On Stochastic Bilevel Programs

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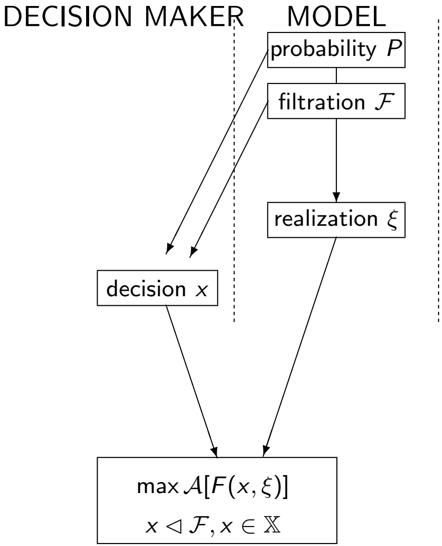
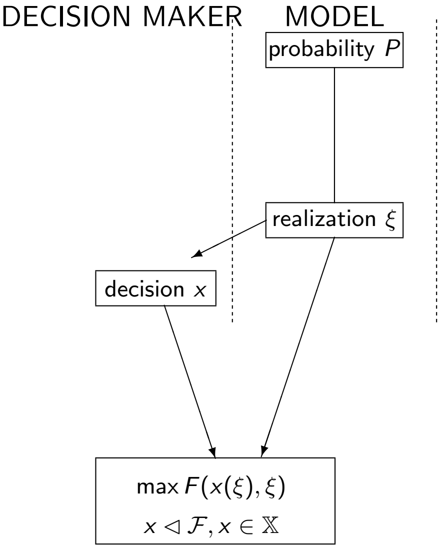
Stochastic optimization-here and now



Stochastic optimization - wait and see (clairvoyant)



Bilevel problems

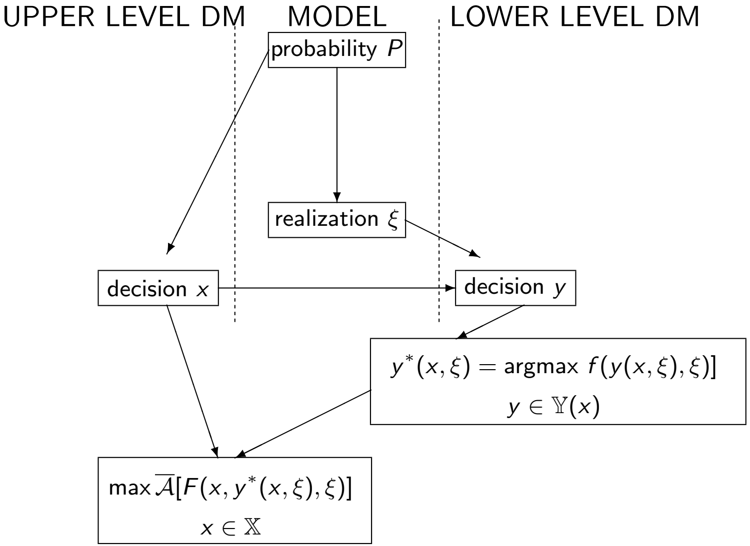
In bilevel problems, there are two independent decision makers (DMs): The typical situation is that the upper level DM sets the price of a contract while the lower level DM decides about to accept the contract and to exercise the rights out of the contract.

In stochastic bilevel problems, there are random parameters, which are only known by their distribution.

Example: Electricity swing options

- Give the buyer the right to get electricity at a predetermined price K per MWh during the contract period (a month, a quarter, a year, ...). The price is set way before the delivery period starts.
- The actual schedule of hourly demand y_t may deviate from some baseline schedule: Usually the cumulative demand must lie in some interval $[\underline{E}, \overline{E}]$ and the demand in hour t must lie within $[\underline{e}_t, \overline{e}_t]$.
- The buyer has to announce its actual demand for each hour with 1 day ahead notice.
- The seller can use forward contracts, own production and the spot market to meet the demand of the buyer.
- The buyer can use the spot market as an alternative for his demand.
- The problem is to determine a fair offered price K for the contract.

UL: here and now, LL: (nearly) wait and see



Deterministic bilevel programs

- UL problem:

$$\max_{x,y} \{F(x,y) : G(x,y) \leq 0, y \in \Psi(x)\}$$

- LL problem:

$$\Psi(x) = \arg\min_{y} \{f(x,y) : g(x,y) \leq 0\}$$

This is the optimistic version (which can be often rewritten as an MPEC problem). A variant is the pessimistic version:

- $\max_x \min_y \{F(x,y) : G(x,y) \leq 0, y \in \Psi(x)\}$

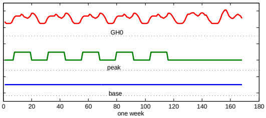
Deterministic bilevel problems

- Already deterministic bilevel problems are hard to solve:
- The feasible set $\{(x, y) : \ G(x, y) \leq 0, \ y \in \Psi(x)\}$
    - is usually nonconvex,
    - can be disconnected or empty,
    - may fail to be compact.
- Many local optima might exist
- It has been proved that even the linear bilevel problem is a strongly NP-hard problem: Any linear [0,1]- integer problem can be reformulated as a continuous variable linear bilevel problem.
- Optimality conditions have been established by e.g. Dempe, Outrata, Fukushima, Gferer and others.

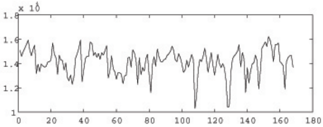
Electricity contracts: Upper level decisions

The upper level DM (contract seller) has to decide about the offered price for the contract as well as about production and the hedges to buy from future markets. Examples of available hedging profiles:

The future market offers hedges with given profiles:



However, the contract buyer has an irregular demand:



Full hedge is not possible: A basic risk remains with the contract seller.

There are M hedging instruments available. A hedging instrument is characterized by its price F_m and its delivery pattern τ, where $\tau(m, t)$ is the amount delivered in time period (hour) t.
The decision to be made by the option seller consists of

- the number $\tilde{x}_m$ of units of future contract m to be bought,
- the ask price K.

We collect the hedge amounts in a hedge vector $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_M)$ and all upper level decision variables in

$$x = (\tilde{x}, K).$$

The unmatched surplus/shortage in time period t is

$$\sum_{m=1}^{M} \tilde{x}_m \tau(m,t) - y_t.$$

This amount will be sold/bought on the spot market. The spot-prices are random processes ξ_t with a given distribution.
The input data of the optimal hedging problem are

$S = (S_t^\omega)$	the spot price scenario model
$p = (p^\omega)$	the scenario probabilities
$F = (F_m)$	the prices of the hedging instruments (future contracts)
$\tau(m,t)$	the delivery pattern of hedge m
$y = (y_t^\omega)$	the demands (which are decided by the LL DM)

The revenue as a function of $\tilde{x}$, K and y

For consistency reasons, we assume that the spot price model is calibrated to the known future prices π_m is such a way, that *expectation-neutrality* holds

$$F_m = \sum_{t=1}^{T} \tau(m,t)\mathbb{E}[\xi_t]$$

Denote by W the profit/loss variable of this contract seen from the option seller. Given the hedge x and the price per unit K, W takes the values

$$W^\omega(x,K,y) = K\sum_{t=1}^{T} y_t^\omega + \sum_{t=1}^{T} S_t^\omega \left[\sum_{m=1}^{M} \tilde{x}_m \tau(m,t) - y_t^\omega\right] - \sum_{m=1}^{M} F_m \tilde{x}_m$$

with probability p^ω.

Acceptability

A revenue variable W is called *acceptable*, if the probability that it falls below some a is less than α (e.g. $\alpha = 0.15$):

$$P\{W(\tilde{x}, K) < a\} \leq \alpha. \qquad \textit{quantile constraint.}$$

The minimal price such that $P\{W(\tilde{x}, K) < 0\} \leq \alpha$ is called the *quantile price*.
Introduce the *Avarage Value-at-Risk* at level α ($\mathbb{A}\mathrm{V@R}_\alpha$)

$$\mathbb{A}\mathrm{V@R}_\alpha(W) = \sup\{\mathbb{E}[WZ] : 0 \leq Z \leq 1/\alpha, \mathbb{E}Z = 1\}.$$

Then

$$\mathbb{A}\mathrm{V@R}_\alpha(W(\tilde{x}, K)) \geq 0$$

implies that

$$P\{W(\tilde{x}, K) < 0\} \leq \alpha.$$

The $\mathbb{A}\mathrm{V@R}$-constraint is a convexification of the quantile constraint.

The lower level LP

The discretized lower level problem is an LP: The contact holder maximizes his expected profit, which depends on the difference between the strike price K and the actual spot price ξ.

$$
\begin{aligned}
[LL] \max_{y,s} \; & \sum_{n=2}^{N} p_n y_{n-} \left(\xi_n - K\right) \\
s.t. \; & \underline{e}_{t(n)} \leq y_n \leq \overline{e}_{t(n)}, \quad \forall n \in \{1, \ldots, N\} \setminus \Omega \\
& s_n = s_{n_-} + y_{n_-}, \quad \forall n \in \{1, \ldots, N\} \setminus \Omega \\
& \underline{E} \leq s_\omega \leq \overline{E}, \quad \forall \omega \in \Omega
\end{aligned}
$$

This is a multistage stochastic optimization problem defined on a scenario tree with N nodes. Node $n-$ is the predecessor of node n and $t(n)$ is the stage of n.

Summarizing the swing option bilevel problem

UL

The contract seller sets the price K and finds appropriate hedges such that under the anticipated demand pattern y_t of the contract buyer (1) her expected profit is maximal or (2) the price K is minimal, given that her profit/loss distribution is acceptable.

LL

The contract buyer determines the demand pattern y_t given the price K such that

- the exercise constraints are satisfied,
- the expected profit is maximized,
- his profit/loss distribution is acceptable.

The two models for the upper level

(1) The monopolistic case

$$[ULM] \left| \begin{array}{l} \max_{K,\tilde{x},y} \mathbb{E}[Y_x] = \mathbb{E}[\bar{y}K - \delta(y) + \sum_{m=1}^{M} \tilde{x}_m[\phi_m - F_m]] \\ \text{subject to} \\ y \in \Psi(x) \end{array} \right.$$

(2) The competitive case

$$[ULC] \left| \begin{array}{l} \min_{K,\tilde{x},y} K \\ \mathbb{A}\mathrm{V@R}_\alpha[\bar{y}K - \delta(y) + \sum_{m=1}^{M} \tilde{x}_m[\phi_m - F_m] \geq 0 \\ \text{subject to} \\ y \in \Psi(x) \end{array} \right.$$

Here $\bar{y} = \sum_{t=0}^{T} y_t$ is the total demand and $\delta(y) = \sum_{t=0}^{T-1} y_t S_t$ is the spot-price value of the demand .

The bilinear structure of the problem

- The monopolistic case

$$[ULM] \left| \begin{array}{l} \max_{x,y} x^\top (d + Dy) \\ \text{subject to} \\ EX \geq \ell \\ y \in \Psi(x) \end{array} \right.$$

- The competitive case

$$[ULC] \left| \begin{array}{l} \max_{x,y} x^\top (d + Dy) \\ \text{subject to} \\ q_x^{i\top} x + q_y^{i\top} y + x^\top Q^i y \geq r^i \\ Ex \geq \ell \\ y \in \Psi(x) \end{array} \right.$$

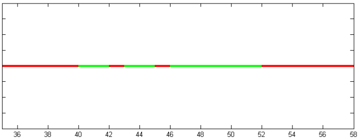
In both cases

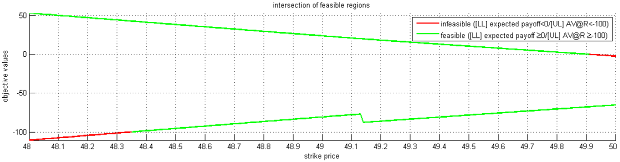
$$[LL] \left| \begin{array}{l} \Psi(x) = \text{ argmax }_y (c^\top + x^\top C) y \\ Ay \leq b \end{array} \right.$$

More abut these models in the talk by Raimund Kovecevic.

Acceptability conditions in the LL

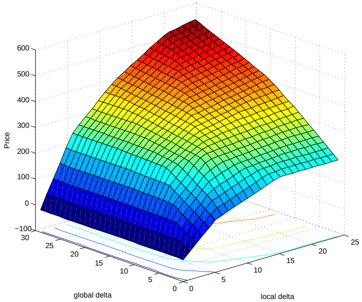
If the LL contains some acceptability conditions (e.g. that the expected profit is nonnegative), the LL problem may become unfeasible if the strike price K is too high. On the other hand, the UL problem may become infeasible if the strike price K is too low. Thus the total feasibility region lies between an lower and an upper bound (but may be a union of non-intersecting intervals). In the competivie case, we search for the lowest point of the total feasibility region.




intersection of feasible regions
infeasible ([LL] expected payoff<0/[UL] AV@R<-100)
feasible ([LL] expected payoff ≥0/[UL] AV@R ≥-100)
objective values
strike price


intersection of regions with positive expected payoffs
infeasible (expected payoff <0)
feasible (expected payoff ≥0
expected payoff
strike price

The costs for flexibility



Flexibility for the contract buyer = costs for the contract seller

Conclusions

Stochastic bilevel programs add an additional complexity to deterministic bilevel programs (which are already hard problems).
The swing option problem has the following peculiarities:

- The correct pricing of a swing option requires to find a behavioral model for the contract holder, in particular his price sensitivity (full reseller, partial reseller, no reseller).
- A worst case can be found by considering the optimizing strategy for the buyer assuming he is a reseller (as we did it).
- The optimal hedging strategies for fixed contracts and swing options may be quite different.
- Higher flexibility of requires a higher price (the costs of flexibility).
- Similar remarks can be made about other flexible contracts, such as life insurance with lapse right, pension insurance with rights of withdrawal from the accumulated capital, insurance models which incorporate moral hazard, etc.