Decomposition in adjustable robust optimization: illustrations on network design problems

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Outline

1. Introduction
2. Models
3. Robust network design: basic models
4. Exact solutions with fixed recourse
5. Affine decision rules and random recourse

Outline

Knapsack problem under uncertainty

$$\begin{aligned} \min \quad & \sum_{i \in N} c_i x_i \\ \text{s.t.} \quad & \sum_{i \in N} a_i x_i \leq b \\ & x \in \{0,1\}^n \end{aligned}$$

Suppose that the parameters (c, a) are uncertain:

- They vary over time
- They must be predicted from historical data
- They cannot be measured with enough accuracy
- ...

Let's do something clever (and useful)!

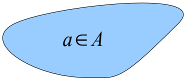
How much do we know?

$$\underbrace{\text{Stochastic programming}}_{\text{A lot}} \quad \Leftrightarrow \quad \underbrace{\text{Robust programming}}_{\text{A little}}$$

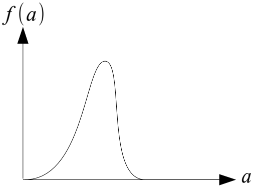
Mean value (Deterministic)

$\bullet\ \boldsymbol{E}[a]$

Robust



Stochastic



How much do we know?

Robust pr. Uncertain parameters are merely assumed to belong to an uncertainty set Ξ $\Rightarrow$ one wishes to optimize some worst-case objective over the uncertainty set

Stochastic pr. Uncertain parameters are precisely described by probability distributions $\Rightarrow$ one wishes to optimize some expectation, variance, Value-at-risk, ...

How much do we know?

Robust pr. Uncertain parameters are merely assumed to belong to an uncertainty set $\Xi \Rightarrow$ one wishes to optimize some worst-case objective over the uncertainty set

Distributionally robust pr. Uncertain parameters are described by classes of probability distributions $\mathcal{F} \Rightarrow$ one wishes to optimize some worst-case objective over the ambiguity classes

- $\mathcal{F}$ is a singleton Stochastic programming
- $\mathcal{F}$ contains all distributions over Ξ Robust optimization

Stochastic pr. Uncertain parameters are precisely described by probability distributions $\Rightarrow$ one wishes to optimize some expectation, variance, Value-at-risk, ...

When do we take decisions?

Now All decisions must be taken before the uncertainty is known with precision ⇒ probability constraints, (static) robust optimization

Delayed Some decisions may be delayed until the uncertainty is revealed ⇒ multi-stage stochastic programming, adjustable robust optimization

Why Robust Optimization (RO)

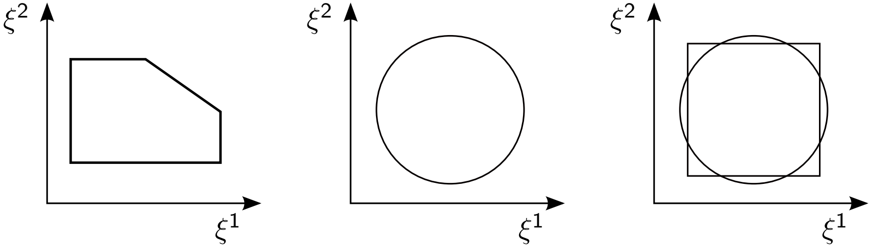
Stochastic programming suffers from two drawbacks:

- Precise information about the distribution is required.
- The resulting optimization problems are usually very large-scale (and intractable).

Robust Optimization (and distributionally RO):

- Require less information about how parameters vary.
- Leads to tractable optimization problems (less true for distributionally RO).

Uncertainty sets



(a) $\|\xi\|_\infty \leq 1, \|\xi\|_1 \leq \kappa_1$ (b) $\|\xi\|_2 \leq \kappa_2$ (c) $\|\xi\|_\infty \leq 1, \|\xi\|_2 \leq \kappa_2$

Outline

Consider a static robust linear optimization problem

$$
\begin{aligned}
\min \quad & c^T \mathbf{x} \\
\text{s.t.} \quad & \mathbf{x} \in \mathcal{X} \\
& A(\xi)\mathbf{x} \le b \quad \xi \in \Xi.
\end{aligned} \tag{1}
$$

with the following notation

ξ : uncertain parameter
$\Xi \subset \mathbb{R}^K$: uncertainty polytope
$\mathcal{X} \subset \mathbb{R}^I$: intersection of a polyhedron with integer restrictions
A : affine function of ξ, i.e., $A(\xi) := A^0 + \sum_k A^{1k}\xi^k$.

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Can be solved in two ways:

Cutting-planes Let $\Xi^* \subset \Xi$. We relax (1) to $A(\xi)x \le b \quad \xi \in \Xi^*$. We generate additional constraints on demand by solving $\max_{\xi \in \Xi} A_i(\xi)x$ for each constraint i.

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Dualization Using duality in linear programming, we can reformulate (1) as a polynomial number of linear constraints.

Dualization

Theorem

Let $\Xi := \left\{ \xi \in \mathbb{R}^K : \xi \geq 0, \sum_{k \in K} e_j^k \xi^k \leq d_j,\ j \in J \right\}$. *A vector* $\mathbf{x}$ *is feasible for*

$$a^T(\xi)\mathbf{x} \leq b(\xi), \quad \xi \in \Xi$$

if and only if there exists dual variables $\mathbf{z} \in \mathbb{R}^J$ *such that* $\mathbf{x}$ *is feasible for the system of constraints*

$$\sum_{i \in I} a_i^0 \mathbf{x_i} + \sum_{j \in J} d_j \mathbf{z_j} \leq b^0$$

$$\sum_{j \in J} e_j^k \mathbf{z_j} \geq \sum_{i \in I} a_i^k \mathbf{x_i} - b^k, \quad k \in K$$

$$\mathbf{z} \geq 0.$$

Proof.

$$
\begin{aligned}
a^T(\xi)x \le b(\xi), \quad \xi \in \Xi \quad &\Leftrightarrow \quad \sum_{i\in I}(a_i^0 + \sum_{k\in K} a_i^k \xi^k)x_i \le b^0 + \sum_{k\in K} b^k \xi^k, \quad \xi \in \Xi \\
&\Leftrightarrow \quad \sum_{i\in I} a_i^0 x_i + \max_{\xi\in\Xi} \sum_{k\in K}(\sum_{i\in I} a_i^k x_i - b^k)\xi^k \le b^0 \\
&\Leftrightarrow \quad \sum_{i\in I} a_i^0 x_i + \min_{z\ge 0} \sum_{j\in J} d_j z_j \le b^0 \\
&\qquad\qquad \text{s.t.} \sum_{j\in J} e_j^k z_j \ge \sum_{i\in I} a_i^k x_i - b^k, \quad k \in K \\
&\Leftrightarrow \quad \sum_{i\in I} a_i^0 x_i + \sum_{j\in J} d_j z_j \le b^0 \\
&\qquad \sum_{j\in J} e_j^k z_j \ge \sum_{i\in I} a_i^k x_i - b^k, \quad k \in K \\
&\qquad z \ge 0.
\end{aligned}
$$

Which approach is best ?

We refer to the recent paper:

Dimitris Bertsimas, Iain Dunning, and Miles Lubin: "Reformulations versus cutting planes for robust optimization: A computational and machine learning perspective". Available at Optimization Online.

	Polyhedral	Ellipsoidal
Reformulation	1.56	**1.43**
Cutting-plane	**1.49**	1.67

Table : Results for linear programs.

	Polyhedral	Ellipsoidal
1. Reformulation	**1.66**	2.24
2. Cutting-plane	2.96	2.65
3. Cutting-plane & root	1.74	**1.74**
4. Cutting-plane & heur.	3.02	2.66
5. Cutting-plane & root & heur.	1.82	1.79

Table : Results for mixed-integer linear programs.

Adjustable situation

$$
\begin{aligned}
& \min && c^T \mathbf{x} \\
& \text{s.t.} && \mathbf{x} \in \mathcal{X} \\
(P) \quad & && A(\xi)\mathbf{x} + E(\xi)\mathbf{y}(\xi) \leq b \quad \xi \in \Xi.
\end{aligned}
$$

where y represents the vector of adjustable variables.

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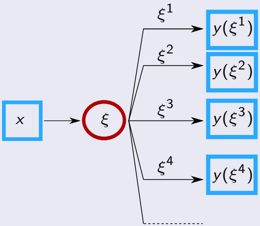
Remark

We consider 2-stage problems only: network design problems, facility location problems, ...



Remark

We suppose that all recourse variables y are continuous. Very recent works from Bertsimas et al. (2014) and Hanasusanto et al. (2014) consider the case of integer recourse (both available at Optimization Online):

- *Dimitris Bertsimas and Angelos Georghiou "Design of Near Optimal Decision Rules in Multistage Adaptive Mixed-Integer Optimization".*
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We present next approaches for two types of problems:

- Exact solution algorithms for (P) when E is constant.
- Solution algorithms based on affine decision rules when E depends affinely on ξ.

Affine decision rules

A classical approach for (P) is to restrict y to affine functions of ξ:

$$\mathbf{y}(\xi) = \mathbf{y^0} + \sum_{k \in K} \mathbf{y^k} \xi^k.$$

when E is constant, (P) becomes

$$\begin{aligned} \min \quad & c^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X} \\ & \left(A^0 + \sum_k A^{1k} \xi^k\right) \mathbf{x} + E\left(\mathbf{y^0} + \sum_{k \in K} \mathbf{y^k} \xi^k\right) \leq b \quad \xi \in \Xi. \end{aligned}$$

Disadvantage Provides a conservative solution. When Ξ is a simplex, the conservative solution is optimal.

Advantage The resulting optimization problems have the same structure as static RO problems.

Affine decision rules

Theorem

Let Ξ be an uncertainty polytope, A be an affine function of ξ, $\mathbf{x}$ be a vector of optimization variables, and

$$\mathbf{y}(\xi) = \mathbf{y^0} + \sum_{k \in K} \mathbf{y^k} \xi^k$$

be a vector of affine adjustable optimization variables. Then, robust constraint

$$A(\xi)\mathbf{x} + E(\xi)\mathbf{y}(\xi) \leq b \quad \xi \in \Xi$$

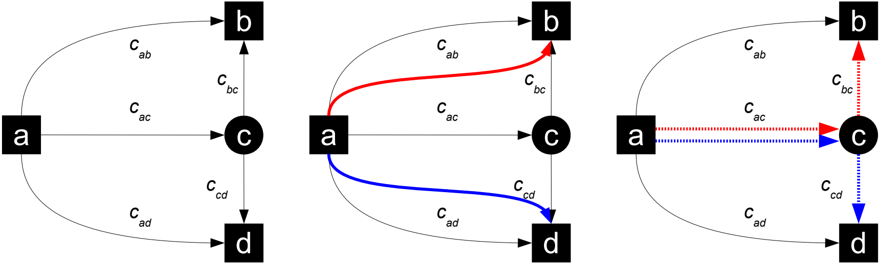
is equivalent to

$$\hat{A}(\xi)\hat{\mathbf{x}} \leq b \quad \xi \in \Xi,$$

where $\hat{A}$ is an affine function and $\hat{\mathbf{x}}$ is a vector of optimization variables.

Outline

Classical network design problem



Commodities are routed according to shortest paths. In the example, we assume $c_{ac} + c_{bc} \leq c_{ab}$ and $c_{ac} + c_{cd} \leq c_{ad}$.
Solution cost: $\xi^{ab}(c_{ac} + c_{bc}) + \xi^{ad}(c_{ac} + c_{cd})$.

Robust network design problem: **dynamic** recourse/routing

Demands vectors $\xi_1, \ldots, \xi_n$ must be routed **non-simultaneously**.
The problem becomes a two-stages program:

1. decide of the capacities
2. decide of the routing.

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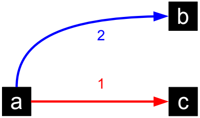
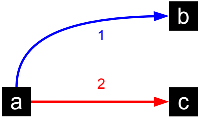
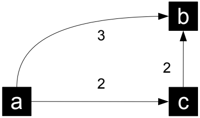


Demands for scenario 1



Demands for scenario 2



Capacity cost per unit

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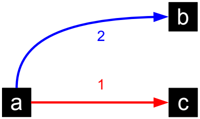

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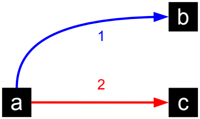
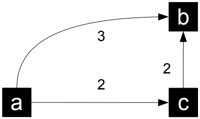


Demands for scenario 2



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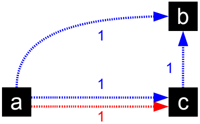
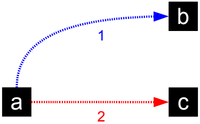
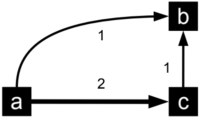


Routing for scenario 1



Routing for scenario 2



Capacity installation

Robust network design problem: **static** recourse/routing

Allowing for dynamic/arbitrary recourse may not be a good idea:

- Yield NP-hard optimization problems, see Chekuri et al. (2007), Gupta et al. (2001), Mattia (2010).
- Unpractical to change completely the routing according to the demand variations.

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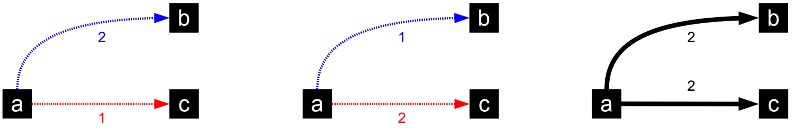
$\Rightarrow$ Introduction of static routing described by routing templates.



Routing for scenario 1 | Routing for scenario 2 | Capacity installation

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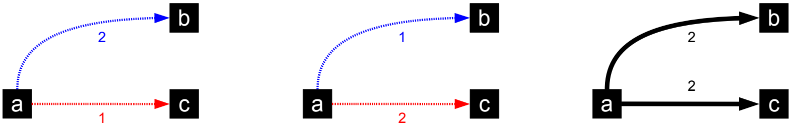

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$$\text{Cost\_Static} = 10 > 9 = \text{Cost\_Dynamic}$$

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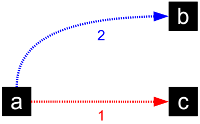

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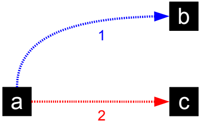
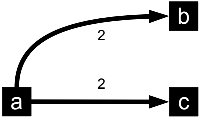


Routing for scenario 2



Capacity installation

$$\text{Cost\_Static} = 10 > 9 = \text{Cost\_Dynamic}$$

See Ben-Ameur (2007) and Scutellà (2009,2010) for intermediary (NP-hard) routing frameworks.

Using the parlance of robust optimization

First stage: capacity variables $\mathbf{x}$

Second stage: flow variables $\mathbf{y}$

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First stage: capacity variables $\mathbf{x}$

Second stage: flow variables $\mathbf{y}$

$$\min \sum_{a \in A} \kappa_a \mathbf{x_a}$$

$$\text{s.t.} \sum_{a \in \delta^+(v)} \mathbf{y_a^k}(\xi) - \sum_{a \in \delta^-(v)} \mathbf{y_a^k}(\xi) = \begin{cases} -\xi^k & v = s(k) \\ \xi^k & v = t(k), \\ 0 & \text{else} \end{cases} \qquad v \in V, k \in K, \ \xi \in \Xi$$

$$\sum_{k \in K} \mathbf{y_a^k}(\xi) \leq \mathbf{x_a}, \qquad a \in A, \xi \in \Xi$$

$$\mathbf{y_a^k}(\xi) \geq 0, \qquad a \in A, k \in K, \ \xi \in \Xi$$

$$\mathbf{x_a} \geq 0, \qquad a \in A,$$

Routing function

Dynamic routing: flow function $y^k : \Xi \to \mathbb{R}^{A \times K}$ is **arbitrary**.

Routing function

Static routing: flow functions f^k are **independent linear** functions for each commodity:

$$y_a^k(\xi) := \mathbf{y_a^k}\xi^k, \qquad a \in A,\ k \in K$$

for some routing template vector $y^k \in \mathbb{R}^A$.

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Affine routing: flow functions y^k are **affine** functions:

$$y_a^k(\xi) := \mathbf{y_a^{0k}} + \sum_{h \in K} \mathbf{y_a^{kh}}\xi^h, \qquad a \in A,\ k \in K$$

Dynamic routing: flow function $y^k : \Xi \to \mathbb{R}^{A \times K}$ is **arbitrary**.

Outline

Exact solution procedure

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\begin{aligned}
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&\text{s.t.} \quad \mathbf{x} \in \mathcal{X} \\
(P) \qquad & A(\xi)\mathbf{x} + E\mathbf{y}(\xi) \leq b \quad \xi \in \Xi. \qquad (2)
\end{aligned}
$$

Let $\mathcal{K}(\Xi)$ be the set defined by (2).

Lemma

It holds that $\mathcal{K}(\Xi) = \mathcal{K}(\text{vert}(\Xi))$.

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Idea of the proof:

$$A(\xi^*)x^* + Ey(\xi^*) \leq b \Leftrightarrow \sum_{s=1}^{\text{vert}(\Xi)} \lambda_s \left(A(\xi_s)x^* + Ey(\xi_s)\right) \leq \sum_{s=1}^{\text{vert}(\Xi)} \lambda_s b.$$

Finite reformulation

$$\begin{aligned} \min \quad & c^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X} \\ & A(\xi)\mathbf{x} + E\mathbf{y}(\xi) \leq b \quad \xi \in \text{vert}(\Xi). \end{aligned}$$

We shall consider an initial subset Ξ^* of Ξ and generate additional scenarios during decomposition algorithms:

Master problem

$$(MP') \quad \begin{aligned} \min \quad & c^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X}. \\ & \text{Constraints corresponding to } \xi \in \Xi^* \end{aligned}$$

Separation

Using the Farkas' lemma, we obtain:

Theorem

Let $x^* \in \mathbb{R}^n$ be given. Vector x^* belongs to $\mathcal{K}(\Xi)$ if and only if the optimal solution of the following optimization problem is non-positive

$$
(SP) \quad \begin{array}{ll} \max & (b - A(\xi)x^*)^T \pi \\ s.t. & \xi \in \Xi \\ & E^T \pi = 0 \\ & \mathbf{1}^T \pi = 1 \\ & \pi \geq 0. \end{array}
$$

Bilinear optimization problems are very difficult to solve. Mattia (2013) proposed a MILP reformulation based on complementary slackness conditions $\Rightarrow$ still very hard to solve.

We restrict ourselves to polytopes that can be obtained as affine projection of $(0,1)$ polytopes (polytopes whose extreme points are binary vectors).

Assumption

There exists a $(0,1)$ polytope Ω and an affine projection $\mathcal{A}$ such that $\Xi = \mathcal{A}(\Omega)$.

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Example

Let $\text{vert}(\Xi) = \{\xi_1, \ldots, \xi_s\}$ and let Ω be the unit simplex in $\mathbb{R}^s$. Then, for any $\omega \in \Omega$,

$$\mathcal{A}(\omega) = \left(\sum_{j=1}^{s} \xi_j^1 \omega_j, \ldots, \sum_{j=1}^{s} \xi_j^K \omega_j \right)$$

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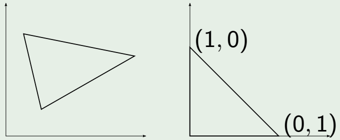
$$\mathcal{A}(\omega) = \left(\sum_{j=1}^{s} \xi_j^1 \omega_j, \ldots, \sum_{j=1}^{s} \xi_j^K \omega_j \right)$$

Example



(a) Ξ (b) Ω

Figure : Example of polytope Ξ affinely equivalent to a $(0,1)$ polytope.

Separation 2

Theorem

Let $x^* \in \mathbb{R}^n$ be given. Vector x^* belongs to $\mathcal{K}(\Xi)$ if and only if the optimal solution of the following optimization problem is non-positive

$$
\begin{aligned}
& & \max \quad & (b - A^0 x^*)^T \pi - \sum_{k \in K} (A^{1k} x^*)^T \mathbf{v^k} & \\
&(SPL) & s.t. \quad & \xi \in \Xi & \\
& & & E^T \pi = 0 & \\
& & & \mathbf{1}^T \pi = 1 & \\
& & & \mathbf{v_m^k} \geq \pi_{\mathbf{m}} - (1 - \xi^{\mathbf{k}}) & k \in K, m \in M \\
& & & \mathbf{v_m^k} \leq \xi^{\mathbf{k}} & k \in K, m \in M \\
& & & \pi, \mathbf{v_m^k} \geq 0, & \\
& & & \xi \in \{0, 1\}^K. &
\end{aligned}
$$

Two different approaches

Benders
$$(b - A(\xi^*)x)^T \pi^* \leq 0. \quad (3)$$

Row and column generation
$$A(\xi^*)x + Ey(\xi^*) \leq b. \quad (4)$$

Two different approaches

Benders	$(b - A(\xi^*)x)^T \pi^* \leq 0.$	(3)
Row and column generation	$A(\xi^*)x + Ey(\xi^*) \leq b.$	(4)

Algorithm 2: *RG* and *RCG*

repeat
- solve (MP');
- let x^* be an optimal solution;
- solve (SPL);
- let (ξ^*, π^*) be an optimal solution and z^* be the optimal solution cost;
- **if** $z^* > 0$ **then**
  - *RG*: add constraint (3) to (MP');
  - *RCG*: add constraint (4) to (MP');

until $z^* > 0$;

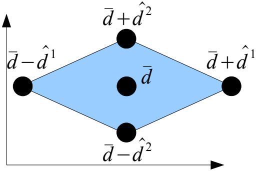
Application to network design

- SNDlib networks: janos-us, sun, and giul39

Application to network design

- SNDlib networks: janos-us, sun, and giul39
- Two kinds of polytopes:

$$\Xi = \begin{cases} \xi^k \in [\bar{\xi}^k - \sigma_-^k \hat{\xi}^k, \bar{\xi}^k + \sigma_+^k \hat{\xi}^k] \text{ for all } k \in K \\ \sum_{k \in K} \sigma_-^k + \sigma_+^k \leq \Gamma \end{cases}$$



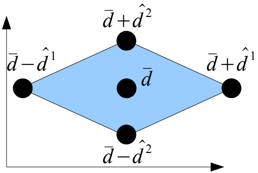
Ξ for $\Gamma = 1$

Details

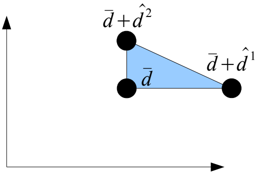
- SNDlib networks: janos-us, sun, and giul39
- Two kinds of polytopes:

$$\Xi = \left\{ \begin{array}{l} \xi^k \in [\bar{\xi}^k - \sigma_-^k \hat{\xi}^k, \bar{\xi}^k + \sigma_+^k \hat{\xi}^k] \text{ for all } k \in K \\ \sum_{k \in K} \sigma_-^k + \sigma_+^k \leq \Gamma \end{array} \right.$$

$$\Xi^+ = \left\{ \begin{array}{l} \xi^k \in [\bar{\xi}^k, \bar{\xi}^k + \sigma_+^k \hat{\xi}^k] \text{ for all } k \in K \\ \sum_{k \in K} \sigma_+^k \leq \Gamma \end{array} \right.$$



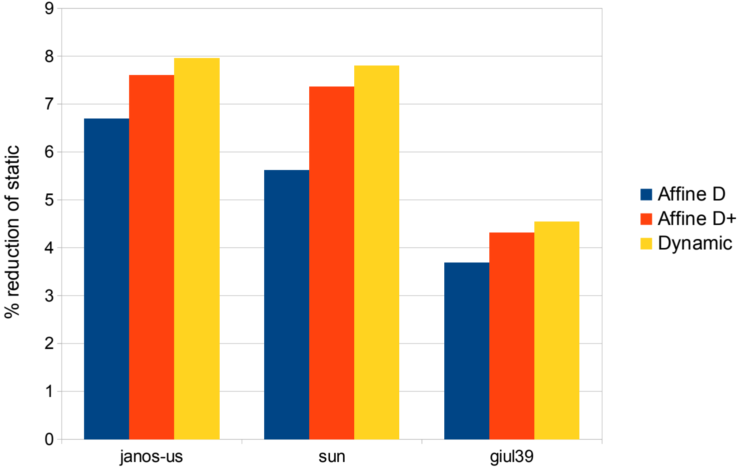
Ξ for $\Gamma = 1$



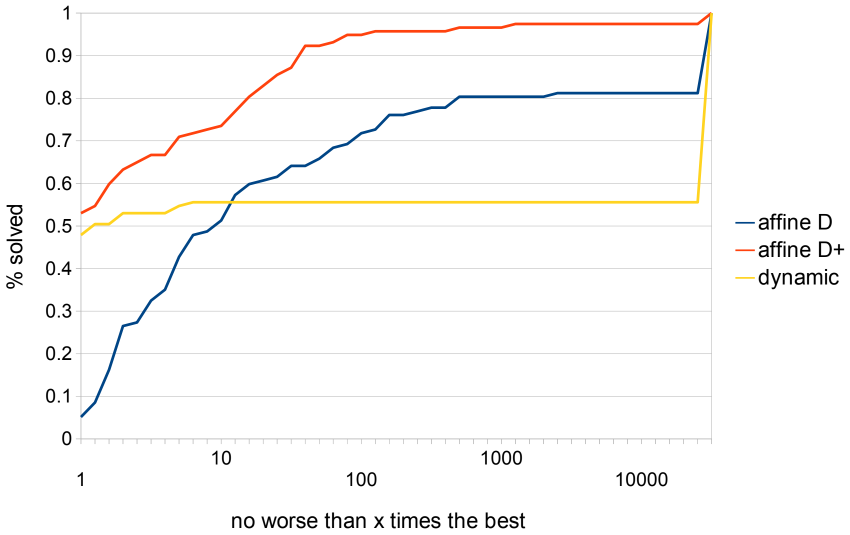
Ξ^+ for $\Gamma = 1$

Average cost reduction

$$|K| \in \{10, 20, 30, 40, 50\} \qquad \Gamma \in \{1, 2, 3, 4, 5, 6\}$$



Solution times: Performance profile



Numerical results

K	Γ	opt_{stat}	$gap_{dyn}(\%)$	t_{RCG}	t_{SPL} (%)	iter	t_{RG}	$t_{P'}$
30	2	672665	8.7	150	64	18	4967	13
30	3	699279	7.0	301	78	19	**T**	213
30	4	699590	2.9	1500	90	27	**T**	**M**
30	5	699590	0.7	1344	91	25	**T**	**M**
40	2	732850	8.8	365	69	21	6523	49
40	3	763505	7.6	1037	88	22	**T**	**M**
40	4	766293	4.1	6879	96	30	**T**	**M**
40	5	766293	1.5	5866	95	31	**T**	**M**
40	6	766293	–	**T**	–	–	**T**	**M**
50	2	793295	8.9	694	73	23	**T**	98
50	3	827405	8.2	4446	94	27	**T**	**M**
50	4	839656	6.0	22645	98	35	**T**	**M**
50	5	841295	–	**T**	–	–	**T**	**M**
50	6	841295	–	**T**	–	–	**T**	**M**

Table : Results for janos-us.

Attention: not applicable to multi-stage problems

ξ^t Information available at period t (i.e. $(\xi_1, \ldots, \xi_{t-1})$)

y_t Decision at period t

$$
\begin{aligned}
\min \quad & c^T \mathbf{x} \\
\text{s.t.} \quad & \mathbf{x} \in \mathcal{X} \\
& A(\xi)\mathbf{x} + E\mathbf{y}(\xi) \le b \quad \xi \in \Xi \\
& y_t(\xi) = y_t(\xi') \quad \text{for all } \xi, \xi' \in \Xi \text{ s.t. } \xi^t = \xi'^t
\end{aligned} \tag{5}
$$

Constraints (5) prevents us from using Farkas' Lemma as before!

Outline

Random recourse: general case

$$\begin{aligned} \min \quad & c^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X} \\ & A(\xi)\mathbf{x} + E(\xi)y(\xi) \le b, \quad \xi \in \Xi, \end{aligned} \tag{6}$$

where $E(\xi)$ and $y(\xi)$ are affine functions of ξ:

$$\begin{aligned} \min \quad & c^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X} \\ & \left(A^0 + \sum_k A^{1k}\xi^k\right)\mathbf{x} + \left(E^0 + \sum_k E^{1k}\xi^k\right)\left(\mathbf{y^0} + \sum_{k\in K} \mathbf{y^k}\xi^k\right) \le b, \quad \xi \in \Xi \end{aligned}$$

Example

Random recourse cost

$$r^T(\xi)y(\xi) \le \theta, \quad \xi \in \Xi.$$

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Example

Random recourse cost

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"Easy" case: Ξ is an ellipsoid

$$
\begin{aligned}
& \left(A^0 + \sum_k A^{1k}\xi^k\right)\mathbf{x} + \left(E^0 + \sum_k E^{1k}\xi^k\right)\left(\mathbf{y^0} + \sum_{k\in K}\mathbf{y^k}\xi^k\right) \le b, && \xi \in \Xi \\
\Leftrightarrow \quad & 0 \le \alpha(\mathbf{x},\mathbf{y}) + 2\xi^T\beta(\mathbf{x},\mathbf{y}) + \xi^T\Gamma(\mathbf{x},\mathbf{y})\xi, && \xi \in \Xi
\end{aligned} \tag{7}
$$

"Easy" case: Ξ is an ellipsoid

$$\left(A^0 + \sum_k A^{1k}\xi^k\right)\mathbf{x} + \left(E^0 + \sum_k E^{1k}\xi^k\right)\left(\mathbf{y^0} + \sum_{k\in K}\mathbf{y^k}\xi^k\right) \le b, \quad \xi \in \Xi$$

$$\Leftrightarrow \quad 0 \le \alpha(\mathbf{x},\mathbf{y}) + 2\xi^T\beta(\mathbf{x},\mathbf{y}) + \xi^T\Gamma(\mathbf{x},\mathbf{y})\xi, \qquad \xi \in \Xi \tag{7}$$

Theorem (Ben-Tal et al.)

Let $\Xi = \{\|\xi\|_2 \le \kappa_2\}$. *Constraint* (7) *is equivalent to*

$$\begin{pmatrix} \Gamma(\mathbf{x},\mathbf{y}) + \kappa_2^{-2}\mathbf{v}\,\mathrm{Id} & \beta(\mathbf{x},\mathbf{y}) \\ \beta^T(\mathbf{x},\mathbf{y}) & \alpha(\mathbf{x},\mathbf{y}) - \mathbf{v} \end{pmatrix} \succeq 0$$

$$\mathbf{v} \ge 0,$$

where Id *is the* $|K| \times |K|$ *identity matrix.*

Conic reformulation

Ellipsoidal uncertainty Semi-definite programming (SDP) reformulation

Other uncertainty No exact reformulation, SDP can provide an upper bound.

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Other uncertainty No exact reformulation, SDP can provide an upper bound.

$\Rightarrow$ Our objective is to address with decomposition algorithms for problems that satisfy the following assumption:

Assumption

$$A(\xi)x + E(\xi)y(\xi) \leq b \Rightarrow \begin{array}{l} r(\xi)g^T y(\xi) \leq \theta \\ A(\xi)x + E'y(\xi) \leq b \end{array}$$

Robust network design problem

Classical problem:

Input

- Directed graph $G = (V, A)$
- Capacity cost c
- Set of point-to-point commodities K
- Demand $d^k = \bar{\xi}^k + \xi^k \hat{\xi}^k$
- Uncertainty set Ξ
- *Outsourcing costs* r

Output

- Network capacities $\mathbf{x}$
- Flow (routing templates) $\mathbf{f}$
- *Outsourced fraction* $\mathbf{g}$

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We consider uncertain outsourcing cost $r(\xi)$

design variables	$\mathbf{x} \geq 0$
flow variables	$\mathbf{f} \geq 0$
outsourcing variables	$\mathbf{g} \geq 0$
outsourcing cost	$\theta \geq 0$

$$\min \quad \sum_{a \in A} \kappa_a \mathbf{x_a} + \theta$$

$$\text{s.t.} \quad \theta \geq r(\xi) \sum_{k \in K} d^k(\xi) \mathbf{g^k} \qquad \xi \in \Xi \qquad \mathit{COST}$$

$$\sum_{a \in \delta^-(v)} \mathbf{y_a^k} - \sum_{a \in \delta^+(v)} \mathbf{y_a^k} = 0, \qquad k \in K, v \in V^* \qquad \mathit{FLOW}$$

$$\sum_{k \in K} d^k(\xi) \mathbf{y_a^k} \leq \mathbf{x_a} \qquad a \in A, \xi \in \Xi \qquad \mathit{CAPACITY}$$

$$\mathbf{g^k} = 1 - \sum_{a \in \delta^-(t(k))} \mathbf{y_a^k} + \sum_{a \in \delta^+(t(k))} \mathbf{y_a^k} \qquad k \in K \qquad \mathit{REJECTION}$$

Master problem

- Ξ_0^* and Ξ_a^* are finite subsets of Ξ
- Relax capacity and cost constraints outside these subsets

$$
\begin{aligned}
& \min && \sum_{a \in A} \kappa_a \mathbf{x_a} + \theta && \\
(MP) \quad & \text{s.t.} && \theta \geq r(\xi) \sum_{k \in K} d^k(\xi) \mathbf{g^k} && \xi \in \Xi_0^* \\
& && \sum_{k \in K} d^k(\xi) \mathbf{y_a^k} \leq \mathbf{x_a} && a \in A, \xi \in \Xi_a^* \\
& && FLOW, REJECTION, NON-NEGATIVITY &&
\end{aligned}
$$

Separation problem

Given a solution $(\theta^*, x^*, f^*, g^*)$ to (MP), check for missing constraints:

$$\text{CAPACITY} \quad -x_a^* + \max_{\xi \in \Xi} \sum_{k \in K} d^k(\xi) f_a^{*k} \quad \Rightarrow \quad \text{OK}$$

$$\text{COST} \quad -\theta^* + \max_{\xi \in \Xi} r(\xi) \sum_{k \in K} d^k(\xi) g^{*k} \quad \Rightarrow \quad ?$$

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$$\sim -\theta^* + \max_{\xi\in\Xi} \ln\left(r(\xi)\right) + \ln\left(\sum_{k\in K} d^k(\xi) g^{*k}\right)$$

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$$\sim -\theta^* + \max_{\xi \in \Xi} \ln \left(r(\xi) \right) + \ln \left(\sum_{k \in K} d^k(\xi) g^{*k} \right)$$

We solve COST by a cutting-plane algorithm (embedded into another cutting plane algorithm)

We compare:

- *CP* Cut generation for cost and capacity constraints
- *CP+D* Dualization of capacity constraint and cut generation for cost constraints
- *SDP* Semi-definite reformulation in the ellipsoidal case

Instances:

	$\|V\|$	$\|A\|$	$\|K\|$
di-yuan	11	84	22
pdh	11	68	24
polska	12	36	66
nobel-us	14	42	91
atlanta	15	44	210
newyork	16	98	240
france	25	90	300
india35	35	160	595
germany50	50	176	662
cost266	37	114	1332

Solution times

Instances	Budgeted		Ellipsoid + box		Ellipsoid		
	CP	*CP+D*	*CP*	*CP+D*	*CP*	*CP+D*	*SDP*
di-yuan	0.1	0.1	0.1	6.1	0.2	10.8	17.3
pdh	0.1	0.1	1.4	114	0.1	4.4	10.9
polska	0.1	0.1	0.4	49.7	0.5	10.9	68.9
nobel-us	0.1	0.2	0.8	193	0.3	12	276
atlanta	1.4	4.1	4	260	4.4	75.8	**T**
newyork	0.9	1.1	2.9	**T**	10.1	450	**M**
France	2.9	4.2	74.3	**T**	18.8	271	**M**
india35	20.9	12.6	76.4	**T**	571	1750	**M**
germany	5.8	13.9	8.7	**T**	138	**T**	**M**
cost266	154	61.8	1480	**T**	**T**	**T**	**M**

Table : Solution times for $\epsilon = 0.05$.

Solution costs

Instances	Best	(*RND*)			(RND^0)		
		Bud.	Ell.+Box	Ell.	Bud.	Ell.+Box	Ell.
di-yuan	4.74+06	0.8	**0**	12.8	11.6	11.6	36.5
pdh	2.45+07	**0**	1.3	16.6	2.2	2.2	45.6
polska	3.74+02	1.0	**0**	7.2	21.4	18.9	23.4
nobel-us	4.82+06	3.0	**0**	7.1	33.7	30.3	35.3
atlanta	1.86+08	12.4	**0**	16.6	50.4	38.7	52.5
newyork	2.67+05	5.0	**0**	9.3	38.6	34.0	39.7
France	2.14+04	5.3	**0**	5.3	31.4	19.8	22.2
india35	3.43+03	2.2	**0**	2.8	39.2	22.2	22.9
germany	6.19+05	7.6	**0**	7.3	41.3	26.9	32.7
cost266	1.39+07	8.8	**0**	*N.A.*	50.5	24.5	24.7

Table : Solution costs for $\epsilon = 0.05$.

Thank's for your attention