Mean-Field Games

Fourth lecture: Regularization, Selection, Learning

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Based on joint works with E. Bayraktar, R. Carmona, P. Cardaliaguet,
A. Cecchin, A. Cohen, R. Foguen, A. Vasileiadis

Part VIII. Restoration of Uniqueness

Restoration of uniqueness

• General purpose is to restore uniqueness by forcing the equilibria by a random noise

• Long history for ODEs

∘ ODE driven by bounded non-Lipschitz velocity field

$$\dot{X}_t = b(t, X_t), \quad \text{with prescribed } X_0$$

$\rightsquigarrow$ b continuous $\Rightarrow$ existence but ~~uniqueness~~

∘ well-known: noise may restore ! [Veretennikov, Krylov...]

∘ perturb the dynamics by a Brownian motion $(B_t)_{t \geq 0}$

$$dX_t = b(t, X_t)dt + dB_t$$

∘ based on smoothing properties of the heat kernel $\rightsquigarrow$ use the fact that the PDE

$$\partial_t u(t,x) + \frac{1}{2}\Delta u(t,x) + b(t,x) \cdot D_x u(t,x) = f(t,x)$$

has a strong generalized solution if f is bounded

Part VIII. Restoration of Uniqueness

a. A toy example

Linear quadratic control problem

- Dynamics of tagged player (in $\mathbb{R}^d$)

$$dX_t = \alpha_t dt + \sigma dW_t$$

  - cost functional of the form

$$J(\alpha) = \mathbb{E}\Big[\frac{1}{2}\big|c_g X_T + g(\bar{\mu}_T)\big|^2 + \int_0^T \Big[\frac{1}{2}\big|c_f X_t + f(\bar{\mu}_t)\big|^2 + \frac{1}{2}\big|\alpha_t\big|^2\Big]dt\Big]$$

  - coefficients c_f, c_g may be arbitrarily chosen (say 1)
  - σ may be 0 or 1 $\rightsquigarrow$ matters from numerical point of view
  - $\bar{\mu}_t$ is the mean of μ_t

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  - $\bar{\mu}_t$ is the mean of μ_t

- General form of the optimizer over α when μ is fixed

$$\alpha_t = -\eta_t X_t - h_t$$

  - η and $h \rightsquigarrow$ deterministic and η independent of μ!
  - optimal trajectories

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$$dX_t = \big(-\eta_t X_t - h_t\big)dt + \sigma dW_t$$

  - X is an O.-U. process $\rightsquigarrow$ conditional on X_0, marginal of X is Gaussian with fixed variance $\rightsquigarrow$ fixed point on the mean only!

Search for equilibria

- Characterization of (η, h) for a given μ
  - equation for $\eta \rightsquigarrow$ Riccati equation

$$\dot{\eta}_t = \eta_t^2 - c_f^2, \quad \eta_T = c_g^2$$

  - equation for $h \rightsquigarrow$ backward linear ODE

$$\dot{h}_t = -\Big(c_f f(\bar{\mu}_t) - \eta_t h_t\Big), \quad h_T = c_g g(\bar{\mu}_T)$$

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- End up with forward backward ODE

$$\dot{\bar{\mu}}_t = \big(-\eta_t \bar{\mu}_t - h_t\big)$$
$$\dot{h}_t = -\big(c_f f(\bar{\mu}_t) - \eta_t h_t\big), \quad h_T = c_g g(\bar{\mu}_T)$$

Uniqueness to the FB system

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  - Cauchy-Lipschitz theory in small time only
  - may loose existence / uniqueness on a given time interval

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- Characteristics system of finite-dimensional master equation

$$\partial_t v(t,x) + (-\eta_t x - v(t,x_t))\partial_x v(t,x) + \big(f(x) - \eta_t v(t,x)\big)$$
$$v(T,x) = g(x)$$

  - if smooth solution $\rightsquigarrow$ $\boxed{h_t = v(t,\bar{\mu}_t)}$

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- Well-posedness if $\bar{b} \equiv 0, \bar{f}, \bar{g} \nearrow \Rightarrow$! of characteristics
  - if not $\Rightarrow$ shocks may emerge in finite time...
- $\sigma = 1$ does not help

Common noise

- Return to the FB system and add a noise

$$
\begin{aligned}
d\bar{\mu}_t &= \Big(-\eta_t\bar{\mu}_t - h_t\Big)dt + \varepsilon dB_t \\
dh_t &= -\Big(f(\bar{\mu}_t) - \eta_t h_t\Big)dt - \varepsilon k_t dB_t \\
h_T &= g(\bar{\mu}_T)
\end{aligned}
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  - roughly speaking, add $\varepsilon^2 \partial^2_{xx}$ in master equation
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  - or Girsanov for decoupling the forward and backward equations
- Interpretation of B in the definition of the equilibria?

$$
dX_t = \alpha_t dt + \sigma dW_t + \varepsilon dB_t
$$

  - fixed point condition $\rightsquigarrow \mu_t = \mathcal{L}(X_t^{\star,\mu} | B)$ and $\bar{\mu}_t = \mathbb{E}[X_t^{\star,\mu} | B]$
  - B is common noise!

Selection of equilibria

- Use vanishing viscosity to select equilibria
  - focus on simpler (but typical of LQ models) case ($X_0 = 0$)

$$dX_t = \alpha_t dt + dW_t, \quad J(\alpha) = \mathbb{E}\Big[X_T g(\mu_T) + c_g g(\mu_T)^2 + \frac{1}{2}\int_0^T \alpha_t^2 dt\Big]$$

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- Same analysis as before $\rightsquigarrow$ ODE system

$$\dot{\bar{\mu}}_t = -h_t, \quad \dot{h}_t = 0, \quad h_T = \bar{g}(\bar{\mu}_T) \quad \left(\bar{\mu}_0 = 0\right)$$

  - choose $\bar{g}(x) = \begin{cases} -x & x \in [-1, 1] \\ -\text{sign}(x) & |x| \geq 1 \end{cases}$

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- Equilibria parametrized by $A = h_T \Leftrightarrow A = \bar{g}(-TA)$
  - $T > 1$ (1 = time to observe a shock) $\Rightarrow A \in \{-1, 0, 1\}$

$$A = 0 \Rightarrow J^{opt} = 0, \quad A = \pm 1 \Rightarrow J^{opt} = -TA^2 + c_g A^2 + \frac{1}{2} TA^2$$

- if c_g large then equilibrium of lower cost is $A = 0$!

Vanishing viscosity

- Restore uniqueness by adding a common noise

$$d\bar{\mu}_t^\epsilon = -h_t^\epsilon dt + \epsilon dB_t,$$
$$dh_t^\epsilon = dM_t^\epsilon, \quad h_T^\epsilon = \bar{g}(\bar{\mu}_T^\epsilon)$$

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- PDE interpretation $\rightsquigarrow h_t^\epsilon = v^\epsilon(t, \bar{\mu}_t^\epsilon)$
  - v^ϵ solves viscous Burgers equation

$$\partial_t v^\epsilon - v^\epsilon \partial_x v^\epsilon + \frac{\epsilon^2}{2} v^\epsilon = 0, \quad v^\epsilon(T, \cdot) = \bar{g}$$

  - known fact: $v^\epsilon(t,x) \to -\mathrm{sign}(x)$ as $\epsilon \searrow 0$ for $t < T - 1$

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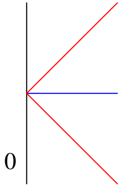
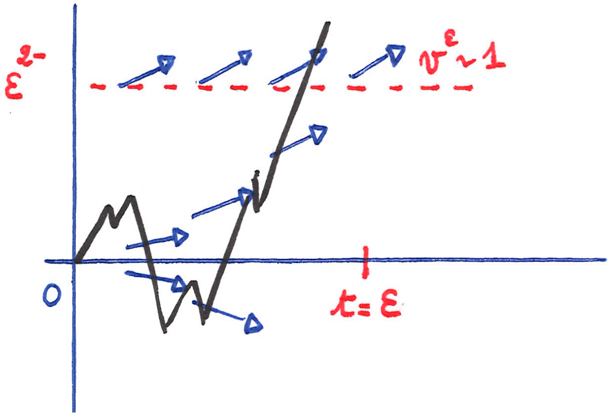
  - known fact: $v^\epsilon(t,x) \to -\text{sign}(x)$ as $\epsilon \searrow 0$ for $t < T - 1$
- Statement: As $\epsilon \searrow 0$ $(\bar{\mu}_t^\epsilon)_t$ converges (in law) to $\frac{1}{2}\delta_{(t)_t} + \frac{1}{2}\delta_{(-t)_t}$
  - do not see $A = 0$!



Sketch of proof



• In time ϵ, the particle should go beyond ϵ^{2-} with high probability

○ then, the drift is very close to $\pm 1 \rightsquigarrow$ the particle follows the drift with very high probability

Part VIII. Restoration of Uniqueness

b. What next?

Other models

• General purpose is to understand the action of the common noise onto uniqueness of equilibria without monotonicity

• Several instances in the Euclidean case

  ◦ $1d$ LQ MFG with common noise [Foguen, 18]

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- Finite state space

  ◦ use a variant of Wright-Fischer/Moran model, see [Bayraktar, Cecchin, Cohen, D., 21]

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  - challenging question in full generality
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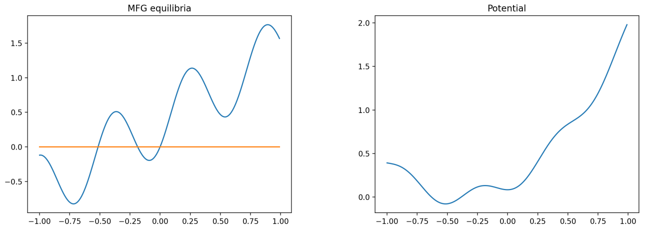
  - $1d$ LQ MFG [Delarue Foguen, 20]
  - MFG with $\{0, 1\}$ as state space [Cecchin, Dai Pra, Fischer, Pellino, 19]
- Both cases share similar features
  - selection is performed by addressing directly the asymptotic behavior of the equilibria of the finite player game
  - selection is connected with the fact that selection principle is also possible for the related master equation (Nash system), which is then a scalar conservation law
- Generalization to finite state MFG with state space of any cardinal
  - ... but POTENTIAL only [Cecchin, D., to appear]

Potential case in LQ setting

• Choose $d = 1$, $c_f = 0, f \equiv 0$ and $c_g = 1$ and $g(x) = \cos(10x + \beta) - 2\beta$ in such way that there are several Nash equilibria including 0



• Call G a primitive of g (second plot) $\rightsquigarrow$ physical equilibria are expected to be given by minima of G!

- ◦ mean-field control problem $\rightsquigarrow$ minimise

$$\mathcal{J}(\alpha) = \mathbb{E}\Big[\frac{1}{2}|X_1|^2 + G(\mathbb{E}(X_1)) + \frac{1}{2}\int_0^1 |\alpha_t|^2 dt\Big],$$

- ◦ over $dX_t = \alpha_t dt + dB_t$

Part VIII. Restoration of Uniqueness

c. Finite State Spaces

MFG with a finite state space

- State space $E = \{1, \cdots, d\}$

  ◦ standard MFG [Gueant et al., Gomes et al., Bensoussan et al., Beyraktar et al.]

  ◦ case $d = 2$ [Cecchin et al.] $\rightsquigarrow$ selection of equilibria

  ◦ MFG with a common noise [Bertucci et al., 2018] $\rightsquigarrow$ force the system to have many jumps at a given time

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• Methodology

◦ restore uniqueness by means of common noise $\rightsquigarrow$ new MFG

◦ let the common noise tend 0 $\rightsquigarrow$ select limiting solutions (very similar to vanishing viscosity method)

◦ even that is difficult $\rightsquigarrow$ focus on potential games!! Strong limitation but contains $d = 2$ case

Simple MFG on E [Guéant; Gomes et al.]

- Tagged player $\rightsquigarrow$ interacting with E-valued population $\boldsymbol{p}$

$$\mathbb{P}(X_{t+dt} = j \,|\, X_t = i) = \alpha_t^{i,j} dt + o(dt), \quad \alpha_t^{i,j} \geq 0, \quad i \neq j$$

$$\mathbb{P}(X_{t+dt} = i \,|\, X_t = i) = 1 + \alpha_t^{i,i} dt + o(dt), \quad \alpha_t^{i,i} = -\sum_{j \neq i} \alpha_t^{j,i}$$

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- Fokker-Planck equation $\rightsquigarrow$ $q_t^i = \mathbb{P}(X_t = i)$

$$dq_t^i = \sum_{j=1}^{d} q_t^j \alpha_t^{j,i} dt$$

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- Cost $\rightsquigarrow$ $\sum_{i=1}^{d} \Big[q_T^i g(i, \boldsymbol{p}_T) + \int_0^T q_s^i \Big(f(i, \boldsymbol{p}_s) + \frac{1}{2} \sum_{j \neq i} |\alpha_s^{i,j}|^2 \Big) ds \Big]$
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- Fixed point / Nash $\rightsquigarrow$ find $(\boldsymbol{p}_t)_t$ and optimal control $((\alpha_t^{\star,i,j})_{i,j})_t$ s.t.

$$p_t^i = \mathbb{P}(X_t^\star = i), \qquad t \in [0, T]$$

MFG with common noise

- Randomize the Fokker-Planck equation directly
- Freeze $(\boldsymbol{p}_t)_t$ as a continuous stochastic path with values in $\mathcal{P}(\{1, \cdots, d\})$ and that is adapted w.r.t. $(W^{i,j})_{i,j}$

$$dq_t^i = \sum_{j=1}^{d} q_t^j \alpha_t^{j,i} dt + \frac{\varepsilon}{\sqrt{2}} \frac{q_t^j}{p_t^i} \sum_{j=1}^{d} \sqrt{p_t^i p_t^j} d[W_t^{i,j} - W_t^{j,i}]$$

$$\alpha_t^{i,j} \geq 0, \quad i \neq j; \quad \alpha_t^{i,i} = -\sum_{j \neq i} \alpha_t^{j,i}$$

  - take $(\alpha_t^{i,j})_t$ adapted w.r.t. noise W
  - makes sense if p stays away from boundary

    $\rightsquigarrow$ solution stays within the orthant $(\mathbb{R}_+)^d$!

    $\rightsquigarrow$ but NOT within the simplex! $\Rightarrow$ allow the total mass to vary... but $\mathbb{E}\left[\sum_{i=1}^d q_t^i\right] = 1$

    $\rightsquigarrow$ does not make a density on E but on $\Omega \times E$

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  - player is willing to minimize

$$\sum_{i=1}^{d} \mathbb{E}\Big[q_T^i g(i, \boldsymbol{p}_T) + \int_0^T q_s^i \Big(f(i, \boldsymbol{p}_s) + \frac{1}{2} \sum_{j \neq i} |\alpha_s^{i,j}|^2 \Big) ds \Big]$$

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$$\sum_{i=1}^d \mathbb{E}\Big[q_T^i g(i, \boldsymbol{p}_T) + \int_0^T q_s^i \Big(f(i, \boldsymbol{p}_s) + \frac{1}{2} \sum_{j \neq i} |\alpha_s^{i,j}|^2 \Big) ds \Big]$$

- Find $(\boldsymbol{p}_t)_t$ and optimal control $((\alpha_t^{\star,i,j})_{i,j})_t$ such that $(\boldsymbol{p}_t)_t = (\boldsymbol{q}_t^\star)_t$

$$dp_t^i = \sum_{j=1}^d p_t^j \alpha_t^{\star,j,i} dt + \frac{\varepsilon}{\sqrt{2}} \sum_{j=1}^d \Big(\sqrt{p_t^i p_t^j} d[W_t^{i,j} - W_t^{j,i}] \Big)$$

  - solution takes values in the simplex!

MFG System

- Best response $\rightsquigarrow$ freeze $(p_t)_t$ as a continuous stochastic path with values in $\mathcal{P}(\{1,\cdots,d\})$ and that is adapted w.r.t. $(W^{i,j})_{i,j}$

- Stochastic HJB $\rightsquigarrow$ Common noise makes HJB stochastic

$$du_t^i = -\Big(\underbrace{H^i(u_t)}_{-\frac{1}{2}\sum_{j=1}^{d}(u_t^i - u_t^j)_+^2} + f^i(\boldsymbol{p}_t)\Big)dt \underbrace{- \frac{\varepsilon}{\sqrt{2}} \sum_{j=1}^{d} \sqrt{p_t^i p_t^j}(v_t^{i,i,j} - v_t^{i,j,i})dt}_{\text{Itô-Wentzell term}} + \sum_{j,k=1}^{d} v_t^{i,j,k} dW_t^{j,k}$$

$$u_T^i = g(i, \boldsymbol{p}_T)$$

$\rightsquigarrow$ yields $\alpha_t^{\star,i,j} = (u_t^i - u_t^j)_+$ as optimal transition rate $\quad i \neq j$

MFG System

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$$du_t^i = -\Big(H(i,u_t) + f(i,\boldsymbol{p}_t)\Big)dt - \frac{\varepsilon}{\sqrt{2}}\sum_{j=1}^{d}\sqrt{p_t^i p_t^j}(v_t^{i,i,j} - v_t^{i,j,i})dt + \sum_{j,k=1}^{d} v_t^{i,j,k}dW_t^{j,k}$$

$$u_T^i = g^i(\boldsymbol{p}_T)$$

$\rightsquigarrow$ yields $\alpha_t^{\star,i,j} = (u_t^i - u_t^j)_+$ as optimal transition rate $\quad i \neq j$

- Coupling $\rightsquigarrow$ solve the MFG by coupling with the forward equation

$$dp_t^i = \sum_{j=1}^{d}\Big[p_t^j(u_t^j - u_t^i)_+ - p_t^i(u_t^i - u_t^j)_+\Big]dt + \frac{\varepsilon}{\sqrt{2}}\sum_{j=1}^{d}\Big(\sqrt{p_t^i p_t^j}d[W_t^{i,j} - W_t^{j,i}]\Big)$$

Master equation

- MFG system as system of characteristics

$$u_t^i = \mathcal{U}^i(t,\boldsymbol{p}_t), \quad t \in [0,T], \quad i = 1,\cdots,d$$

  - $\mathcal{U} = (\mathcal{U}^1,\cdots,\mathcal{U}^d)$ solution of some PDE

- Meta-statement [Cardaliaguet et al.] $\rightsquigarrow$ if classical solution $\Rightarrow$! equilibrium

- $\mathcal{U}$ solves second order $\boxed{\text{master PDE}}$ on simplex

$$\partial_t \mathcal{U}^i(t,\boldsymbol{p}) + \frac{\varepsilon^2}{2} \sum_{j,k=1\cdots d} (p_j\delta_{j,k} - p_j p_k)\partial^2_{p_j p_k}\mathcal{U}^i(t,\boldsymbol{p})$$
$$+ \sum_{j \neq k} p_k \Big((\mathcal{U}^k(t,\boldsymbol{p}) - \mathcal{U}^j(t,\boldsymbol{p}))_+ \Big)\Big(\partial_{p_j}\mathcal{U}^i(t,\boldsymbol{p}) - \partial_{p_k}\mathcal{U}^i(t,\boldsymbol{p}) \Big)$$
$$+ \underbrace{\varepsilon^2 \sum_{j \neq i} p_j \Big(\partial_{p_i}\mathcal{U}^i(t,\boldsymbol{p}) - \partial_{p_j}\mathcal{U}^i(t,\boldsymbol{p}) \Big)}_{\text{pay for stochasticity}} + H^i(\mathcal{U}(t,\boldsymbol{p})) + f^i(\boldsymbol{p}) = 0$$

with the boundary condition $\mathcal{U}^i(T,\boldsymbol{p}) = g^i(\boldsymbol{p})$

Ellipticity at the boundary

- Theory for linear PDEs [Epstein & Mazzeo] but not enough for nonlinear

- Force players to escape from the boundary $\rightsquigarrow$ new dynamics

$$dq_t^i = \sum_{j=1}^{d} q_t^j(\phi(p_t^i) + \alpha_t^{j,i})dt + \frac{\varepsilon}{\sqrt{2}} \frac{q_t^i}{p_t^i} \sum_{j=1}^{d} \Big(\sqrt{p_t^i p_t^j} d[W_t^{i,j} - W_t^{j,i}] \Big)$$

  - with $\phi \searrow$ from $[0, \infty)$ into itself $\phi(r) = \begin{cases} \kappa & \textit{if } r < \delta \\ 0 & \textit{if } r > 2\delta \end{cases}$
  - if $p^j < \delta \Rightarrow$ player may jump to site j with rate κ for free
  - $\alpha_t^{i,i} = -\sum_{j \neq i} \alpha_t^{i,j} - \sum_j \phi(p_t^j)$

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- Keep the same cost functional

$$\sum_{i=1}^d \mathbb{E}\Big[q_T^i g(i, \boldsymbol{p}_T) + \int_0^T q_s^i \Big(f(i, \boldsymbol{p}_s) + \frac{1}{2} \sum_{j \neq i} |\alpha_s^{i,j}|^2 \Big) ds \Big]$$

  - equilibrium $\rightsquigarrow$ find $(\boldsymbol{p}_t)_t$ and optimal control $((\alpha_t^{\star,i,j})_{i,j})_t$ s.t.

$$dp_t^i = \sum_{j=1}^d p_t^j(\phi(p_t^i) + \alpha_t^{\star,j,i})dt + \frac{\varepsilon}{\sqrt{2}} \sum_{j=1}^d \Big(\sqrt{p_t^i p_t^j} d[W_t^{i,j} - W_t^{j,i}] \Big)$$

Main statement

- New master equation

$$\partial_t \mathcal{U}^i(t,\boldsymbol{p}) + \frac{\varepsilon^2}{2} \sum_{j,k=1\cdots d} (x_j\delta_{j,k} - x_j x_k)\partial^2_{p_j p_k} \mathcal{U}^i(t,\boldsymbol{p})$$
$$+ \sum_{j,k=1\cdots d} p^k \Big(\phi(p^j) + (\mathcal{U}^k(t,\boldsymbol{p}) - \mathcal{U}^j(t,\boldsymbol{p}))_+\Big)\Big(\partial_{p_j}\mathcal{U}^i(t,\boldsymbol{p}) - \partial_{p_k}\mathcal{U}^i(t,\boldsymbol{p})\Big)$$
$$+ \varepsilon^2 \sum_{j\neq i} p^j \left(\partial_{p^i}\mathcal{U}^i(t,\boldsymbol{p}) - \partial_{p^j}\mathcal{U}^i(t,\boldsymbol{p})\right) + H^i(\mathcal{U}(t,\boldsymbol{p})) + f^i(\boldsymbol{p}) = 0$$

- Assume that g^i and f^i are sufficiently smooth

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- Assume that g^i and f^i are sufficiently smooth
- Theorem 1 $\rightsquigarrow$ For any $\varepsilon > 0$, for any $\delta > 0$,
  - may choose κ with $\phi(r) = \begin{cases} \kappa & \text{if } r < \delta \\ 0 & \text{if } r > 2\delta \end{cases}$

    such that
  - the master equation has a unique classical solution in a suitable space

Part VIII. Restoration of Uniqueness

d. Selection for Finite State Spaces

Potential case

- Assume that

$$f(i,p) = \frac{\partial F}{\partial p_i}(p), \quad g(i,p) = \frac{\partial G}{\partial p_i}(p)$$

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$$G(p_T) + \int_0^T \Big[F(p_t) + \frac{1}{2}\sum_{i=1}^{d} p_t^i \sum_{j\neq i} |\alpha_t^{i,j}|^2\Big]dt$$

over

$$dp_t^i = \sum_{j\neq i} p_t^j \alpha_t^{j,i} dt$$

◦ any minimizer solves MFG system!

◦ but exist solutions of the MFG system that are not in the set of minimizers! Do they make sense?

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∘ any minimizer solves MFG system!

∘ but exist solutions of the MFG system that are not in the set of minimizers! Do they make sense?

- Statement: Common noise selects minimizers!

Mean field control problem

- With control $((\alpha_t^{i,j})_{i,j})_t$ associate
  - controlled path

$$dp_t^i = \sum_{j=1}^d p_t^j(\phi(p_t^i) + \alpha_t^{j,i})dt + \frac{\varepsilon}{\sqrt{2}} \sum_{j=1}^d \Big(\sqrt{p_t^i p_t^j} d[W_t^{i,j} - W_t^{j,i}] \Big)$$

  - cost functional

$$\mathbb{E}\Big[G(p_T) + \int_0^T \Big[F(p_t) + \frac{1}{2} \sum_{i=1}^d p_t^i \sum_{j \neq i} |\alpha_t^{i,j}|^2 \Big] dt \Big]$$

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- Theorem 2 $\rightsquigarrow$ For any $\varepsilon > 0$,
  - may choose κ with $\phi_\varepsilon(r) = \kappa \varepsilon^{-2}$ if $r < \delta$

    such that, whatever $\delta > 0$, the mean control problem has a unique bounded optimal control and the related HJB equation has a (unique) classical solution $\mathcal{V}^\varepsilon(t, p)$

Potential structure with common noise

- Theorem 3 The unique optimizer of the mean field control problem with common noise is the unique equilibrium of a new MFG!

  - same dynamics as before but new cost functional

$$\sum_{i=1}^{d} \mathbb{E}\Big[q_T^i g(i, \boldsymbol{p}_T) + \int_0^T q_s^i \Big(f(i, \boldsymbol{p}_s) + \vartheta_{\varepsilon,\phi}(i, s, \boldsymbol{p}_s) + \frac{1}{2} \sum_{j \neq i} |\alpha_s^{i,j}|^2 \Big) ds \Big]$$

  - master equation has a unique classical solution $\mathcal{U}^{\varepsilon,i}(t, p)$

$$\mathcal{U}^{\varepsilon,i}(t, p) - \mathcal{U}^{\varepsilon,j}(t, p) = \frac{\partial \mathcal{V}^{\varepsilon}}{\partial p_i}(t, p) - \frac{\partial \mathcal{V}^{\varepsilon}}{\partial p_j}(t, p)$$

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Theorem 4 We can cook ϕ_ε converging to 0 inside the simplex s.t.

1. additional cost $\vartheta_{\varepsilon,\phi}(i, s, \boldsymbol{p}_s)$ has vanishing contribution along the equilibria
2. equilibria of the new MFG are tight; weak limits are supported by minimizers of the original mean field control problem

Master equation for original MFG

• Back to the case without common noise: the value function $\mathcal{V}$ is Lipschitz in time and space

◦ a.e. differentiable in $(t, p) \Rightarrow$ uniqueness of the minimizer a.e. [Cannarsa and Sinestrari] and hence unique selected equilibrium

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• Theorem 5 With the same choice of ϕ_ε as in Theorem 4, we have a.e.

$$\mathcal{U}^{\varepsilon,i}(t,p) - \mathcal{U}^{\varepsilon,j}(t,p) \underset{\varepsilon \searrow 0}{\longrightarrow} \frac{\partial \mathcal{V}}{\partial p_i}(t,p) - \frac{\partial \mathcal{V}}{\partial p_j}(t,p)$$

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- Theorem 6 Characterization [Kruzkov] of the limit as unique weak solution to the master equation deriving from a semiconcave potential

$$\begin{aligned}
&\partial_t \mathcal{U}^i(t,\boldsymbol{p}) + H^i(\mathcal{U}(t,\boldsymbol{p})) + f^i(\boldsymbol{p}) \\
&\quad + \sum_{j \neq k} p_k \underbrace{\Big((\mathcal{U}^k(t,\boldsymbol{p}) - \mathcal{U}^j(t,\boldsymbol{p}))_+\Big)\Big(\partial_{p_j}\mathcal{U}^i(t,\boldsymbol{p}) - \partial_{p_k}\mathcal{U}^i(t,\boldsymbol{p})\Big)}_{-\frac{1}{2}\partial_{p_i}[(\mathcal{U}^k(t,\boldsymbol{p}) - \mathcal{U}^j(t,\boldsymbol{p}))_+^2]} = 0
\end{aligned}$$

  ◦ ! result even if non smooth solution and non-unique equilibria

Part IX. Learning

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a. General philosophy

General objective

- Learning equilibria in mean-field games
    - with a numerical method...
    - ... or without exhaustive knowledge of what is inside the MFG

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General objective

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  - with a numerical method...
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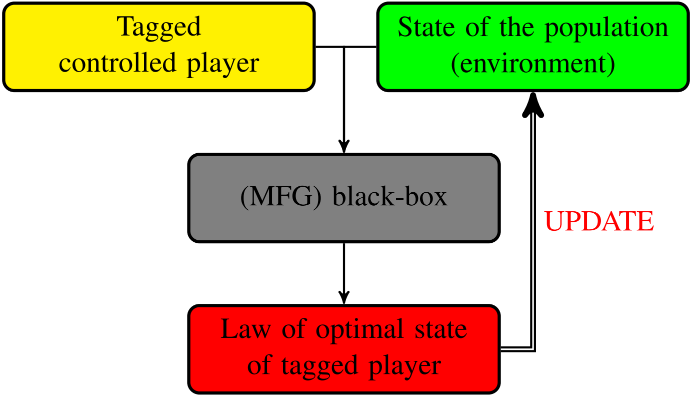
- General strategy



What does UPDATE mean?

- Adapt the fixed point problem

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(2) solve the stochastic optimal control problem in the environment $(\mu_t)_{0 \le t \le T}$

$$dX_t = b(X_t, \mu_t, \alpha_t)dt + \sigma(X_t, \mu_t)dW_t$$

∘ with $X_0 = \xi$ being fixed on some set-up $(\Omega, \mathbb{F}, \mathbb{P})$ with a d-dimensional B.M.

∘ with cost $J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t)dt\Big]$

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∘ with cost $J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t)dt\Big]$

(3) let $(X_t^{\star,\mu})_{0 \le t \le T}$ be the unique optimizer (under nice assumptions) $\rightsquigarrow$ let

$$\Phi_t(\mu) = \mathcal{L}(X_t^{\star,\mu}), \quad t \in [0, T]$$

- Use Φ to update!

Part IX. Learning

b. Which updates?

Picard does NOT work

- Describe state of the population as $(\mu_t)_{0 \le t \le T}$
  - μ_t is a probability measure describing statistical state at time t

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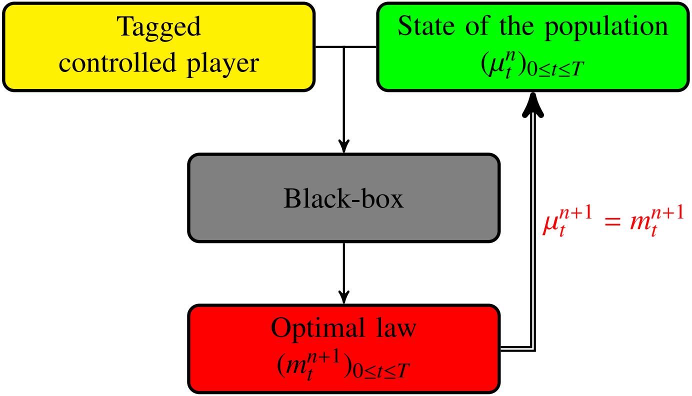

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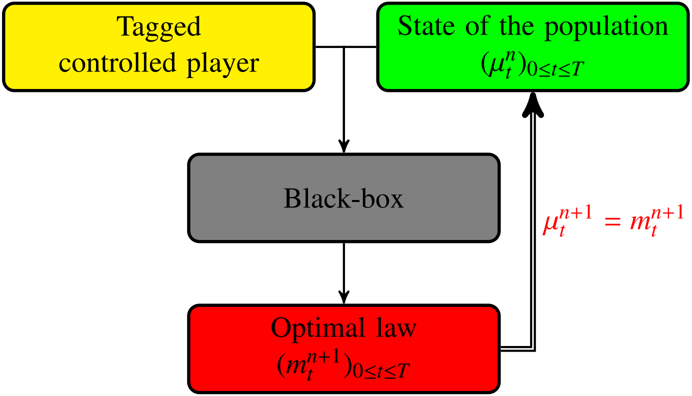
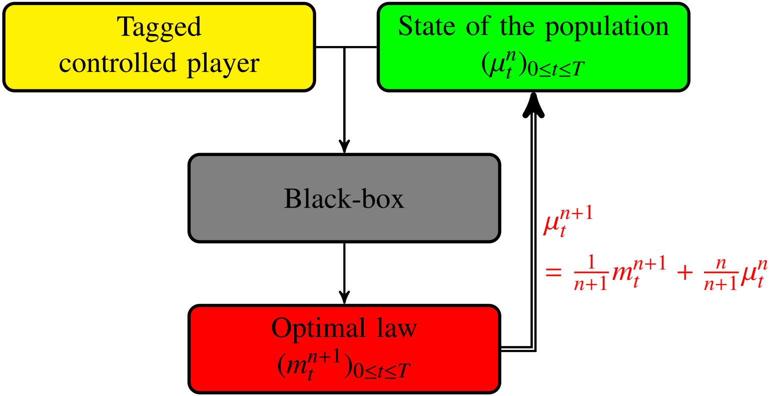


  - fails unless T small: forward-backward problem behind!!!

Fictitious play

- Describe state of the population as $(\mu_t)_{0\le t\le T}$
  - μ_t is a probability measure describing statistical state at time t
- Good idea : Fictitious play



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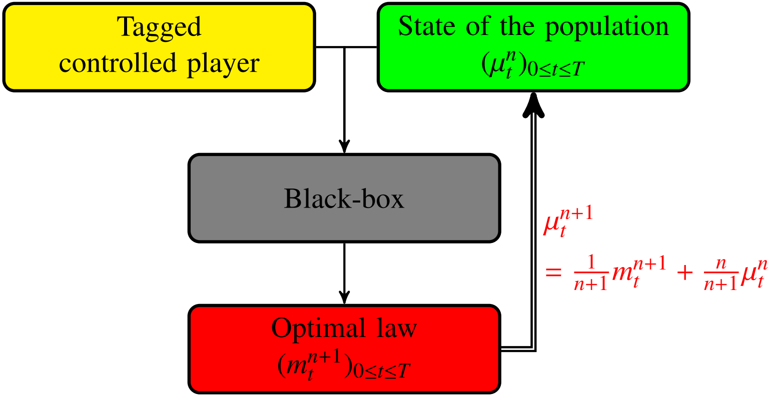


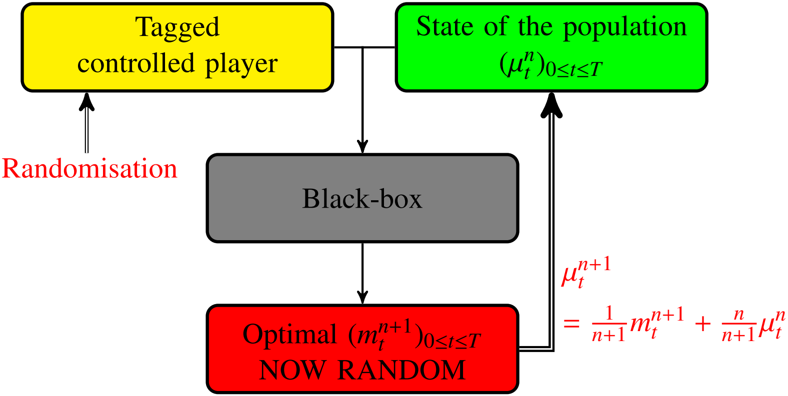
  - in few cases! [Cardaliaguet,Hadikhanloo, Silva, Elie, Laurière]

Part IX. Learning

c. Exploration

Randomisation

- Describe state of the population as $(\mu_t)_{0\le t\le T}$
  - μ_t is a probability measure describing statistical state at time t
- Good idea: Fictitious play



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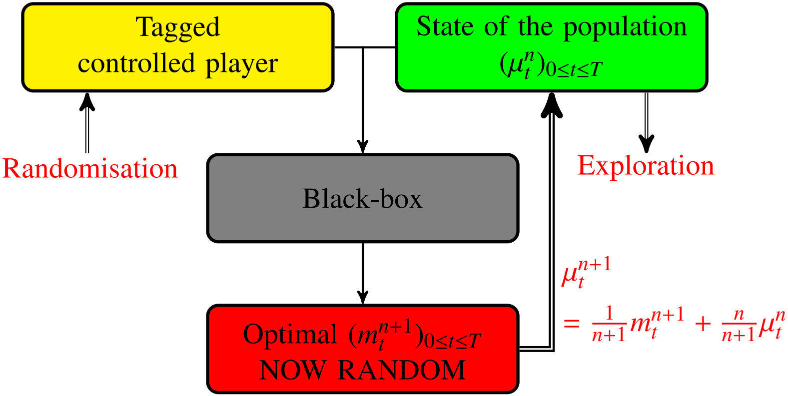

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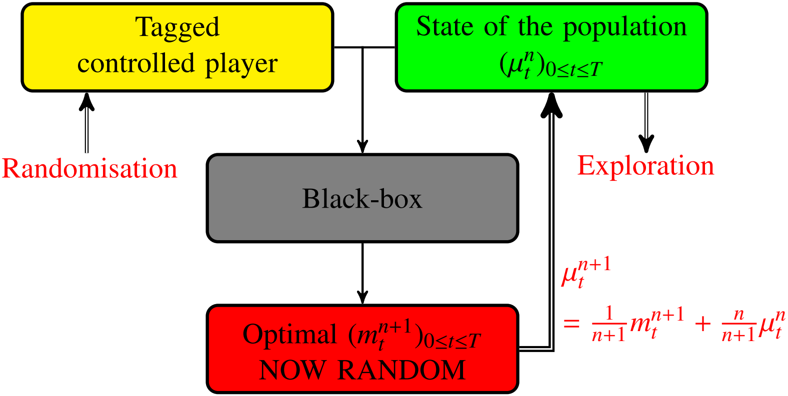

  - Does it help for convergence?

How to use this additional noise? ($\varepsilon = 1$)

- Fictitious play for new optimisation problem
  - proxy $\bar{\boldsymbol{m}}^n = (\bar{m}_t^n)_{0 \le t \le T}$ for RANDOM mean state of population
  - same cost functional but over dynamics with common noise

$$J(\alpha) = \mathbb{E}\Big[\tfrac{1}{2}\big|c_g X_T + g(\bar{m}_T^n)\big|^2 + \int_0^T \big[\tfrac{1}{2}\big|c_f X_t + f(\bar{m}_t^n)\big|^2 + \tfrac{1}{2}\big|\alpha_t\big|^2\big]dt\Big]$$

over

$$dX_t = \alpha_t dt + \sigma dW_t + \varepsilon dB_t$$

  - conditional mean of optimal mean state

$$dm_t^{n+1} = -(\eta_t m_t^{n+1} + h_t^{n+1})dt + dB_t, \quad m_0^{n+1} = \mathbb{E}(X_0)$$
$$dh_t^{n+1} = -(c_f f(\overline{m}_t^n) - \eta_t h_t^{n+1})dt + k_t^{n+1} dB_t, \quad h_T^{n+1} = c_g g(\overline{m}_T^n)$$

  - update proxy of the environment

$$\overline{m}_t^{n+1} = \frac{1}{n+1} m_t^{n+1} + \frac{n}{n+1} \overline{m}_t^n$$

- Not able to prove convergence!

Scheme that forces decoupling

- Two proxies
  - proxy $\bar{\boldsymbol{m}}^n = (\bar{m}^n_t)_{0 \le t \le T}$ for RANDOM mean state of population
  - proxy $\boldsymbol{h}^n = (h^n_t)_{0 \le t \le T}$ for RANDOM intercept of feedback
- New dynamics
  - tilt the common noise

$$dX_t = \alpha_t dt + \sigma dW_t + d\left(B_t + \int_0^t h^n_s ds\right)$$

  - new cost functional

$$\mathbb{E}\left[\mathcal{E}(\boldsymbol{h}^n)\left(\frac{1}{2}\left|c_g X_T + g(\bar{m}^n_T)\right|^2 + \int_0^T \left[\frac{1}{2}\left|c_f X_t + f(\bar{m}^n_t)\right|^2 + \frac{1}{2}\left|\alpha_t\right|^2\right] dt\right)\right]$$

  - with $\mathcal{E}(\boldsymbol{h}^n) = \exp\left(-\int_0^T h^n_s dB_s - \frac{1}{2}\int_0^T |h^n_s|^2 ds\right)$
- Same fictitious play as before. It works!

Why does it work? ($c_f = c_g = 1$ to simplify)

- Just replace B_t by $B_t + \int_0^t h_s^n ds$

$$dm_t^{n+1} = -(\eta_t m_t^{n+1} + h_t^{n+1} - h_t^n)dt + dB_t, \quad m_0^{n+1} = \mathbb{E}(X_0)$$
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- Implement $\bar{m}_t^{n+1} = \dfrac{1}{n+1} \sum_{j=1}^{n+1} m_t^j$

- Get

$$d\bar{m}_t^{n+1} = -(\eta_t \bar{m}_t^{n+1} + O(1/n))dt + dB_t, \quad \bar{m}_0^{n+1} = \mathbb{E}(X_0)$$
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- Equations decouple... to limiting equations

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- Statement: for F 1-bounded and 1-Lipschitz

$$\left| \mathbb{E}\left[\mathcal{E}\left(\boldsymbol{h}^n\right) F(\bar{\boldsymbol{m}}^n, \boldsymbol{h}^n)\right] - \mathbb{E}\left[\mathcal{E}\left(\boldsymbol{h}\right) F(\bar{\boldsymbol{m}}, \boldsymbol{h})\right] \right| \leq \frac{C}{n}$$

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- Statement: for F 1-bounded and 1-Lipschitz and restore ε

$$\left| \mathbb{E}\left[\mathcal{E}\left(\frac{\boldsymbol{h}^n}{\varepsilon}\right) F(\bar{\boldsymbol{m}}^n, \boldsymbol{h}^n) \right] - \mathbb{E}\left[\mathcal{E}\left(\frac{\boldsymbol{h}}{\varepsilon}\right) F(\bar{\boldsymbol{m}}, \boldsymbol{h}) \right] \right| \leq \frac{C}{n\varepsilon}$$

6. Back to the original problem

Two drawbacks

- Provides a solution of the mean-field game with common noise!
- Does not fit within the inputs of the black-box!

Two drawbacks

- Provides a solution of the mean-field game with common noise!

  - solution to mean-field game with ε common noise gives

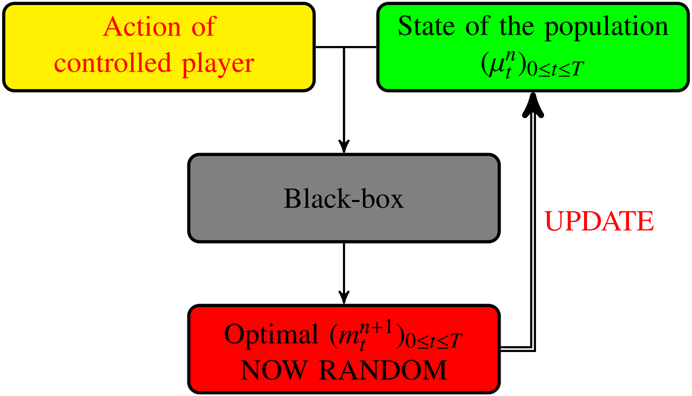
$C\varepsilon$-Nash equilibrium to original mean-field game

$$\inf_{\boldsymbol{\alpha}} \mathbb{E}^B\Big[J^{\text{original}}(\boldsymbol{\alpha}, \boldsymbol{m}^\varepsilon)\Big] \geq \inf_{\boldsymbol{\alpha}} \mathbb{E}^B\Big[J^{\text{original}}(\boldsymbol{\alpha}^{\star,\varepsilon}, \boldsymbol{m}^\varepsilon)\Big] - C\varepsilon$$

  - can choose ε as small as we want!

Two drawbacks

- Does not fit within the inputs of the black-box!



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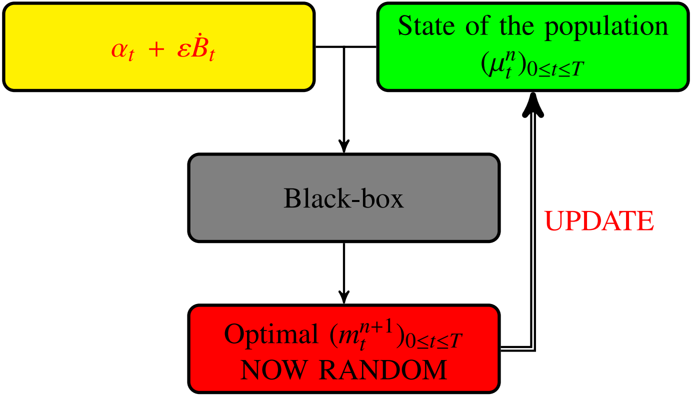
- Does not fit within the inputs of the black-box!

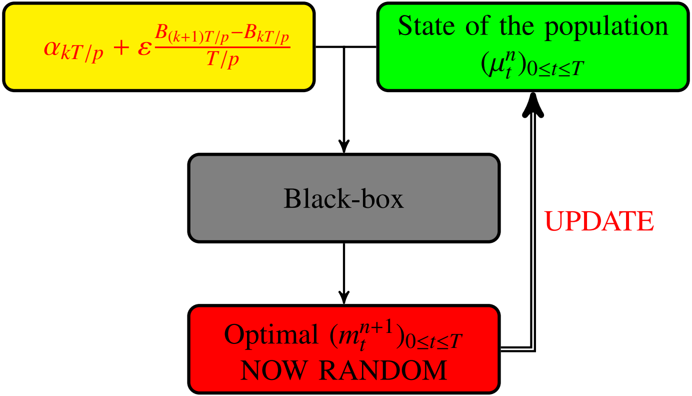


◦ Require action process $(\alpha_t)_{0 \le t \le T}$ to be constant on $[(k/p)T, (k+1)T/p)$ and corrupt $\alpha_{kT/p}$ by $\varepsilon p/T(B_{(k+1)T/p} - B_{kT/p})$

◦ correct cost by $-\varepsilon^2 p/(2T)$.

Two drawbacks

- Does not fit within the inputs of the black-box!



- Statement: for F 1-bounded and 1-Lipschitz

$$\left| \mathbb{E}\Big[\mathcal{E}\Big(\frac{\boldsymbol{h}^n}{\varepsilon}\Big)F(\bar{\boldsymbol{m}}^n, \boldsymbol{h}^n)\Big] - \mathbb{E}\Big[\mathcal{E}\Big(\frac{\boldsymbol{h}}{\varepsilon}\Big)F(\bar{\boldsymbol{m}}, \boldsymbol{h})\Big] \right| \le \frac{C}{(n + p^{1/2})\varepsilon}$$

7. Numerical experiments

Discretization

- Cost

$$\min\Big\{\mathcal{E}^{n,(j)}\frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N}\Big[\frac{1}{2p}\sum_{k=1}^{p}|\alpha_{t_k}^{(i,j)}|^2+\frac{1}{2}\Big|x_1^{(i,j)}+g\big(\overline{m}_1^{n,(j)}\big)\Big|^2\Big]\Big\},$$

where

$$\mathcal{E}^{n,(j)}=\exp\Big(-\sqrt{\frac{1}{p}}\sum_{k=0}^{p-1}h_{t_k}^{n,(j)}\cdot\Delta_{t_{k+1}}w^{(j)}-\frac{1}{2p}\sum_{k=0}^{p-1}|h_{t_k}^{n,(j)}|^2\Big).$$

- Dynamics

$$x_{t_k}^{(i,j)}=x_{t_{k-1}}^{(i,j)}+\frac{1}{p}\alpha_{t_k}^{(i,j)}+\frac{1}{\sqrt{p}}\Delta_{t_k}b^{(i)},\quad \ell=1,\cdots,p\,;\quad x_0^{(i,j)}=x_0,$$

$$\alpha_{t_k}^{(i,j)}=a_{t_{k-1}}x_{t_{k-1}}^{(i,j)}+C_{t_{k-1}}^{(j)}+h_{t_{k-1}}^{n,(j)}+\sqrt{p}\Delta_{t_k}w^{(j)},\quad k=1,\cdots,p.$$

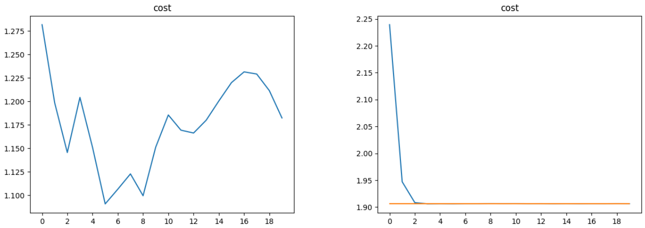
- Semi-feedback form

$$C_{t_k}^{(j)}=\sum_{|\ell|\le D}c_{t_k}(\ell)H_\ell^d\Big((U_{t_k}^n)^{-1}\Big(\overline{m}_{t_k}^{n,(j)}-\frac{1}{N}\sum_{r=1}^{N}\overline{m}_k^{n,(r)}\Big)\Big),$$

where $U_{t_k}^n$ is root of empirical covariance and H^d is Hermite

2d example

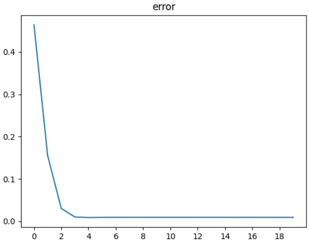
- Choose $d = 2$, $T = 1$, $c_f = 0$, $f \equiv 0$ and $c_g = 1$ and $g_1(x_1, x_2) = \cos(10x_1)\cos(10x_2)$, $g_2(x_1, x_2) = \sin(10x_1)\sin(10x_2)$
- Learnt cost without common noise and with common noise



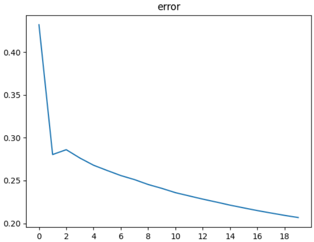
($p = 30$, $\#i = 4E5$ and no common noise in the left, no independent noise but $\#j = 4E5$ in the right)

2d example

- Choose $d = 2$, $T = 1$, $c_f = 0, f \equiv 0$ and $c_g = 1$ and
$g_1(x_1, x_2) = \cos(10x_1)\cos(10x_2)$, $g_2(x_1, x_2) = \sin(10x_1)\sin(10x_2)$
- L^2 error to the solution



(a) Riccati known



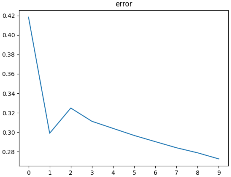
(b) Riccati unknown

The experiments are computed with: $n = 20$ learning iterations, $p = 30$ time steps, $\#i = 1$, $\#j = 5 \times 10^4$, $\sigma = 0$ and $\varepsilon = 1$

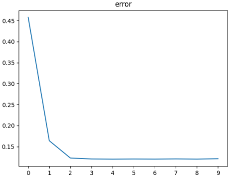
2d example

• Choose $d = 2$, $T = 1$, $c_f = 0, f \equiv 0$ and $c_g = 1$ and $g_1(x_1, x_2) = \cos(10x_1)\cos(10x_2)$, $g_2(x_1, x_2) = \sin(10x_1)\sin(10x_2)$

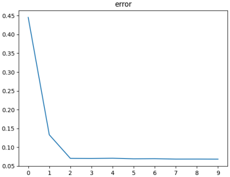
• L^2 error to the solution without the solution of Riccati (without and with independent noise)



(a) $\#j = 4 \times 10^4$, $\#i = 1$



(b) $\#j = 2 \times 10^4$, $\#i = 20$



(c) $\#j = 10^4$, $\#i = 40$

Simulated annealing

• Decrease step by the step the intensity of the common noise starting from the wrong equilibrium 0!

$$\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_q$$

• For viscosity ε_{q+1}, tilt the common noise using return $\boldsymbol{h}^{\infty,q}$ of algorithm with previous viscosity

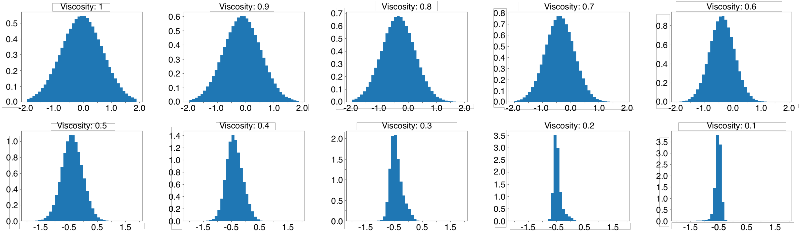
$$dX_t = \alpha_t dt + \sigma dW_t + \varepsilon_{q+1} d\left(B_t + \int_0^t \frac{1}{\varepsilon_{q+1}}(h_s^n - h_s^{\infty,q})ds\right)$$

∘ new cost functional

$$\mathbb{E}\left[\mathcal{E}\Big(\frac{\boldsymbol{h}^n - \boldsymbol{h}^{\infty,q}}{\varepsilon_{q+1}}\Big)\left(\tfrac{1}{2}\left|c_g X_T + g(\bar{m}_T^n)\right|^2 + \int_0^T \left[\tfrac{1}{2}\left|c_f X_t + f(\bar{m}_t^n)\right|^2 + \tfrac{1}{2}\left|\alpha_t\right|^2\right]dt\right)\right]$$

∘ with

$$\mathcal{E}\Big(\frac{\boldsymbol{h}^n - \boldsymbol{h}^{\infty,q}}{\varepsilon_{q+1}}\Big) = \exp\left(-\frac{1}{\varepsilon_{q+1}}\int_0^T (h_s^n - h_s^{\infty,q})dB_s - \frac{1}{2\varepsilon_{q+1}^2}\int_0^T |h_s^n - h_s^{\infty,q}|^2 ds\right)$$


Viscosity: 1
Viscosity: 0.9
Viscosity: 0.8
Viscosity: 0.7
Viscosity: 0.6
Viscosity: 0.5
Viscosity: 0.4
Viscosity: 0.3
Viscosity: 0.2
Viscosity: 0.1