Programme Gaspard Monge pour l'Optimisation et la Recherche Opérationnelle

Programmation linéaire colorée : bases, polytope et algorithmes

Pauline Sarrabezolles – École des Ponts

Travail commun avec Antoine Deza (McMaster University, Hamilton) et Frédéric Meunier (École des Ponts)

Plan

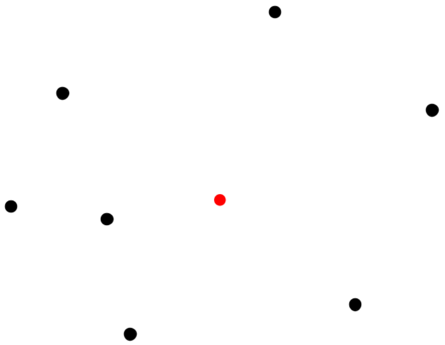
1. Colourful linear programming

2. Some algorithms

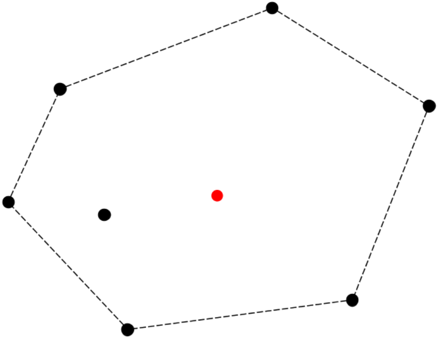
3. Counting questions
   - A geometrical problem : the colourful simplicial depth
   - A combinatorial approach : Octahedral systems

Colourful linear programming

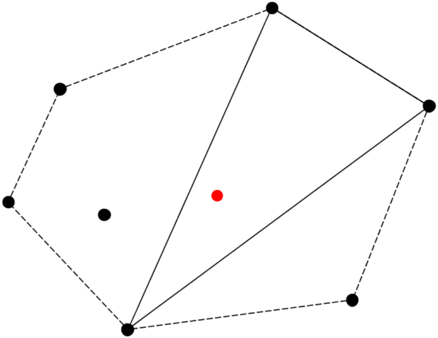
The Carathéodory Theorem in dimension two



The Carathéodory Theorem in dimension two



The Carathéodory Theorem in dimension two



Linear programming

The linear programming problem.

Input : A set $S \subset \mathbb{Q}^d$ and a point $p \in \mathbb{Q}^d$.
Output : *Decide* whether there is a $T \subseteq S$, $|T| \leq d + 1$, such that $p \in \text{conv}(T)$. If “yes”, *find* it.

Carathéodory Theorem $\Longrightarrow$

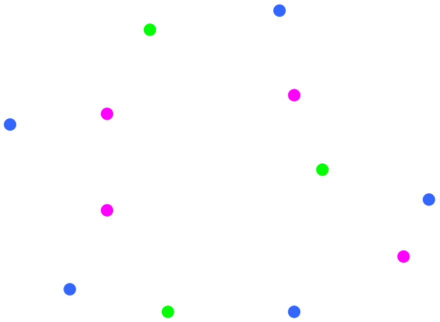
If $p \in \text{conv}(S)$, there is such a T.

Complexity status

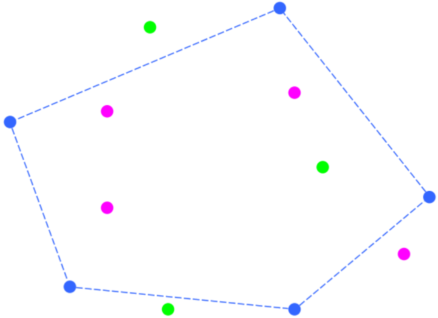
Theorem

Linear programming is in $\mathcal{P}$.

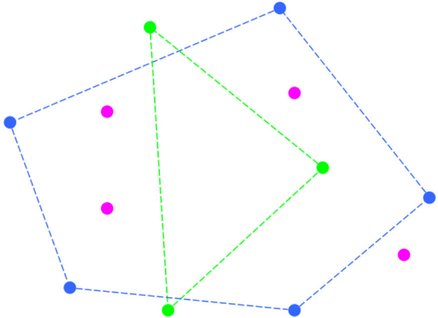
The colourful Carathéodory Theorem in dimension two



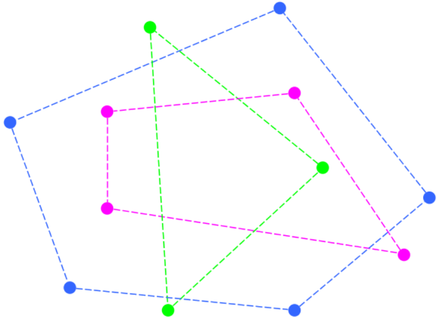
The colourful Carathéodory Theorem in dimension two



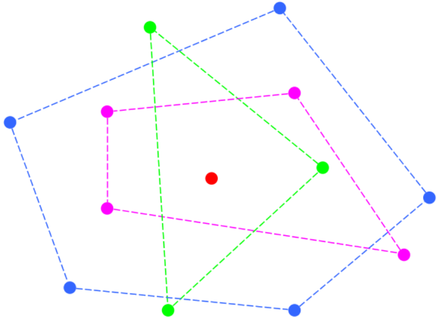
The colourful Carathéodory Theorem in dimension two



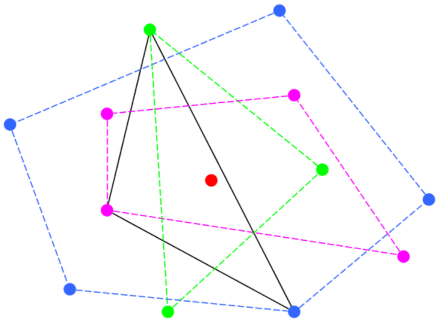
The colourful Carathéodory Theorem in dimension two



The colourful Carathéodory Theorem in dimension two



The colourful Carathéodory Theorem in dimension two



The colourful Carathéodory Theorem [Bárány 1982]

Given a set of points $S = S_1 \cup \ldots \cup S_{d+1}$ *and a point* p *in* $\mathbb{Q}^d$ *such that* $p \in \bigcap_{i=1}^{d+1} \operatorname{conv}(S_i)$*, there is a* $T \subseteq \bigcup_{i=1}^{d+1} S_i$ *such that* $|T \cap S_i| \leq 1$ *for* $i = 1, \ldots, d+1$ *and* $p \in \operatorname{conv}(T)$*.*

$T \subseteq \bigcup_{i=1}^{d+1} S_i$ such that $|T \cap S_i| \leq 1$ for $i = 1, \ldots, d+1$ is *colourful*.

Colourful linear programming [Bárány and Onn in 1997]

The colourful linear programming problem.

Input : k sets, or *colours*, $S_1, \ldots, S_k \subset \mathbb{Q}^d$ and a point $p \in \mathbb{Q}^d$.
Output : *Decide* whether there is a colourful $T = \{s_1, \ldots, s_k\}$ such that $p \in \text{conv}(T)$. If "yes", *find* it.

Colourful Carathéodory Theorem $\Longrightarrow$

If $k = d + 1$ and $p \in \bigcap_{i=1}^{k} \text{conv}(S_i)$, there is such a T.

Complexity status

Theorem (Bárány and Onn, 1997)

Colourful linear programming is strongly $\mathcal{NP}$-complete.

Other colourful linear programming problems

Colourful feasibility problem.

Input : $d+1$ sets $S_1, \ldots, S_{d+1} \subset \mathbb{Q}^d$ and a point $p \in \mathbb{Q}^d$ such that, $p \in \bigcap_{i=1}^{d+1} \operatorname{conv} S_i$.
Output : Find a colourful simplex containing p.

Complexity : open question.

Other colourful linear programming problems

Lemma (Octahedral Lemma)

Let $X_1, \ldots, X_{d+1}$ *be sets of points, with* $|X_i| = 2$*, and a point* p*. There is an* even *number of colourful simplices generated by* $\bigcup_{i=1}^{d+1} X_i$ *containing* p*.*

In particular, if there is one there is another.

Other colourful linear programming problems

Another colourful simplex.

Input : A colourful simplex σ containing p and a colourful simplex σ' disjoint from σ.
Output : Find another colourful simplex containing p generated by points of $\sigma \cup \sigma'$.

Complexity : It belongs to the $\mathcal{PPAD}$ class. It is an open question whether it is $\mathcal{PPAD}$-complete.

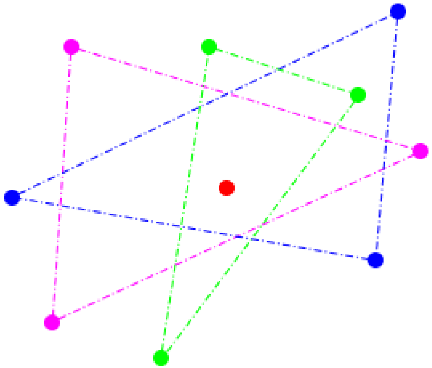
Some algorithms [1]

[1] In this section, p is the origin $\mathbf{0}$.

An algorithm for the colourful feasibility problem.

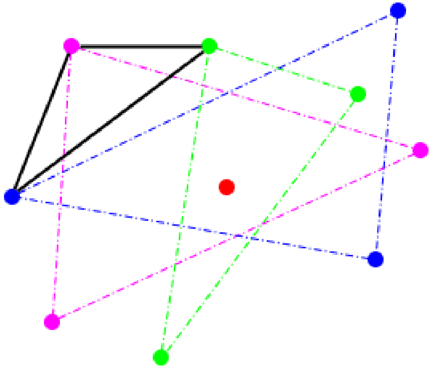
Bárány's algorithm (1982)

Consider $S_1, \ldots, S_{d+1}$, sets of points, each containing $\mathbf{0}$.



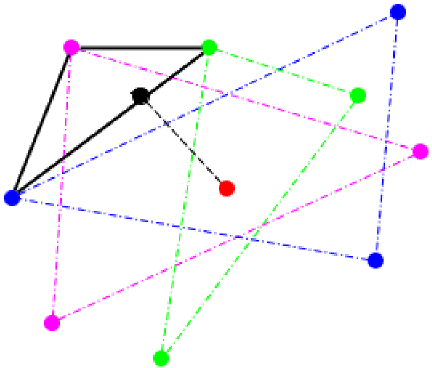
Bárány's algorithm (1982)

Consider a colourful simplex.



Bárány's algorithm (1982)

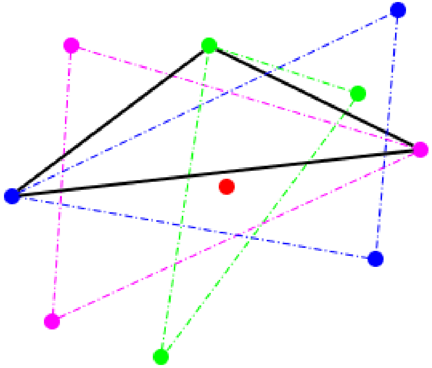
Consider the closest point to the **0** in this simplex.



This point lies on a facet of the colourful simplex. A colour i is missing on this facet.

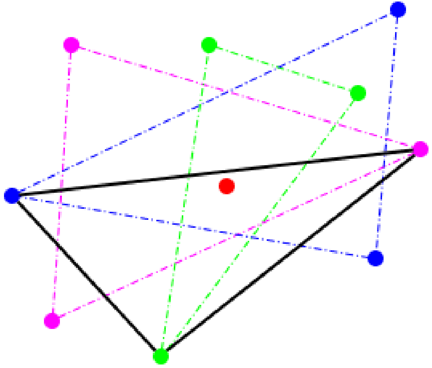
Bárány's algorithm (1982)

Replace the vertex of colour *i* with another vertex of the same colour, getting a point closer to $\mathbf{0}$



Bárány's algorithm (1982)

Iterate...



Bárány's algorithm

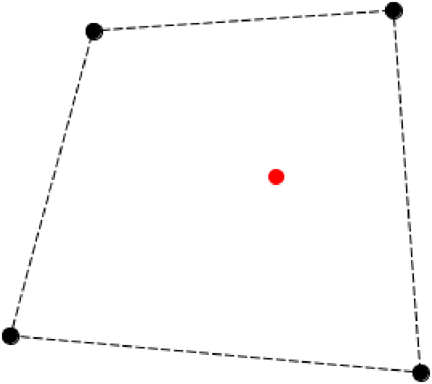
Complexity for rational data :
Given $\rho > 0$ and $S_1, \ldots, S_{d+1} \subset \mathbb{Q}^d$ of bit size L, with $B(0, \rho) \subset \text{conv}(S_i)$.

This algorithm find a colourful simplex containing **0** in polynomial time in L and $1/\rho$.

An algorithm for the problem another colourful simplex

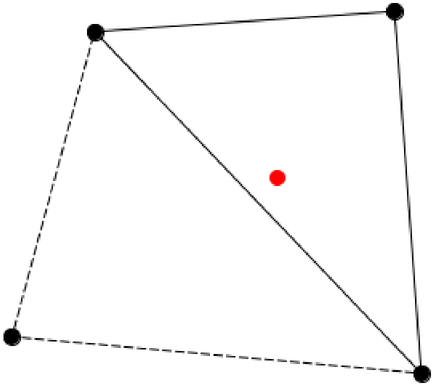
Reminder : the simplex algorithm

A point in the convex hull of $d + 2$ points in $\mathbb{R}^d$ is in exactly two simplices generated by those points.



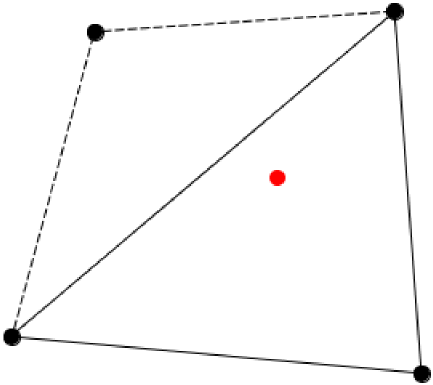
Reminder : the simplex algorithm

A point in the convex hull of $d + 2$ points in $\mathbb{R}^d$ is in exactly two simplices generated by those points.



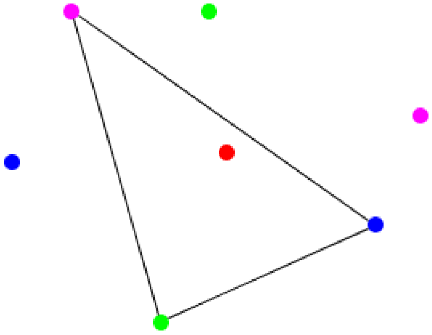
Reminder : the simplex algorithm

A point in the convex hull of $d+2$ points in $\mathbb{R}^d$ is in exactly two simplices generated by those points.



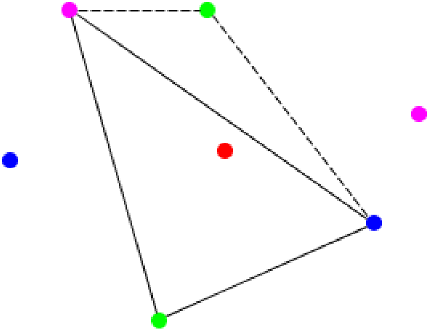
Pivoting algorithm

Consider a colourful simplex σ containing $\mathbf{0}$, and a disjoint colourful simplex *i.e. one point of each colour not in* σ.



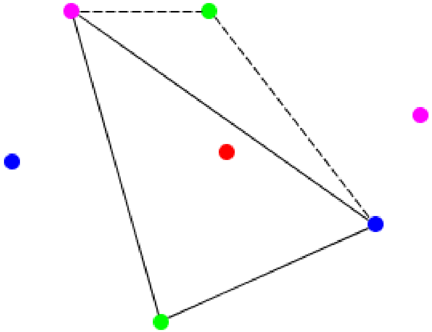
Pivoting algorithm

Consider a colour, called the pivoting colour.



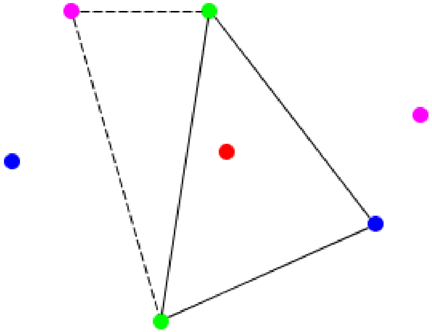
Pivoting algorithm

Apply the argument of the simplex algorithm.



Pivoting algorithm

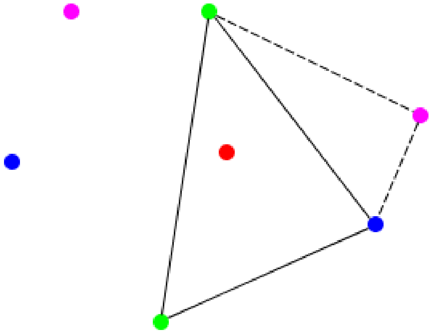
Consider the other simplex containing the origin.



This simplex is "almost" colourful. The pivoting colour is duplicated, and a colour i is missing.

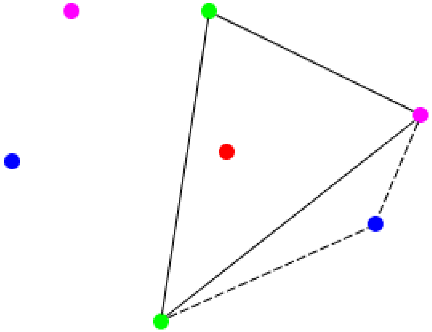
Pivoting algorithm

Add the vertex of colour i not in σ, and obtain a new simplex containing $\mathbf{0}$.



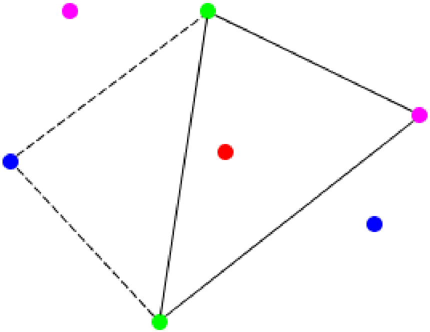
Pivoting algorithm

Iterate...



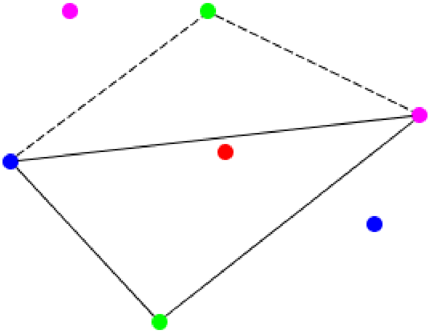
Pivoting algorithm

Iterate...



Pivoting algorithm

Iterate...



Pivoting algorithm

Open questions :

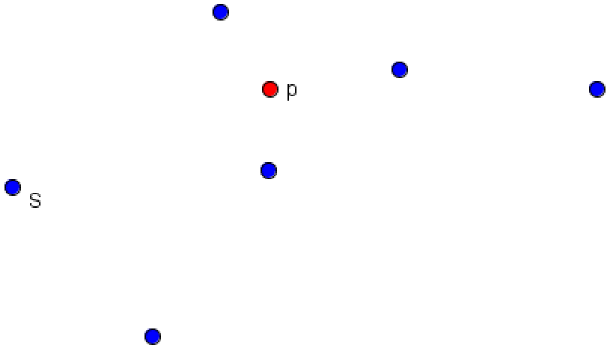
- How many steps in the pivoting algorithm ?
- Where do the colourful and almost colourful solutions lie on the polytope ?

Counting feasible bases

Original motivation : simplicial depth

Let S be a set of points in $\mathbb{R}^d$.

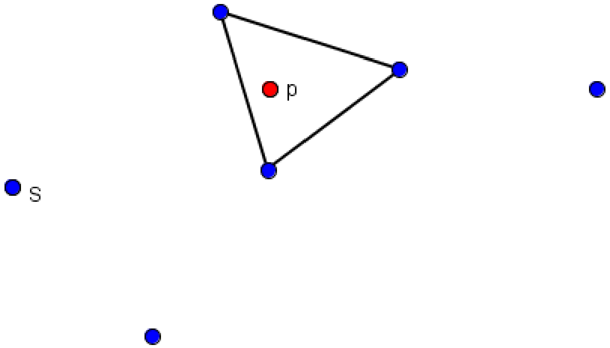
Simplicial depth of a point p = number of d-simplices generated by S and containing p.



Original motivation : simplicial depth

Let S be a set of points in $\mathbb{R}^d$.

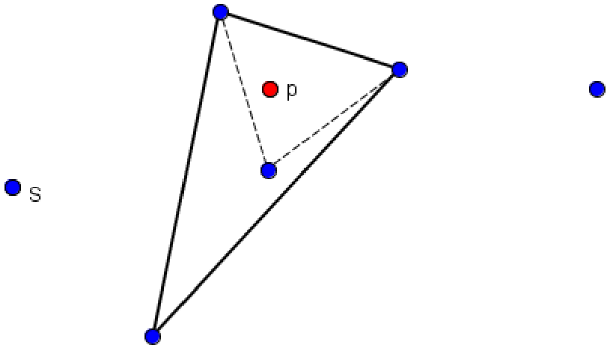
Simplicial depth of a point p = number of d-simplices generated by S and containing p.



Original motivation : simplicial depth

Let S be a set of points in $\mathbb{R}^d$.

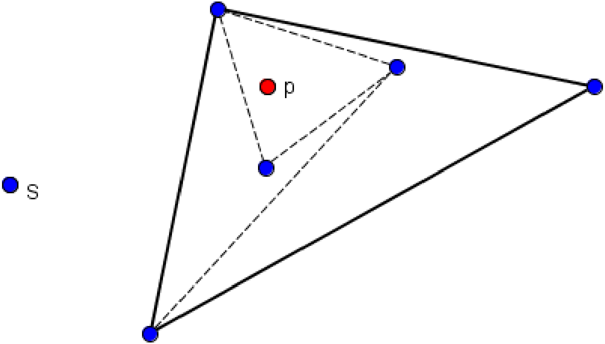
Simplicial depth of a point p = number of d-simplices generated by S and containing p.



Original motivation : simplicial depth

Let S be a set of points in $\mathbb{R}^d$.

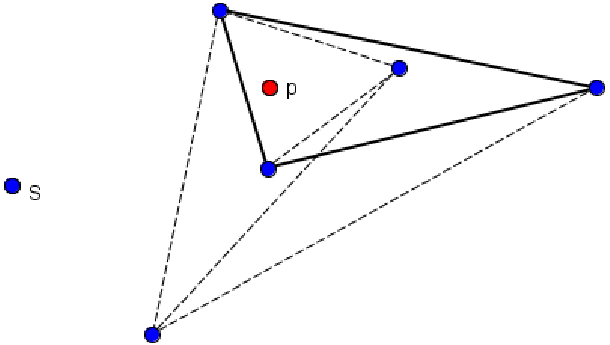
Simplicial depth of a point p = number of d-simplices generated by S and containing p.



Original motivation : simplicial depth

Let S be a set of points in $\mathbb{R}^d$.

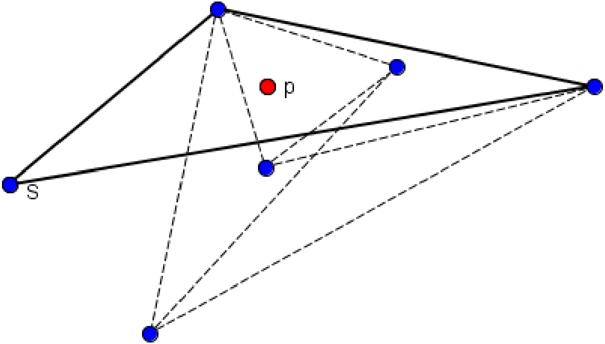
Simplicial depth of a point p = number of d-simplices generated by S and containing p.



Original motivation : simplicial depth

Let S be a set of points in $\mathbb{R}^d$.

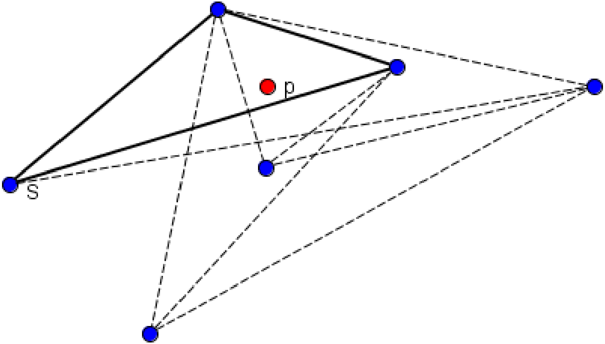
Simplicial depth of a point p = number of d-simplices generated by S and containing p.



Original motivation : simplicial depth

Let S be a set of points in $\mathbb{R}^d$.

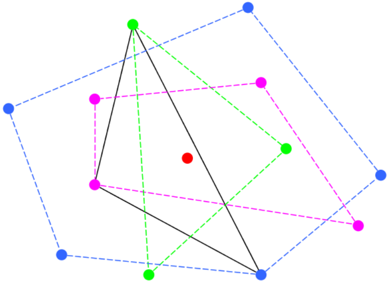
Simplicial depth of a point p = number of d-simplices generated by S and containing p.



Original motivation : simplicial depth

Let $S_1, \ldots, S_{d+1}$ be $(d+1)$ sets of points in $\mathbb{R}^d$.

Colourful simplicial depth of a point p is :
depth$_{S_1,\ldots,S_{d+1}}(p)$ = number of colourful d-simplices generated by $\bigcup_{i=1}^{d+1} S_i$ and containing p.

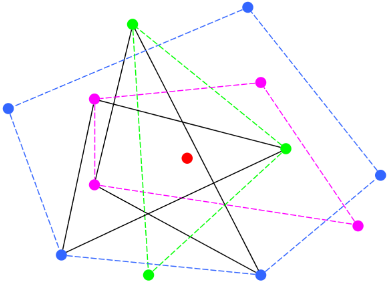


Original motivation : simplicial depth

Let $S_1, \ldots, S_{d+1}$ be $(d+1)$ sets of points in $\mathbb{R}^d$.

Colourful simplicial depth of a point p is :

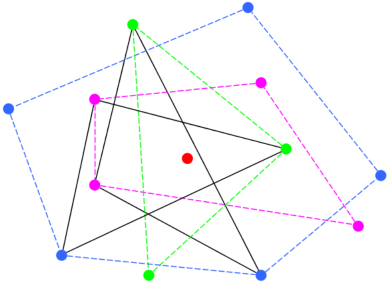
depth$_{S_1,\ldots,S_{d+1}}(p)$ = number of colourful d-simplices generated by $\bigcup_{i=1}^{d+1} S_i$ and containing p.



Original motivation : simplicial depth

Let $S_1, \ldots, S_{d+1}$ be $(d+1)$ sets of points in $\mathbb{R}^d$.

Colourful simplicial depth of a point p is :
$\mathbf{depth}_{S_1,\ldots,S_{d+1}}(p)$ = number of colourful d-simplices generated by $\bigcup_{i=1}^{d+1} S_i$ and containing p.



$$\mu(d) = \min_{S_1,\ldots,S_{d+1},p} \mathbf{depth}_{S_1,\ldots,S_{d+1}}(p).$$

A lower bound on simplicial depth

For $S \cup \{p\}$ in general position

[Bárány1982]

$$\max_p \operatorname{depth}_S(p) \geq \frac{1}{(d+1)^{d+1}} \binom{n}{d+1} \quad \text{with } n = |S|.$$

Proof combines the Tverberg theorem and the colourful Carathéodory theorem.

A lower bound on simplicial depth

For $S \cup \{p\}$ in general position

[Bárány1982]

$$\max_p \operatorname{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} \quad \text{with } n = |S|.$$

Proof combines the Tverberg theorem and the colourful Carathéodory theorem.

A new lower bound for simplicial depth

$$\mu(d) = \min_{\substack{S_1,\dots,S_{d+1} \\ p \in \bigcap_{i=1}^{d+1} \mathrm{conv}(S_i)}} \#\{T : T \text{ colourful and } p \in \mathrm{conv}(T)\}.$$

Strong version of Colourful Carathéodory Theorem : each point in $\bigcup_{i=1}^{d+1} S_i$ is part of a colourful simplex containing the origin.

$$\max_p \mathrm{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{(d+1)}} \binom{n}{d+1} \quad \text{with } n = |S|.$$

What is the exact value of $\mu(d)$?

A new lower bound for simplicial depth

$$\mu(d) = \min_{\substack{S_1,\ldots,S_{d+1} \\ p \in \bigcap_{i=1}^{d+1} \mathrm{conv}(S_i)}} \#\{T : T \text{ colourful and } p \in \mathrm{conv}(T)\}.$$

Strong version of Colourful Carathéodory Theorem : each point in $\bigcup_{i=1}^{d+1} S_i$ is part of a colourful simplex containing the origin.

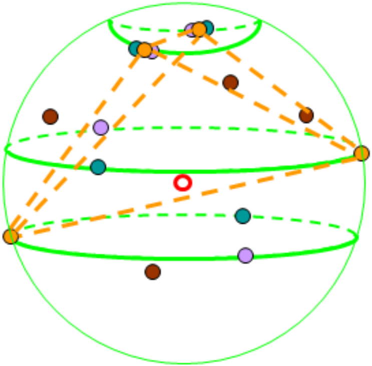
$$\max_p \mathrm{depth}_S(p) \geq \frac{d+1}{(d+1)^{(d+1)}} \binom{n}{d+1} \quad \text{with } n = |S|.$$

What is the exact value of $\mu(d)$?

Upper bound on the colourful simplicial depth

Unfortunately,
[Deza et al., 2006]

$$\mu(d) \leq d^2 + 1.$$



Gromov's bound

$$\max_p \operatorname{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{(d+1)}} \binom{n}{d+1} \quad \text{with } n = |S|,$$

with $\mu(d) = d^2 + 1$ at best.

[Gromov, 2010]

$$\max_p \operatorname{depth}_S(p) \geq \frac{2d}{(d+1)!(d+1)} \binom{n}{d+1} \quad \text{with } n = |S|.$$

(simplification by Karasev, 2012).

The conjecture

Conjecture.

$$\mu(d) = d^2 + 1.$$

The successive improvements

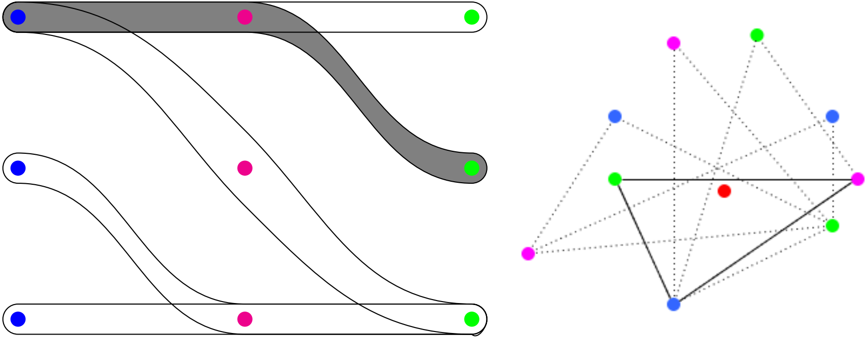
	Lower bound for $\mu(d)$	Conjecture true for d up to
Bárány, 1982	$d+1$	1
Deza et al., 2006	$2d$	2
Bárány and Matoušek, 2007	$\max(3d, \frac{1}{5}d^2+\frac{1}{5}d)$	3
Stephen and Thomas, 2008	$\frac{1}{4}d^2+d+1$	$\emptyset$
Deza, Stephen, and Xie, 2011	$\frac{1}{2}d^2+d+\frac{1}{2}$	$\emptyset$
Deza, Meunier, and S., 2012	$\frac{1}{2}d^2+\frac{7}{2}d-8$	4

A combinatorial counterpart : octahedral systems

An *octahedral system* Ω in an n-partite hypergraph $(V_1, \ldots, V_n, E)$ satisfying *parity condition :* for any $X \subseteq \bigcup_{i=1}^{n} V_i$ such that $|X \cap V_i| = 2$ for all i, the number of edges of Ω induced by X is even.

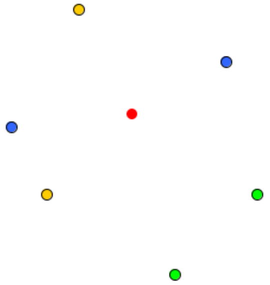
Octahedral systems *without isolated vertex* generalize colourful configurations.

An octahedral system



Two main properties for the geometrical approach

Octahedral Lemma



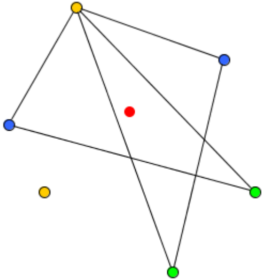
$X \subseteq S$, $|X \cap S_i| = 2$ for all $i \longrightarrow$ an even number of colourful simplices.

Strong colourful Carathéodory Theorem

If $\mathbf{0} \in \text{conv}(S_i)$ for all i, each point is part of some colourful simplices containing the origin.

Two main properties for the geometrical approach

Octahedral Lemma



$X \subseteq S$, $|X \cap S_i| = 2$ for all i $\longrightarrow$ an even number of colourful simplices.

Strong colourful Carathéodory Theorem

If $\mathbf{0} \in \text{conv}(S_i)$ for all i, each point is part of some colourful simplices containing the origin.

Combinatorial approach

Vertex set :
$V = V_1 \cup \cdots \cup V_{d+1}$.

Edge set : E.

Parity condition : The number of edges induced by X, with $|X \cap V_i| = 2$ for all i, is even.

Octahedral systems without isolated vertex : Every point in $\bigcup_{i=1}^{d+1} V_i$ is in at least one edge.

Geometrical approach

A colourful configuration
$S = S_1 \cup \cdots \cup S_{d+1}$.

Colourful simplices containing the origin.

Octahedral Lemma : The number of colourful simplices containing the origin generated by points in X, with $|X \cap S_i| = 2$ for all i, is even.

Strong Colourful Carathéodory Theorem : Every point in $\bigcup_{i=1}^{d+1} S$ is part of some colourful simplex containing the origin.

If Ω realizes a colourful configuration, **the number of edges $|E|$ is the number of colourful simplices containing the origin.**

Definition (ν)

$\nu(d)$ is the minimal number of edges of an octahedral system without isolated vertex with $|V_i| = d + 1$ for $i = 1, \ldots, d + 1$.

$$\nu(d) \leq \mu(d)$$

Lower bounds

Theorem (Deza, Meunier, S.)

$$\nu(d) \geq \frac{1}{2}d^2 + \frac{7}{2}d - 8$$

Lower bounds

Theorem (Deza, Meunier, S.)

$$\nu(d) \geq \frac{1}{2}d^2 + \frac{7}{2}d - 8$$

$$\mu(d) \geq \frac{1}{2}d^2 + \frac{7}{2}d - 8$$

Idea of the proof : induction

Inductive approach.

Given an octahedral system $\Omega = (V, E)$ without isolated vertex and one of its vertices v, use the bound for $\Omega' = (V', E') = \Omega \setminus \{v\}$:

$$|E| = |E'| + \deg_{\Omega}(v).$$

For any such Ω', parity condition automatically satisfied.

We would like to ensure that Ω' is again without isolated vertex.

Main Idea. Delete the vertices one after another until reaching an octahedral system whose number of edges can be estimated.

Small instances

An octahedral system with $n = 5$, $|V_1| = \ldots = |V_5| = 5$ and without isolated vertex has at least 17 edges.

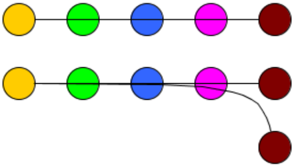
Proposition

$$\mu(4) = 17.$$

Computational approach “branch-and-bound” $\mu(4) \geq 14$, (Deza, Stephen, and Xie, 2012).

Small instances

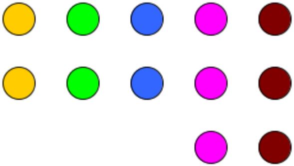
$$n = 5,\ |V_1| = \ldots = |V_5| = 5 \implies |E| \geq 17.$$



$|E| \geq 3$

Small instances

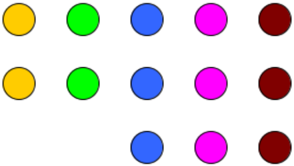
$$n = 5,\ |V_1| = \ldots = |V_5| = 5 \implies |E| \geq 17.$$



$|E| \geq 4$

Small instances

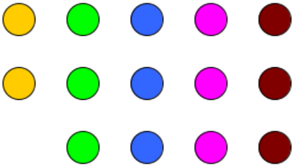
$$n = 5,\ |V_1| = \ldots = |V_5| = 5 \implies |E| \geq 17.$$



$|E| \geq 5$

Small instances

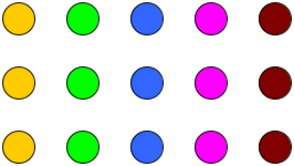
$$n = 5,\ |V_1| = \ldots = |V_5| = 5 \implies |E| \geq 17.$$



$|E| \geq 6$

Small instances

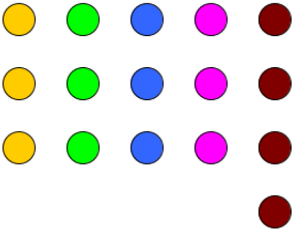
$$n = 5,\ |V_1| = \ldots = |V_5| = 5 \implies |E| \geq 17.$$



$|E| \geq 7$

Small instances

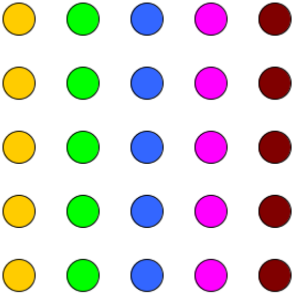
$$n = 5,\ |V_1| = \ldots = |V_5| = 5 \implies |E| \geq 17.$$



$|E| \geq 8$

Small instances

$$n = 5,\ |V_1| = \ldots = |V_5| = 5 \implies |E| \geq 17.$$



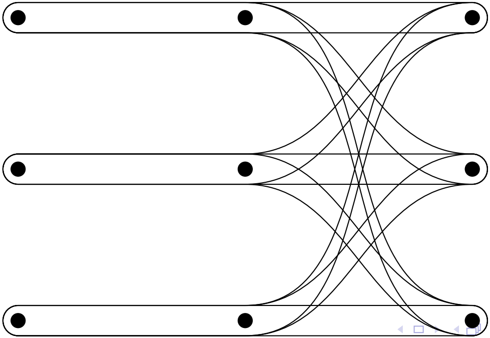
$|E| \geq 17$

Realisability

Is any octahedral system Ω with $|V_i| = d + 1$ for $i = 1, \ldots, d + 1$ and without isolated vertex the combinatorial counterpart of sets of points $S_1, \ldots, S_{d+1}$ in $\mathbb{R}^d$?

No.

Counterexample.



Consequence

It might be possible that the conjecture $\mu(d) = d^2 + 1$ cannot be proven using octahedral systems...

Open questions

- Complexity status of colourful linear programming under the condition $p \in \bigcap_{i=1}^{d+1} \operatorname{conv} S_i$.
- Number of steps in the pivoting algorithm.
- $\mu(d) \stackrel{?}{=} d^2 + 1$.

Thank you.