Chapter 2

Examples, Benjamini–Schramm limits

In this chapter, we give the first examples of convergences of deterministic and random pointed graphs. The choice of the origin vertex is essential, as convergence is defined with respect to the local structure around this base point. A fundamental idea, introduced by Benjamini and Schramm, is to consider a finite graph rooted at a vertex chosen uniformly at random. This gives rise to a random rooted graph that encodes the local geometry of the original graph from the perspective of a typical vertex. The distributional limits of such random rooted graphs, known as **Benjamini–Schramm limits**, provide a rigorous framework for analyzing local properties of large sparse graphs, see Part III.

2.1 Deterministic limits

2.1.1 First examples

We illustrate by pictures a few convergences of deterministic random graphs:

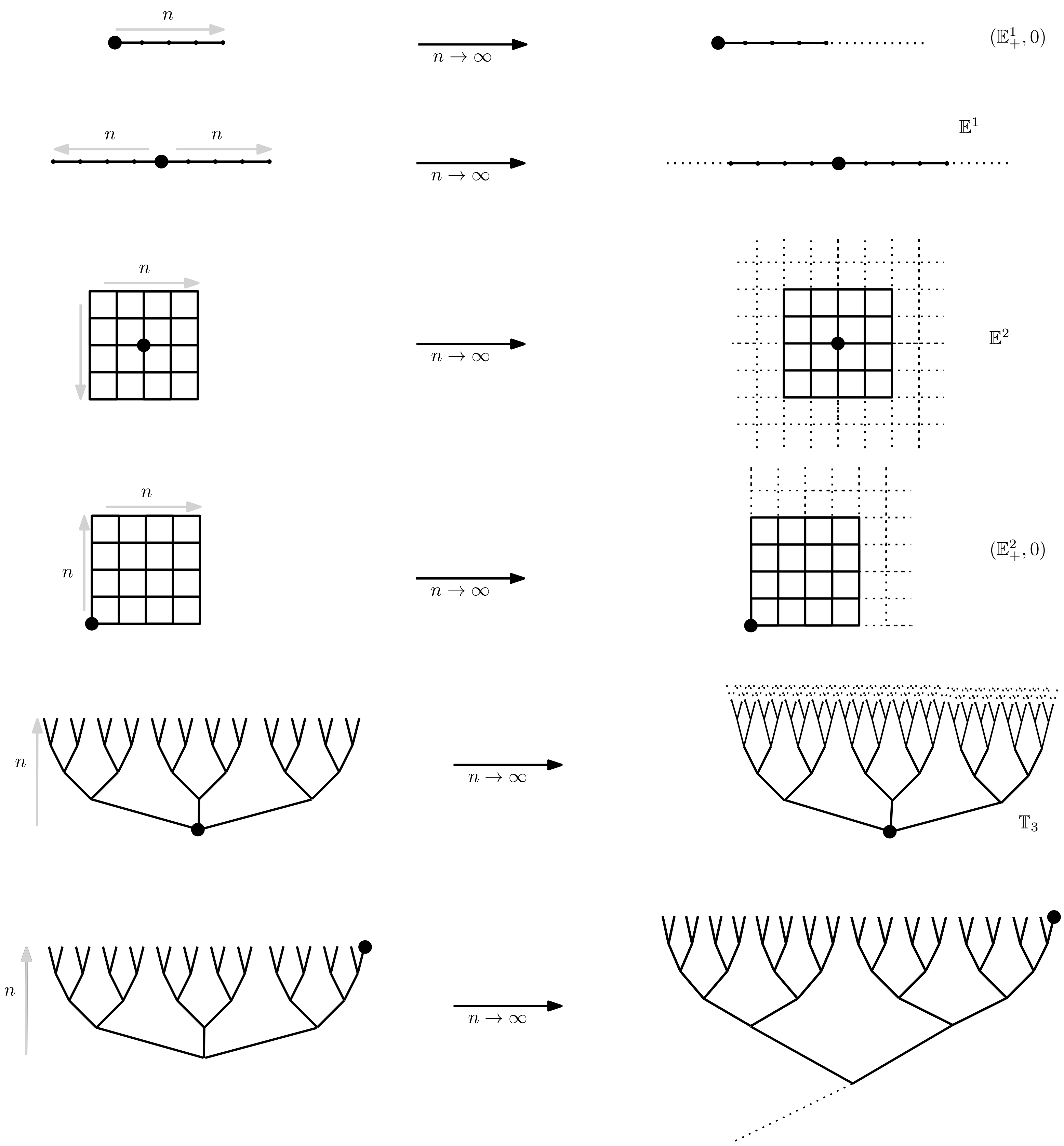


Figure 2.1: A few examples of convergence of deterministic pointed graphs for the local topology. In particular, the last two examples are seen as pointed graphs and not plane trees. The last limit graph is called the canopy tree, it has a unique infinite branch.

To the preceding examples, one can add the d-dimensional torus equipped with the standard edges (the graph pointed at any vertex is transitive) which converges towards the d-dimensional Euclidean lattice as the side length tends to $+\infty$.

2.1.2 Large girth graphs

The last two examples illustrated the fact that the location of the based point may dramatically change the limiting graph. As a first challenge to the reader we propose to think about the following exercise: Construct a sequence of d-regular graphs g_n so that for any $\rho_n \in \mathrm{V}(\mathsf{g}_n)$ we have the local convergence $(\mathsf{g}_n, \rho_n) \to (\mathbb{T}_d, \rho)$ as $n \to \infty$, where $\mathbb{T}_d$ is the infinite d-regular tree. Equivalently, this boils down to constructing a d-regular graph which is locally tree-like around each vertex, that is, so that the length of the smallest non-trivial cycle, the so-called **girth** of the graph, tends to infinity. This is however a very difficult exercise, and the first explicit construction of such graphs (for $d = 4$ and along some particular values of n) was famously provided by Margulis [56]. In fact, nowadays the easiest way to prove existence of such graphs is via the probabilistic method.

Theorem 5 (*Construction of large girth graphs*)

For each prime number $p \geq 3$, consider the Cayley graph g_p of the subgroup of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ with generators

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

We denote by $\mathsf{Girth}(\mathsf{g}_p)$ the girth of g_p, i.e, the length of the shortest non-trivial cycle in g_p. There exist constants $c < C \in (0, \infty)$ such that

$$c \cdot \log(p) \leq \mathsf{Girth}(\mathsf{g}_p) \leq C \cdot \log(p) \quad \textit{for all sufficiently large } p.$$

In particular, as $p \to \infty$, we have $\mathsf{g}_p \to \mathbb{T}_4$ for the local topology on $\mathcal{G}^\bullet$ (these graphs are transitive).

Proof. First, let us prove a logarithmic upper bound on the girth of g_p. In fact, for any 4-regular graph g with n vertices, we have

$$\mathsf{Girth}(\mathsf{g}) \leq 2 \cdot \left(\log_3 \left(\frac{n+1}{2} \right) + 1 \right). \tag{2.1}$$

Indeed, fix $\rho \in \mathsf{g}$, and set $r = \lceil \mathsf{Girth}(\mathsf{g})/2 \rceil - 1$. Since $2r < \mathsf{Girth}(\mathsf{g})$, there is a one-to-one correspondence between the vertices of $[(\mathsf{g}, \rho)]_r$ and non-backtracking paths starting from ρ and with length at most r. Indeed, for each vertex $x \in [(\mathrm{g}, \rho)]_r$ there exists a non-backtracking path from ρ to x which has length at most r, and any two non-backtracking paths starting from ρ and with length at most r must lead to different vertices of $((\mathrm{g}, \rho)]_r$, for otherwise there would be a non-trivial cycle in g with length at most $2r$. In particular, the pointed graph $[(\mathrm{g}, \rho)]_r$ is isomorphic to $[(\mathbb{T}_4, \rho)]_r$, and we have

$$\#[(\mathrm{g}, \rho)]_r = 1 + 4 + 4 \cdot 3 + \ldots + 4 \cdot 3^{r-1} = 1 + 2 \cdot (3^r - 1) = 2 \cdot 3^r - 1.$$

Since on the other hand we have $\#([)\mathrm{g}, \rho)]_r \leq n$, we obtain $r \leq \log_3((n+1)/2)$, which yields (2.1). Specialising (2.1) to g_p and using the crude upper bound $\#\mathrm{V}(\mathrm{g}_p) \leq p^4$, we obtain

$$\mathsf{Girth}(\mathrm{g}_p) \leq 2 \cdot \left(\log_3 \left(\frac{p^4+1}{2}\right) + 1\right),$$

which shows that $\mathsf{Girth}(\mathrm{g}_p)$ is at most of order $\log(p)$ as $p \to \infty$.

Now, let us provide a matching lower bound. To this end, observe that the girth of g_p is at least $\kappa(p)$, the smallest integer $k \geq 1$ for which there exists $M_1, \ldots, M_k \in \left\{A, A^{-1}, B, B^{-1}\right\}$ with $M_{i+1} \neq M_i^{-1}$ for all $i \in \{1, \ldots, k-1\}$ such that $M_1 \ldots M_k = I$. This is because a non-backtracking cycle of length k in g_p corresponds to an irreducible word of length k over A, A^{-1}, B, B^{-1} which is trivial. Therefore, it suffices to prove that $\kappa(p)$ is at least of order $\log(p)$ as $p \to \infty$. Let us denote by $\pi_p : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ the group homomorphism which reduces coefficients modulo p. We will denote by $\mathbf{A}, \mathbf{A}^{-1}, \mathbf{B}, \mathbf{B}^{-1}$ the same matrices as A, A^{-1}, B, B^{-1} but in $\mathrm{SL}_2(\mathbb{Z})$. To make our lives easier, we will take for granted that there is no non-trivial irreducible word over $\mathbf{A}, \mathbf{A}^{-1}, \mathbf{B}, \mathbf{B}^{-1}$. This means that for any $\mathbf{M}_1, \ldots, \mathbf{M}_k \in \left\{\mathbf{A}, \mathbf{A}^{-1}, \mathbf{B}, \mathbf{B}^{-1}\right\}$ with $\mathbf{M}_{i+1} \neq \mathbf{M}_i^{-1}$ for all $i \in \{1, \ldots, k-1\}$, we have $\mathbf{M}_1 \ldots \mathbf{M}_k \neq I$ (that is, the group generated by $\mathbf{A}$ and $\mathbf{B}$ in $\mathrm{SL}_2(\mathbb{Z})$ is a free group). This is a consequence of the so-called Ping-Pong lemma, we refer the interested reader to the associated wikipedia page, or directly to [56] and references therein. Now, by the definition of $\kappa(p)$, there exists $M_1, \ldots, M_{\kappa(p)} \in \left\{A, A^{-1}, B, B^{-1}\right\}$ with $M_{i+1} \neq M_i^{-1}$ for all $i \in \{1, \ldots, \kappa(p)-1\}$ such that $M_1 \ldots M_{\kappa(p)} = I$. For each $i \in \{1, \ldots, \kappa(p)\}$, let $\mathbf{M}_i$ be the unique element of $\left\{\mathbf{A}, \mathbf{A}^{-1}, \mathbf{B}, \mathbf{B}^{-1}\right\}$ such that $\pi_p(\mathbf{M}_i) = M_i$ (we use here that $p \geq 3$). We get that $\mathbf{M}_{i+1} \neq \mathbf{M}_i^{-1}$ for all $i \in \{1, \ldots, \kappa(p)-1\}$, hence $\mathbf{M}_1 \ldots \mathbf{M}_{\kappa(p)} \neq I$. Since $\pi_p(\mathbf{M}_1 \ldots \mathbf{M}_{\kappa(p)}) = M_1 \ldots M_{\kappa(p)} = I$, we deduce that one of the coefficients of $\mathbf{M}_1 \ldots \mathbf{M}_{\kappa(p)}$ is a non-zero multiple of p. In particular, we obtain $\|\mathbf{M}_1 \ldots \mathbf{M}_{\kappa(p)}\| \geq p$, where $\|\cdot\|$ is the Frobenius norm:

$$\left\|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right\| = \left(a^2 + b^2 + c^2 + d^2\right)^{1/2} \geq \max(|a|, |b|, |c|, |d|).$$

Since $\|\cdot\|$ is submultiplicative (which can be checked using Cauchy–Schwarz), we have on the other hand $\|\mathbf{M}_1 \ldots \mathbf{M}_{\kappa(p)}\| \leq C^{\kappa(p)}$, where $C = \max\left(\|A\|, \left\|A^{-1}\right\|, \|B\|, \left\|B^{-1}\right\|\right) > 1$, and we conclude that $\kappa(p) \geq \log_C(p)$, as desired.

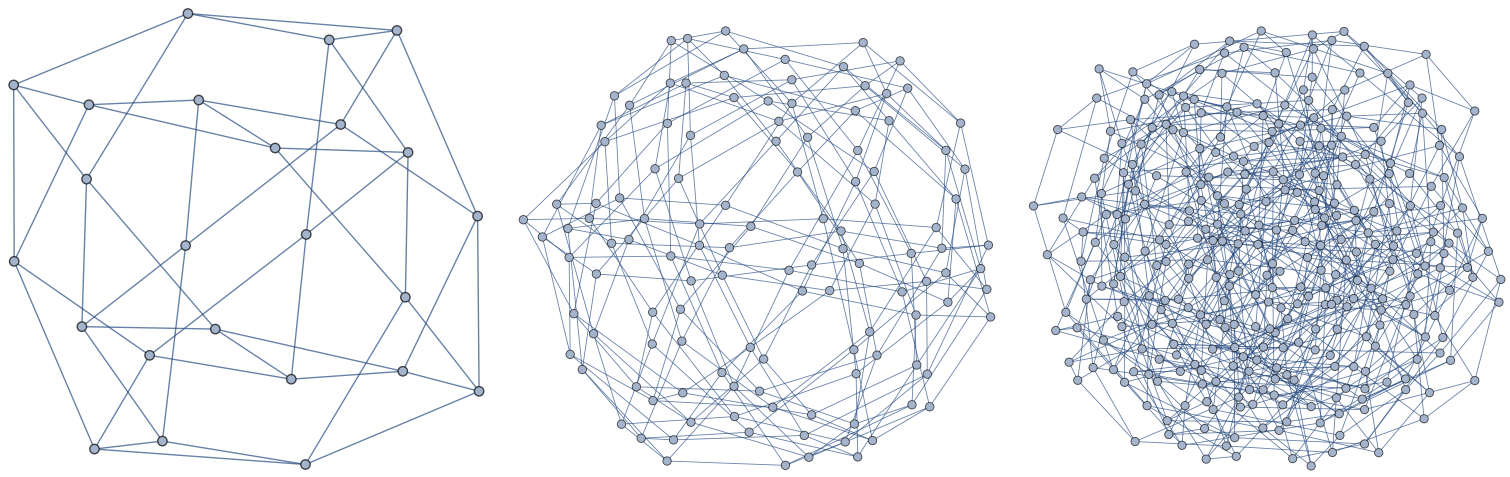


Figure 2.2: The group generated by $A^{\pm 1}, B^{\pm 1}$ in $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}$ for $p = 3, 5$ and 7.

Finally, now that we know that $\mathsf{Girth}(\mathsf{g}_p) \to \infty$ as $p \to \infty$, the local convergence of g_p to $\mathbb{T}_4$ is immediate: for each $r \geq 0$, we have $2r < \mathsf{Girth}(\mathsf{g}_p)$ for all sufficiently large p, and we have seen in the proof of the upper bound on $\mathsf{Girth}(\mathsf{g}_p)$ that as soon as this holds the restrictions $[(\mathsf{g}_p, \rho)]_r$ and $[(\mathbb{T}_4, \rho)]_r$ are isomorphic. □

Remark 1. The above theorem construct large 4-regular graphs of size $n \leq p^4$ with girth at least c $\log p$. A trivial upper bound for the maximal girth of a d-regular graph of size n is $(2 + o(1)) \log_{d-1}(n)$ and it is still open as of today to see whether this trivial upper bound is asymptotically optimal.

2.2 Benjamini–Schramm limits

Let us now turn attention to convergence of *random* pointed graphs. A first and actually already non-trivial way to get a random pointed graph is to start from a deterministic finite graph g_n and sample its pointed vertex $\rho_n \in \mathrm{V}(\mathsf{g}_n)$ uniformly at random. This procedure is sometimes called taking its Benjamini–Schramm version in reference to [20].

2.2.1 Randomization of deterministic graphs

Definition 5. *If* g *is a finite graph, its uniformly pointed version is the random pointed graph* $\mathrm{U}^\bullet(\mathsf{g})$ *whose law is* $\frac{1}{\#\mathrm{V}(\mathsf{g})} \sum_{x \in \mathrm{V}(\mathsf{g})} \delta_{(\mathsf{g},x)}$ *or equivalently, for any measurable function* $F : \mathcal{G}^\bullet \to \mathbb{R}$ *we have*

$$\mathbb{E}[F(\mathrm{U}^\bullet(\mathsf{g}))] = \frac{1}{\#\mathrm{V}(\mathsf{g})} \sum_{x \in \mathrm{V}(\mathsf{g})} F(\mathsf{g}, x).$$

Such graphs will be called **uniformly pointed** in the next chapter. Let us emphasize once, that $\mathrm{U}^\bullet(\mathsf{g})$ is a random equivalence class of pointed graph, and that although we use the labeling of the vertices of g in the definition, it is easy to check that actually $\mathrm{U}^\bullet(\mathsf{g})$

only depend on the equivalence class of $\mathbf{g}$ as an unpointed graph. These identifications will be implicit in most of the lecture notes. Let us go over the three examples of graphs of Figure 2.1 and describe the random infinite pointed graphs we obtained this way.

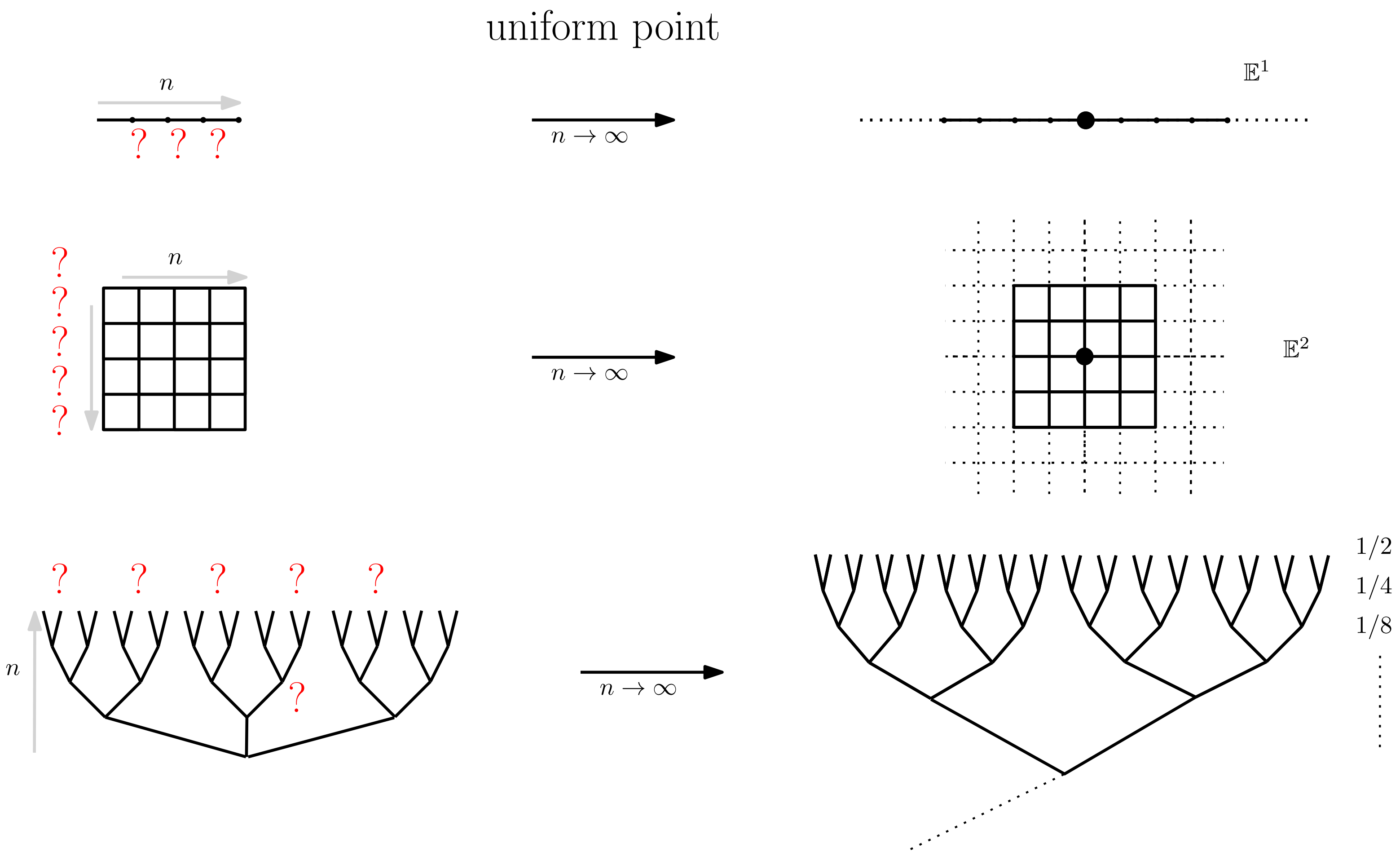


Figure 2.3: Randomization of the pointed graphs presented in Figure 2.1.

- In the first case, the obvious intuition is that if we sample the pointed vertex uniformly at random in a discrete segment with $n+1$ vertices, then it will fall "in the bulk" with large probability. More precisely, for $r \geq 0$, the pointed vertex will be at distance larger than or equal to r from the two extremities with probability
$$\frac{n+1-2r}{n+1},$$
as long as $2r \leq n+1$. Obviously, for r fixed the above probability tends to 1 as $n \to \infty$ and we deduce that $[\mathrm{U}^\bullet(\mathrm{g}_n)]_r$ is equal to ball of radius r around 0 in the bi-infinite line graph $\mathbb{E}^1$ with a probability tending to 1. This entails the convergence in probability of the random pointed graph towards the bi-infinite discrete line (the limit is thus different from the deterministic pointing at one extremity).

- The second case is similar, the probability that the pointed vertex is at distance at least r from the boundary of the grid is equal to
$$\frac{(n+1)^2 - 4n - 4(n-2)\ldots - 4(n-2(r-1))}{(n+1)^2}$$

and tends to 1 as $n \to \infty$ for fixed r. In such case $[\mathrm{U}^\bullet(\mathrm{g}_n)]_r$ is the r-neighborhood of the origin in the infinite grid. It follows that $\mathrm{U}^\bullet(\mathrm{g}_n)$ converges in probability towards the 2-dimensional grid $\mathbb{E}_2$.

- The last case is our first encounter of a random limit. When pointing the three-regular tree truncated at height n, due to exponential growth, the boundary effects are not negligible anymore. In particular, the probability that a uniform random vertex of g_n lands at height $n-h \geq 1$ is equal to

$$\frac{3 \cdot 2^{n-h-1}}{1+3(1+2+\ldots+2^{n-1})} \xrightarrow[n\to\infty]{} 2^{-h-1}.$$

In this case, (and when $n \geq 2r$), the r-neighborhood of this vertex coincides with the neighborhood of a vertex at level h in the canopy tree (level 0 being the highest level). It follows that, if we consider the random rooted graph (Can, ρ) obtained from the canopy tree by rooting it at a vertex of level h with probability 2^{-h-1}, then we have the following convergence in distribution for the local topology:

$$\mathrm{U}^\bullet(\mathrm{g}_n) \xrightarrow[n\to\infty]{(d)} (\mathrm{Can}, \rho).$$

2.2.2 Benjamini–Schramm convergences

Notice that we can consider $\mathrm{U}^\bullet(G_n)$ even if the underlying graph G_n is itself random[1]. That is, we first sample the graph G_n and then consider its uniformly pointed version $\mathrm{U}^\bullet(G_n)$, which is a random pointed finite graph. Let us apply it to this example:

In the first case, when pointing those graphs uniformly at random, the point will fall with probability roughly 1/2 in the "line" part of the graph, and with the complement probability in the "grid" part. By the above discussion, when the point falls in these parts it does so far enough from the boundary. In particular we get a random limit:

$$\mathrm{U}^\bullet(\mathrm{g}_n) \xrightarrow[n\to\infty]{(d)} \frac{1}{2}(\delta_{\mathbb{E}^1} + \delta_{\mathbb{E}^2}).$$

We obtain the same limit for the uniformly pointed version of the random graph G_n.

Definition 6 (Benjamini–Schramm convergence). *A sequence of unpointed random finite graphs $(G_n : n \geq 1)$ is said to converge in the Benjamini–Schramm sense towards $G_\infty^\bullet = (G_\infty, \rho_\infty)$ if we have the following convergence in distribution for* $\mathrm{d_{loc}}$*:*

$$\mathrm{U}^\bullet(G_n) \xrightarrow[n\to\infty]{(d)} G_\infty^\bullet.$$

[1]Although it is hard to make sense of a random infinite unpointed graph G_∞, there is no problem in the finite case.

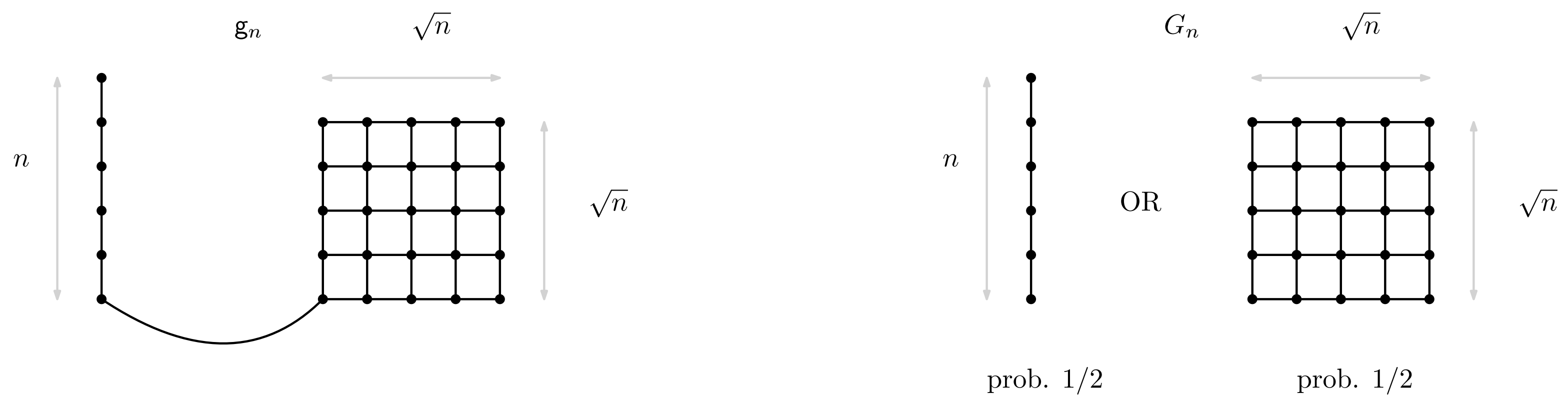


Figure 2.4: The first sequence g_n is a deterministic graph made of a discrete line and a discrete square of size roughly n. The second sequence is a random graph G_n made either of the discrete line of size n or the discrete square of size n with equal probability. Their uniformly pointed versions both converge to the same random pointed graph.

The Benjamini–Schramm limit of a sequence of random graphs enables us to capture the "average" local geometry of the random graphs G_n as $n \to \infty$. Establishing the Benjamini–Schramm convergence of some natural examples of random graphs is usually a highly non-trivial task and this will occupy us in Chapters 3, 4, 5 and 6 for various models of random graphs. Down-to-earth, a sequence of random finite graphs converges in the Benjamini–Schramm sense towards the random pointed graph $G_\infty^\bullet$ if for any bounded continuous $\phi : \mathcal{G}^\bullet \to \mathbb{R}_+$, we have

$$\mathbb{E}\left[\frac{1}{\#\mathrm{V}(G_n)} \sum_{x \in \mathrm{V}(G_n)} \phi(G_n, x)\right] \xrightarrow[n\to\infty]{} \mathbb{E}[\phi(G_\infty^\bullet)]. \tag{2.2}$$

In many examples, this convergence is actually stronger:

Definition 7 (quenched Benjamini–Schramm)**.** *A sequence $(G_n)_{n\geq 0}$ of (unpointed) random finite graphs is said to converge in the Benjamini–Schramm quenched sense towards $G_\infty^\bullet$ if any of the following equivalent conditions are satisfied:*

(i) For any bounded continuous $\phi, \psi : \mathcal{G}^\bullet \to \mathbb{R}_+$ we have

$$\mathbb{E}\left[\frac{1}{(\#\mathrm{V}(G_n))^2} \sum_{x,y \in \mathrm{V}(G_n)} \phi(G_n, x)\psi(G_n, y)\right] \xrightarrow[n\to\infty]{} \mathbb{E}[\phi(G_\infty^\bullet)] \cdot \mathbb{E}[\psi(G_\infty^\bullet)].$$

(ii) For any bounded continuous $\phi : \mathcal{G}^\bullet \to \mathbb{R}_+$ we have

$$\mathbb{E}\left[\phi(\mathrm{U}^\bullet(G_n)) \mid G_n\right] = \frac{1}{\#\mathrm{V}(G_n)} \sum_{x \in \mathrm{V}(G_n)} \phi(G_n, x) \xrightarrow[n\to\infty]{(\mathbb{P})} \mathbb{E}[\phi(G_\infty^\bullet)].$$

Because of the second item, this convergence is sometimes called "local convergence in probability", but we prefer the former terminology to avoid confusion.

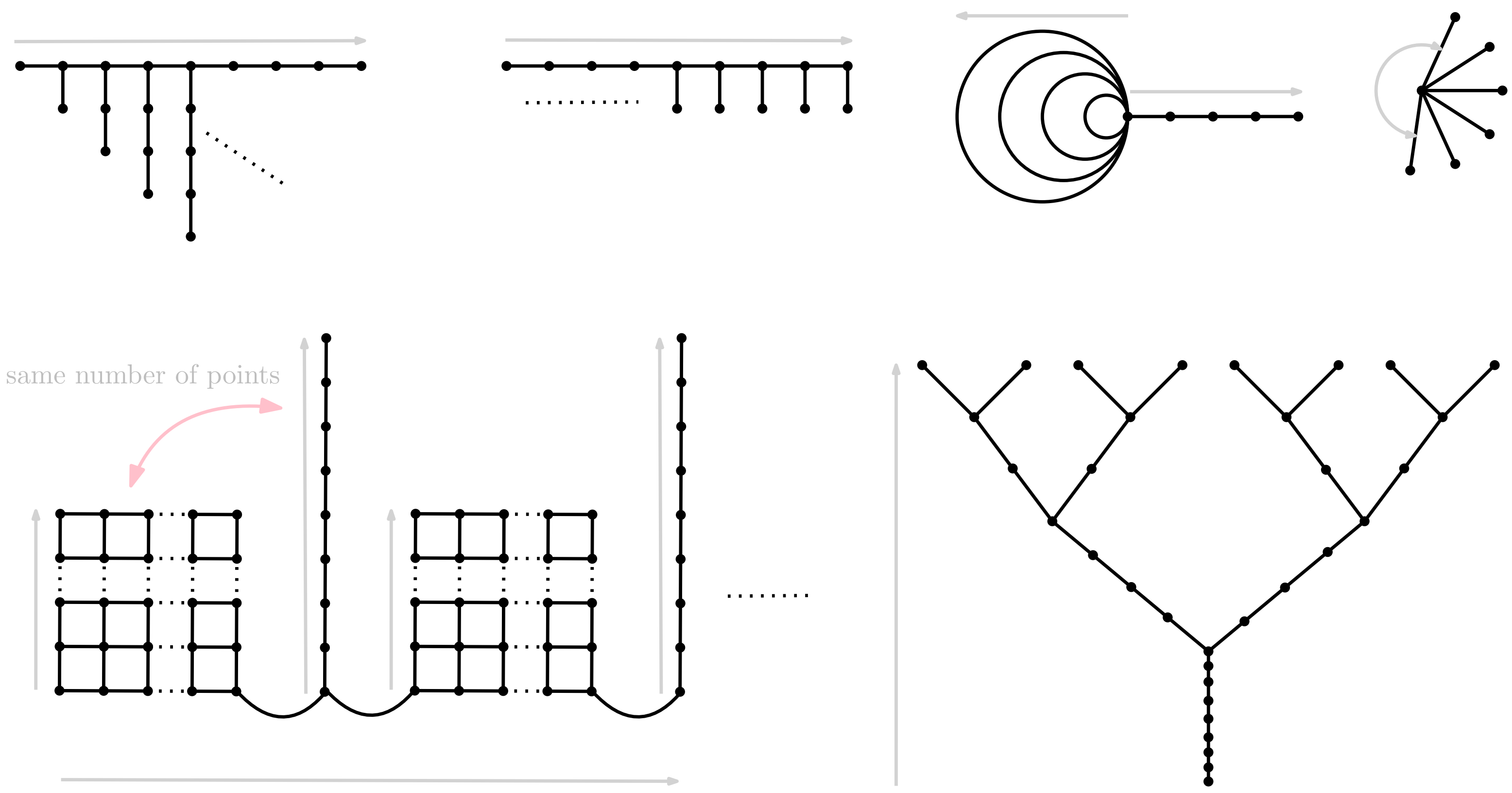
Proof of the equivalence. We introduce the shorthand notation

$$Z_\phi(n) := \frac{1}{\#\mathrm{V}(G_n)} \sum_{x \in \mathrm{V}(G_n)} \phi(G_n, x),$$

where $Z_\phi(n)$ is a random variable depending on the unpointed graph G_n only. Assume (i) and specialize it to $\psi = 1$ and $\psi = \phi$ to obtain that $\mathbb{E}[Z_\phi(n)] \to \mathbb{E}[\phi(G_\infty^\bullet)]$ and $\mathbb{E}[(Z_\phi(n))^2] \to (\mathbb{E}[\phi(G_\infty^\bullet)])^2$. The convergence in probability of the bounded variable Z_n then follows from Chebytchev's inequality (a.k.a. second moment method). Assuming (ii), then $Z_\phi(n)$ and $Z_\psi(n)$ converge in probability, so does their product. An application of the dominated convergence theorem implies (i). $\square$

Clearly a sequence of deterministic graphs $(\mathrm{g}_n)_{n\geq 0}$ which converges in the Benjamini–Schramm sense also converges in the quenched sense. The difference only pops up for random graphs $(G_n)_{n\geq 0}$. To understand the differences between the two notions, let us come back to the two examples considered in Figure 2.4: in the first case (the deterministic sequence of graphs on the left of the figure), one can check that the Benjamini–Schramm quenched convergence holds. However, it is not the case for the second case (right part of the figure) where the geometry of the graph G_n is itself random. We shall see later a stronger property, namely the fact that the limit (G_∞, ρ_∞) is ergodic.

Exercise 10. Compute, if they exist, the local limits of the uniformly pointed versions of the following graphs, where the size of the gray arrows tends to infinity. Show that quenched convergences hold (although in the fifth case, the limit is not ergodic):



Exercise 11 (From [57]). Consider $\mathbb{Z}^2$ and color each site $(p,q) \in \mathbb{Z}^2$ in black if p and q are coprime. Consider the random coloring obtained from the box $[[-n,n]]^2$ by sampling the origin uniformly at random. Show that the random coloring converges to a random coloring of $\mathbb{Z}^2$ (specify the topology).

Bibliographical notes. The concept of Benjamini–Schramm convergence of a sequence of (random) graphs has been popularized by [20] although it had precursors such as [7] in the case of random trees. The quenched Benjamini–Schramm convergence is e.g. considered in [27]. See [66, Chapter 2] for more references. Theorem 5 is due to Margulis [56].