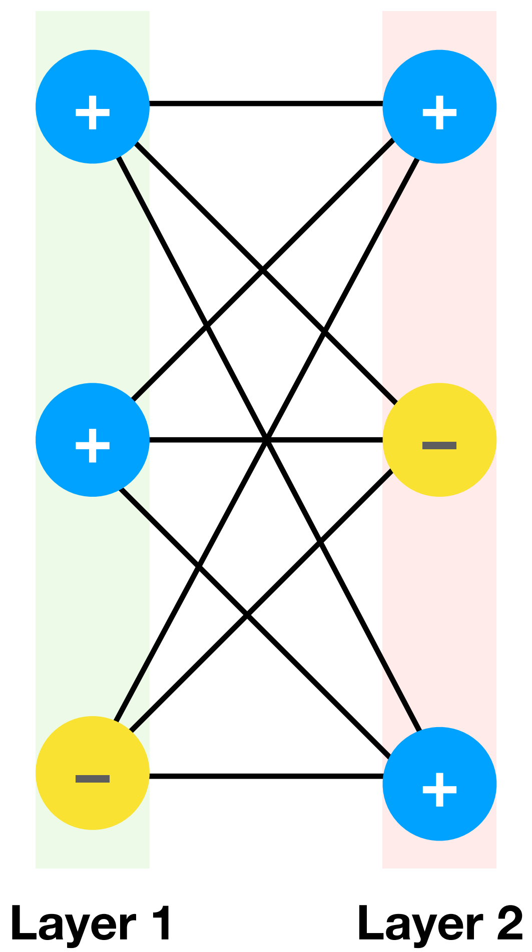

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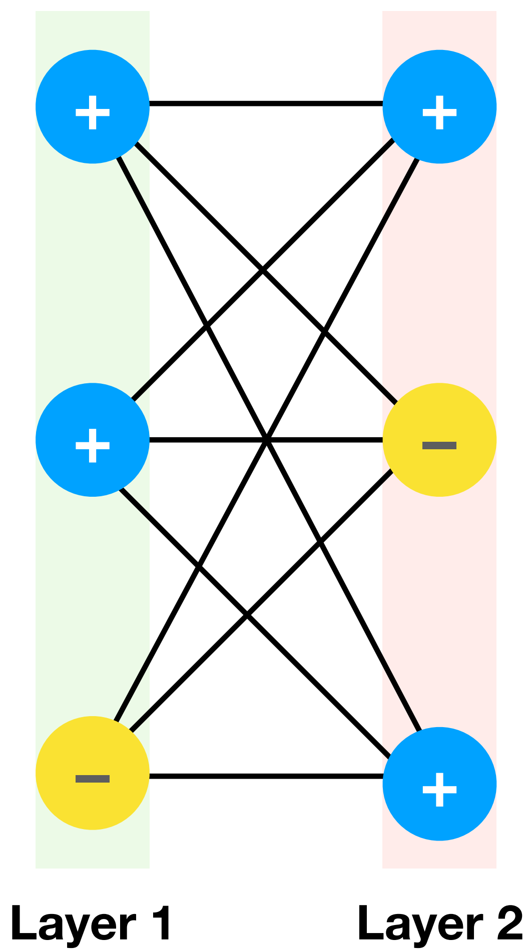

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Guerra $\lim F_N(\beta) \leq \inf \mathscr{P}_\beta(\mu)$ **Breaks down**

Talagrand/Panchenko $\lim F_N(\beta) \geq \inf \mathscr{P}_\beta(\mu)$ **Maybe not sharp**

General spin glass

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No prediction for the limit

A PDE perspective

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Math: Mourrat (2019+)

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Gibbs measure $\langle\,\cdot\,\rangle = \langle\,\cdot\,\rangle_{t,h}$

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The Cauchy problem of **(HJ)** *is well-posed.*

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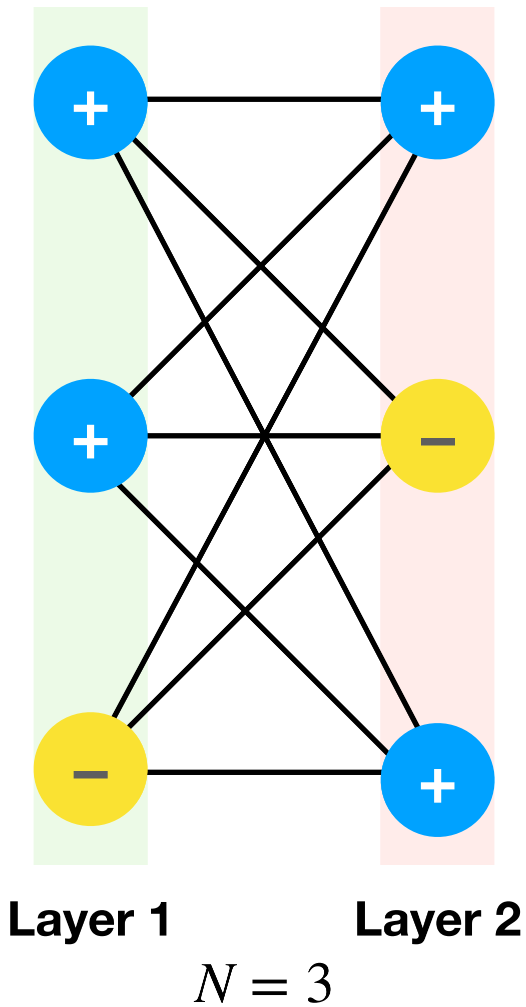
Conclusions

$$\partial_t f + \xi(-\partial_h f) = 0 \quad \textbf{(HJ)}$$



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