Maxplus basis methods for high dimensional optimal control problem: introduction and perspectives

Zheng QU

CMAP and INRIA, École Polytechnique

PGMO project: Tropical methods in optimization

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Talk based on the works of several authors, including W.H.Fleming, W.McEneaney, M.Akian, S.Gaubert and Equipe Maxplus INRIA.

Key words

- Dynamic programming
- HJB equations
- Grid-based methods (Curse Of Dimensionality)
- Max-plus basis methods
  [Fleming,McEneaney 00], [Akian,Gaubert,Lakhoua06], [McEneaney,Deshpande,Gaubert 08], [Sridharan,James,McEneaney 10], [Dower,McEneaney 11]...
- A *possibly infinite* set of basis functions
- Max-plus linearity of Lax-Oleinik semigroup
- Possibly *"high"* dimensional optimal control problem (6 to 15)

Maxplus algebra

- Maxplus semiring

$$\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$$
$$a \oplus b := \max(a, b), \quad a \otimes b := a + b$$

- Maxplus semimodule ("vector space") of functions

$$X \subset \mathbb{R}^d \quad \text{(state space)}$$
$$\mathcal{F} := \{f : X \to \mathbb{R}_{max}\} \quad \text{(value function space)}$$
$$(f_1, f_2) \to f_1 \oplus f_2, \quad (\lambda, f) \to \lambda \otimes f$$

where

$$f_1 \oplus f_2(x) := \max(f_1(x), f_2(x)), \;\; x \in X$$
$$\lambda \otimes f(x) = \lambda + f(x), \;\; x \in X.$$

Maxplus projector

- Basis functions

$$\mathcal{B} = \{\mathbf{w}_i : X \to \mathbb{R}_{\max}\}_{i \in I}.$$

- Subsemimodule (subspace) generated by $\mathcal{B}$

$$\text{Span}\,\mathcal{B} := \{\bigoplus_{i \in I} \lambda_i \otimes \mathbf{w}_i : \lambda \in \mathbb{R}^I_{\max}\} .$$

- Maxplus projector on Span $\mathcal{B}$

$$\mathscr{P}_{\mathcal{B}}[f] := \sup\{\tilde{f} \in \text{Span}\,\mathcal{B} : \tilde{f} \leq f\} = \sup_{i \in I} \lambda_i + \mathbf{w}_i$$

where

$$\lambda_i = \inf_{x \in X} f(x) - \mathbf{w}_i(x).$$

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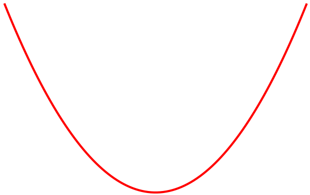
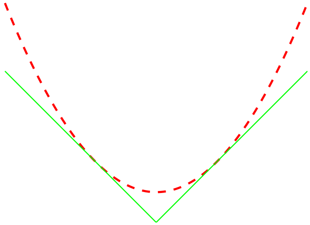
Examples of max-plus projector



$$f(x) = \frac{x^2}{4}$$
$$\mathcal{B} = \{x, -x\}$$



$$\mathscr{P}_{\mathcal{B}}[f] = \sup(x-1, -x-1)$$

Basis functions of linear forms:

$$\mathcal{B} = \{\langle q_i, x\rangle\}_{i=1,\dots,p}$$

Adapted to convex proper lsc functions.

Examples of max-plus projector

- Basis functions of quadratic form:

$$\mathcal{B} = \{-\frac{1}{2}(x - x_i)^\top C(x - x_i)\}_{i=1,\dots,p}$$

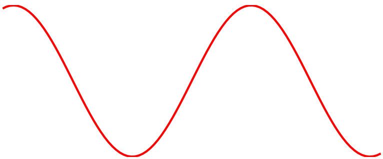
  where C is a symmetric positive-definite matrix.
- Adapted to C-semiconvex functions ($f(x) + \frac{1}{2}x^\top Cx$ is convex).

Examples of max-plus projector

$$\mathcal{B} = \{-\frac{x^2}{2}, -\frac{(x-1.8)^2}{2}, -\frac{(x-3.5)^2}{2}, -\frac{(x+4)^2}{2}\}$$

$$f(x) = \sin(x) : \quad 1-\text{semiconvex}$$

f :



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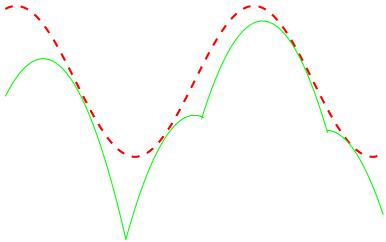
$\mathscr{P}_{\mathcal{B}}[f]$:



Max-plus algebra

- Max-plus semiring

$$\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$$
$$a \oplus b := \max(a, b), \quad a \otimes b := a + b$$

- Max-plus semimodule of functions

$$\mathcal{F} := \{f : X \to \mathbb{R}_{max}\}, \quad X \subset \mathbb{R}^d$$
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- Min-plus semimodule of functions

$$\mathcal{G} := \{g : X \to \mathbb{R}_{min}\}, \quad X \subset \mathbb{R}^d$$
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$$\mathcal{G} := \{g : X \to \mathbb{R}_{min}\}, \quad X \subset \mathbb{R}^d$$
$$(g_1, g_2) \to g_1 \wedge g_2, \quad (s, g) \to s \otimes g$$

where

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Max-plus projector

- Basis functions

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- Test functions

$$\mathcal{Z} = \{\mathbf{z}_j : X \to \mathbb{R}_{\min}\}_{j \in J}.$$

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- Min-plus projector on $\underline{\mathrm{Span}}\mathcal{Z}$

$$\mathscr{P}^{\mathcal{Z}}[g] := \inf\{\tilde{g} \in \underline{\mathrm{Span}}\mathcal{Z} : \tilde{g} \geq g\} = \inf_{j\in J} s_j + \mathbf{z}_j$$

where

$$s_j = \sup_{x\in X} g(x) - \mathbf{z}_j(x).$$

Example of min-plus projector

Lipschitz finite element test functions : [Akian,Gaubert,Lakhoua06]

$$\mathcal{Z} = \{a\|x - y_j\|_1\}_{j=1,\dots,q}.$$

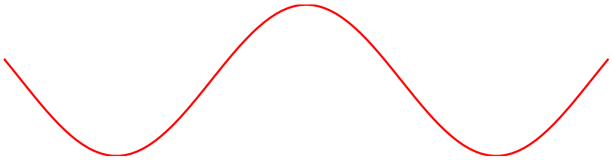
Adapted to Lipschitz continuous functions with Lipschitz constant $L \leq a$.

Example of min-plus projector

$$\mathcal{Z} = \{|x|, |x-\pi|, |x-\frac{\pi}{2}|, |x+\pi|, |x+\frac{\pi}{2}|\}$$

$$g(x) = \cos(x) : \quad 1-\text{Lipschitz}$$

g :



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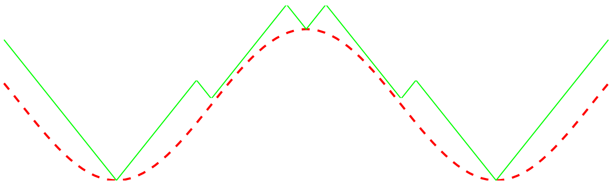
$\mathscr{P}^{\mathcal{Z}}[g]$:



Maxplus finite element projector

- Define the max-plus scalar product

$$\langle g|f\rangle := \sup_{x\in X} f(x) - g(x), \quad g\in\mathcal{G}, f\in\mathcal{F}.$$

- Two functions f and $\tilde{f}$

$$f \le \tilde{f} \Leftrightarrow \langle g|f\rangle \le \langle g|\tilde{f}\rangle, \ \forall g\in\mathcal{G}.$$

Maxplus finite element projector

- The Max-plus projector equals to

$$\mathscr{P}_{\mathcal{B}}[f] = \sup\{\tilde{f} \in \operatorname{span} \mathcal{B} : f \leq \tilde{f}\}$$
$$= \sup\{\tilde{f} \in \operatorname{span} \mathcal{B} : \langle g|\tilde{f}\ \rangle \leq \langle g|f\rangle, g \in \mathcal{G}\}.$$

- Analogous as in Petrov-Galerkin Method
  A finite set of basis functions $\mathcal{B} \subset \mathcal{F}$, a finite set of test functions $\mathcal{Z} \subset \mathcal{G}$

$$\Pi_{\mathcal{B}}^{\mathcal{Z}}[f] := \sup\{\tilde{f} \in \operatorname{Span} \mathcal{B} : \langle z|\tilde{f}\rangle \leq \langle z|f\rangle, \forall z \in \mathcal{Z}\}$$

Theorem ([Cohen,Gaubert,Quadrat 96])

$$\Pi_{\mathcal{B}}^{\mathcal{Z}} = \mathscr{P}_{\mathcal{B}} \circ \mathscr{P}^{\mathcal{Z}}.$$

Example of max-plus finite element projector

Basis functions (quadratic finite element):

$$\mathcal{B} = \{-\frac{1}{2}(x - x_i)^\top C(x - x_i)\}_{i=1,\dots,p}.$$

Test functions (Lipschtiz finite element):

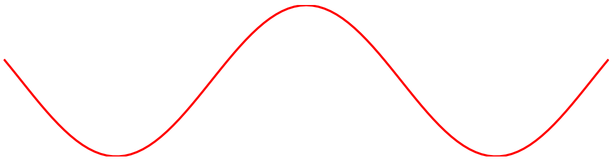
$$\mathcal{Z} = \{a\|x - y_j\|_1\}_{j=1,\dots,q}.$$

Example of max-plus finite element projector

$$\mathcal{B} = \{-|x|^2, -|x \pm 1.5|^2, -|x \pm 3|^2, -|x \pm 4|^2\}$$

$$\mathcal{Z} = \{|x|, |x \pm \pi|, |x \pm \frac{\pi}{2}|\}$$

$$f(x) = \cos(x)$$



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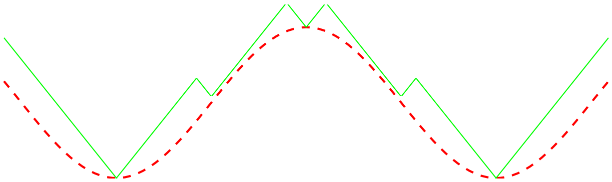
$\mathscr{P}^{\mathcal{Z}}[f]$:



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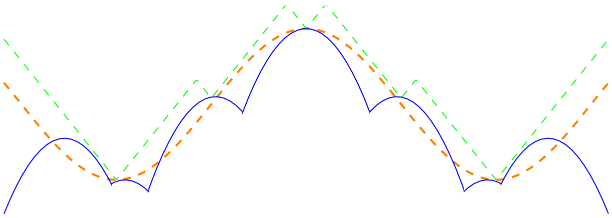
$\mathscr{P}_{\mathcal{B}} \circ \mathscr{P}^{\mathcal{Z}}[f]$:



Deterministic optimal control problem

- State space $X \subset \mathbb{R}^d$, control space U
- Value function at time T:

$$V_T(x) := \sup_{\mathbf{u} \in \mathcal{U}_T} \int_0^T \ell(\mathbf{x}(s), \mathbf{u}(s)) ds + \phi(\mathbf{x}(T))$$

with state dynamic:

$$\dot{\mathbf{x}}(s) = f(\mathbf{x}(s), \mathbf{u}(s)), \quad s \in [0, T)$$
$$\mathbf{x}(0) = x.$$

where

$$\mathcal{U}_T := \{\mathbf{u} \in L^2([0, T]; U)\}.$$

Optimal control problem/Lax-Oleinik semigroup

- Dynamic programming principle:

$$V_{t+\tau} = S^\tau[V_t], \quad \forall t \geq 0, 0 \leq \tau \leq T - t$$

where the Lax-Oleinik semigroup $(S^t)_{t\geq 0} : \mathcal{F} \to \mathcal{F}$ is defined as:

$$S^t[V_0] := V_t = \sup_{\mathbf{u}} \int_0^t \ell(\mathbf{x}(s), \mathbf{u}(s))ds + V_0(\mathbf{x}(t))$$

- Hamiltonian

$$H(x, p) := \sup_{u \in U} < p, f(x, u) > +\ell(x, u).$$

- HJB PDE

$$-\frac{\partial V_t}{\partial t} + H(x, \frac{\partial V_t}{\partial x}) = 0, \qquad V_0 = \phi$$

Optimal control problem/Lax-Oleinik semigroup

- $\forall t \geq 0,\ \lambda \in \mathbb{R}_{\max},\ \phi, \psi \in \mathcal{F}$,

$$S^t[\sup(\phi,\psi)] = \sup(S^t[\phi], S^t[\psi])$$
$$S^t[\lambda + \phi] = \lambda + S^t[\phi]$$

Optimal control problem/Lax-Oleinik semigroup

- $\forall t \geq 0,\ \lambda \in \mathbb{R}_{\max},\ \phi, \psi \in \mathcal{F}$,

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- Maxplus linearity: $\forall t \geq 0,\ \lambda \in \mathbb{R}_{\max},\ \phi, \psi \in \mathcal{F}$,

$$S^t[\phi \oplus \psi] = S^t[\phi] \oplus S^t[\psi]$$
$$S^t[\lambda \otimes \phi] = \lambda \otimes S^t[\phi]$$

Maxplus basis method: general principle

- Chose a set of adapted Basis functions

$$\mathcal{B} = \{\mathbf{w}_i\}_{i \in I}$$

- Discretize over time interval $[0, T]$

$$t = 0, \tau, 2\tau, \ldots, T.$$

- Propagation principle
  - At time t,

$$V_t \simeq \tilde{V}_t = \sup_{i \in I} \lambda_i^t + \mathbf{w}_i \in \operatorname{Span} \mathcal{B}.$$

  - At time $t + \tau$,

$$V_{t+\tau} = S^\tau[V_t] \simeq S^\tau[\tilde{V}_t] = \sup_{i \in I} \lambda_i^t + S^\tau[\mathbf{w}_i]$$

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$$V_{t+\tau} = S^\tau[V_t] \simeq S^\tau[\tilde{V}_t] = \sup_{i \in I} \lambda_i^t + S^\tau[\mathbf{w}_i]$$

- Change the problem of solving $S^T[\phi]$ to solving $S^\tau[\mathbf{w}_i]$
  - Short horizon τ, regularizing function $\mathbf{w}_i$ (ex. quadratic function)
  - Approximation of the semigroup $S^\tau[\mathbf{w}_i]$ by $\tilde{S}^\tau[\mathbf{w}_i]$.

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$$V_t \simeq \tilde{V}_t = \sup_{i \in I} \lambda_i^t + \mathbf{w}_i \in \operatorname{Span} \mathcal{B}.$$

  - At time $t+\tau$,

$$\begin{aligned} V_{t+\tau} \simeq S^\tau[\tilde{V}_t] &= \sup_{i \in I} \lambda_i^t + S^\tau[\mathbf{w}_i] \\ &\simeq \sup_{i \in I} \lambda_i^t + \tilde{S}^\tau[\mathbf{w}_i] \ \text{(approximation)} \\ &\simeq \sup_{i \in I} \lambda_i^{t+\tau} + \mathbf{w}_i \ \text{(projection)} \end{aligned}$$

Maxplus basis method [Fleming,McEneaney 00]

- Finite quadratic basis functions

$$\mathcal{B} = \{\mathbf{w}_i\}_{i=1,\dots,p}$$

where $\mathbf{w}_i = -\frac{1}{2}(x - x_i)^\top C(x - x_i)$.

- Technical assumptions

  A1. V_0 is C-semiconvex.

  A2. $S^\tau[\mathbf{w}_i]$ is C-semiconvex for $i = 1, \dots, p$.

- Propagation principle
  - At time t, $V_t \simeq \tilde{V}_t = \sup_{i \in I} \lambda_i^t + \mathbf{w}_i$.
  - At time $t + \tau$,

$$V_{t+\tau} \simeq \bigoplus_{i=1,\dots,p} \lambda_i^t \otimes S^\tau[\mathbf{w}_i] \simeq \bigoplus_{i=1,\dots,p} \lambda_i^t \otimes (\mathscr{P}_{\mathcal{B}} \circ \tilde{S}^\tau[\mathbf{w}_i])$$

Maxplus basis method [Fleming,McEneaney 00]

- Define a $p \times p$ matrix B by:

$$B_{ji} = \inf_{x \in X} \tilde{S}^{\tau}[\mathbf{w}_i](x) - \mathbf{w}_j(x)$$

so that

$$\mathscr{P}_{\mathcal{B}} \circ \tilde{S}^{\tau}[\mathbf{w}_i] = \bigoplus_{j=1,\dots,p} B_{ji} \otimes \mathbf{w}_j$$

- Recursive form

$$\lambda^{t+\tau} = B \otimes \lambda^{t}$$

Maxplus basis method [Fleming,McEneaney 00]

- Define a $p \times p$ matrix B by:

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so that

$$\mathscr{P}_{\mathcal{B}} \circ \tilde{S}^\tau[\mathbf{w}_i] = \bigoplus_{j=1,\dots,p} B_{ji} \otimes \mathbf{w}_j$$

- Recursive form (Polynomial algebraic operations)

$$\lambda^{t+\tau} = B \otimes \lambda^t$$

Maxplus finite element method [Akian,Gaubert,Lakhoua06]

- Finite quadratic basis functions

$$\mathcal{B} = \{\mathbf{w}_i(x) = -\frac{1}{2}(x - x_i)^\top C(x - x_i)\}_{i=1,\dots,p}$$

- Lipschitz finite element test functions:

$$\mathcal{Z} = \{\mathbf{z}_j(x) = a\|x - y_j\|_1\}_{j=1,\dots,q}$$

- Technical assumptions

A1. V_t is C-semiconvex and Lipschitz continuous of Lipschitz constant $L \leq a$, for all $t = 0, \tau, \dots, T$.

A2. $S^\tau[\mathbf{w}_i]$ is Lipschitz continuous of constant $L \leq a$, for all $i = 1, \dots, p$.

Maxplus finite element method [Akian,Gaubert,Lakhoua06]

- Propagation principle
  - At time t, $V_t \simeq \tilde{V}_t = \sup_{i \in I} \lambda_i^t + \mathbf{w}_i$.
  - At time $t+\tau$,

$$V_{t+\tau} \simeq \bigoplus_{i=1,\dots,p} \lambda_i^t \otimes S^\tau[\mathbf{w}_i] \simeq \Pi_{\mathcal{B}}^{\mathcal{Z}}[\bigoplus_{i=1,\dots,p} \lambda_i^t \otimes \tilde{S}^\tau[\mathbf{w}_i]]$$

- Define two $q \times p$ matrices K and M by:

$$M_{ji} = \langle \mathbf{z}_j | \mathbf{w}_i \rangle, \quad K_{ji} = \langle \mathbf{z}_j | \tilde{S}^\tau[\mathbf{w}_i] \rangle$$

- Recursive form:

$$\lambda_i^{t+\tau} = \min_{j=1,\dots,q} (-M_{ji} + \max_{k=1,\dots,p} (K_{jk} + \lambda_k^t)).$$

Maxplus finite element method [Akian,Gaubert,Lakhoua06]

- Propagation principle
  - At time t, $V_t \simeq \tilde{V}_t = \sup_{i \in I} \lambda_i^t + \mathbf{w}_i$.
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- Recursive form: (Polynomial algebraic operations)

$$\lambda_i^{t+\tau} = \min_{j=1,\dots,q} (-M_{ji} + \max_{k=1,\dots,p} (K_{jk} + \lambda_k^t)).$$

Comparison of the two methods

- [Fleming,McEneaney 00]

$$V_{t+\tau} \simeq \tilde{V}^1_{t+\tau} = \bigoplus_{i=1,\dots,p} \lambda_i^t \otimes \mathscr{P}_{\mathcal{B}} \circ \tilde{S}^\tau[\mathbf{w}_i].$$

- [Akian,Gaubert,Lakhoua06]

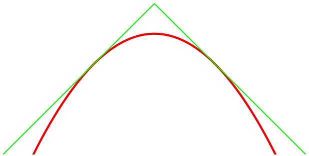
$$V_{t+\tau} \simeq \tilde{V}^2_{t+\tau} = \mathscr{P}_{\mathcal{B}} \circ \mathscr{P}^{\mathcal{Z}}[\bigoplus_{i=1,\dots,p} \lambda_i^t \otimes \tilde{S}^\tau[\mathbf{w}_i]]$$

$\Rightarrow$ When $\mathcal{Z} = \mathcal{G}$, $\tilde{V}_t \leq V_t$ and $\tilde{S}^\tau \leq S^\tau$, we have

$$V_{t+\tau} \geq \tilde{V}^2_{t+\tau} = \mathscr{P}_{\mathcal{B}} \circ \tilde{S}^\tau[\tilde{V}_t] \geq \tilde{V}^1_{t+\tau}.$$

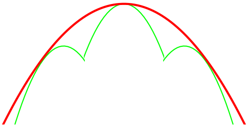
Comparison with dual dynamic programming

Dual Dynamic Programming



- Only for concave (maximisation) optimal control problems
- $\tilde{V}_t \geq V_t$
- Forward $\leftrightarrows$ Backward

Maxplus Basis Method



- Value functions need to be semiconvex (less restricted condition)
- $\tilde{V}_t \leq V_t$
- Single sense propagation

Error estimates

- [Akian,Gaubert,Lakhoua06]

$$\|V_T - \tilde{V}_T^2\|_{\infty,X} \leq (1 + \frac{T}{\tau})\Big(\max_{i=1,\dots,p} \|S^\tau[\mathbf{w}_i] - \tilde{S}^\tau[\mathbf{w}_i]\|_{\infty,X} + \max_{t=0,\tau,\dots,T} \|V_t - \mathscr{P}_{\mathcal{B}}[V_t]\|_{\infty,X} + \max_{t=0,\tau,\dots,T} \|V_t - \mathscr{P}^{\mathcal{Z}}[V_t]\|_{\infty,X}\Big)$$

Error estimates

- [Akian,Gaubert,Lakhoua06]

$$\|V_T - \tilde{V}_T^2\|_{\infty,X} \leq (1+\frac{T}{\tau})\Big(\max_{i=1,\dots,p} \|S^\tau[\mathbf{w}_i] - \tilde{S}^\tau[\mathbf{w}_i]\|_{\infty,X} + \max_{t=0,\tau,\dots,T} \|V_t - \mathscr{P}_{\mathcal{B}}[V_t]\|_{\infty,X} + \max_{t=0,\tau,\dots,T} \|V_t - \mathscr{P}^{\mathcal{Z}}[V_t]\|_{\infty,X}\Big)$$

- Their estimates also apply to [Fleming,McEneaney 00]:

$$\|V_T - \tilde{V}_T^1\|_{\infty,X} \leq (1+\frac{T}{\tau})\Big(\max_{i=1,\dots,p} \|S^\tau[\mathbf{w}_i] - \tilde{S}^\tau[\mathbf{w}_i]\|_{\infty,X} + \max_{i=1,\dots,p} \|\mathscr{P}_{\mathcal{B}} \circ \tilde{S}^\tau[\mathbf{w}_i] - \tilde{S}^\tau[\mathbf{w}_i]\|_{\infty,X}\Big)$$

Error estimates

- Approximation of the semigroup (Euler scheme)

$$S^{\tau}[\mathbf{w}_i] \simeq \tilde{S}^{\tau}[\mathbf{w}_i] = \mathbf{w}_i + \tau H(x, \nabla \mathbf{w}_i).$$

When X is bounded, under some technical assumptions,

$$\max_{i=1,\dots,p} \| S^{\tau}[\mathbf{w}_i] - \tilde{S}^{\tau}[\mathbf{w}_i] \|_{\infty,X} \sim O(\tau^2).$$

- Maxplus projection error of a C-semiconvex function f

$$\| \mathscr{P}_{\mathcal{B}}[f] - f \|_{\infty,X}$$

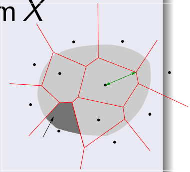
where $\mathcal{B} = \{-\frac{1}{2}(x-x_i)^{\top} C(x-x_i)\}_{i=1,\dots,p}$.

Maxplus projection error: an upper bound

Theorem ([Akian,Gaubert,Lakhoua06])

Let X be a compact convex subset of $\mathbb{R}^d$ and f be a C-semiconvex function and Lipschitz continuous of Lipschitz constant L, then

$$\|\mathscr{P}_{\mathcal{B}}[f] - f\|_{\infty,X} \leq |C|\rho(\hat{X}; x_1, \dots, x_p)\,\mathrm{diam}\, X$$



where $\hat{X} = X + B(0, \frac{L}{|C|})$ and $\rho(\hat{X}; x_1, \dots, x_p)$ is the maximal radius of the Voronoi cells of the space $\hat{X}$ divided by the points $\{x_1, \dots, x_p\}$.

Maxplus projection error: an upper bound

Theorem ([Akian,Gaubert,Lakhoua06])

Let X be a compact convex subset of $\mathbb{R}^d$ and f be a $(C-\alpha)$-semiconvex function and Lipschitz continuous of Lipschitz constant L, then

$$\|\mathscr{P}_{\mathcal{B}}[f]-f\|_{\infty,X} \leq \frac{\rho(\hat{X};x_1,\ldots,x_p)^2}{\alpha}\operatorname{diam} X$$

Maxplus projection error: an upper bound

We have

$$\|\mathscr{P}_{\mathcal{B}}[f] - f\|_{\infty,X} \leq O(\rho(\hat{X}; x_1, \dots, x_p)^2)$$

It is known (covering surface with discs [Hlawka 49, Rogers 64]) that:

$$\min_{x_1,\dots,x_p} \rho(\hat{X}; x_1, \dots, x_p) \sim O(\frac{1}{p^{\frac{1}{d}}}), \text{ as } p \to +\infty$$

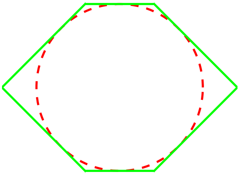
Therefore, the minimal number of basis functions $p(\epsilon)$ needed to obatin an error of order $O(\epsilon)$ is bounded by

$$p(\epsilon) \leq O(\frac{1}{\epsilon^{\frac{d}{2}}})$$

.

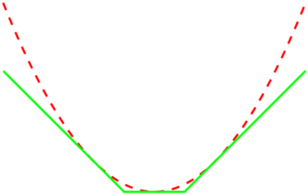
Maxplus projection error: an asymptotic estimates

Strong analogy between



Covering a convex body by a circumscribed polytope with at most p faces

Asymptotic estimates for best approximation of convex bodies by P.M.Gruber .



Projecting a convex function into a max-plus linear subspace generated by at most p linear basis functions

Asymptotic estimates for best max-plus projection

Analogous result of [Gruber 93, Gruber 07]

Theorem ([Gaubert,McEneaney,Qu 11])

Let X be a compact convex set and $f \in \mathcal{C}^2(\mathbb{R}^d : \mathbb{R})$ be a convex function such that $f''(x) > 0$, $\forall x \in \mathbb{R}^d$. Then,

$$\min_{x_1,\dots,x_p} \|\mathscr{P}_{\mathcal{B}}[f] - f\|_{\infty,X} \sim O(\frac{1}{p^{\frac{2}{d}}}), \qquad \text{as } p \to \infty$$

$$\min_{x_1,\dots,x_p} \|\mathscr{P}_{\mathcal{B}}[f] - f\|_{1,X} \sim O(\frac{1}{p^{\frac{2}{d}}}), \qquad \text{as } p \to \infty$$

A negative result

Corollary

Minimal number of linear forms $p(\epsilon)$ to reach an approximation of order $O(\epsilon)$ is of order:

$$p(\epsilon) \sim O(\frac{1}{\epsilon^{\frac{d}{2}}})$$

A negative result for the max-plus basis method

A negative result

Corollary

Minimal number of linear forms $p(\epsilon)$ to reach an approximation of order $O(\epsilon)$ is of order:

$$p(\epsilon) \sim O(\frac{1}{\epsilon^{\frac{d}{2}}})$$

A negative result for the max-plus basis method and for the dual dynamic programming method.

Asymptotic estimates for best max-plus projection

Analogous result of [Gruber 93, Gruber 07]

Theorem ([Gaubert,McEneaney,Qu 11])

Under the same assumptions, as $p \to \infty$,

$$\min_{x_1,\dots,x_p} \|\mathscr{P}_{\mathcal{B}}[f] - f\|_{\infty,X} \sim \frac{C_1}{p^{\frac{2}{d}}}, \min_{x_1,\dots,x_p} \|\mathscr{P}_{\mathcal{B}}[f] - f\|_{1,X} \sim \frac{C_2}{p^{\frac{2}{d}}},$$

where

$$C_1 = \alpha_1 \Big(\int_X (\det(f''(x)))^{\frac{1}{d+2}} \, dx \Big)^{\frac{d+2}{d}}$$

$$C_2 = \alpha_2 \Big(\int_X (\det(f''(x)))^{\frac{1}{2}} dx \Big)^{\frac{2}{d}}$$

Maxplus distributive property

- Finite distributive property

$$(\bigoplus_{i=1,\dots,p} a_i) \otimes (\bigoplus_{j=1,\dots,q} b_j) = \bigoplus_{i,j} a_i \otimes b_j$$

- Infinite distributive property
Let $I = \{1, \dots, p\}$ and $(W, \mathcal{B}(W), \mathbb{P})$ be a probability space. Let $h : W \times I \to \mathbb{R}$ be a measurable function. Under some technical assumptions, it is known that

Theorem (see [McEneaney 09])

$$\int_W \sup_{i \in I} h(w, i) d\mathbb{P} = \sup_{\tilde{i} \in \mathcal{I}} \int_W h(w, \tilde{i}(w)) d\mathbb{P}$$

where $\mathcal{I} = \{\tilde{i} : W \to I \text{ measurable}\}$ *is the strategies.*

Stochastic control problem

- Problem statement

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space and B a $\mathcal{F}_t$-Brownian motion. Let $\mathcal{U}_t$ denote the set of $\mathcal{F}_t$-progressively measurable controls, taking values in U. Consider the following stochastic control problem:

$$V_T(x) := \sup_{u \in \mathcal{U}_T} \mathbb{E}[\int_0^T \ell(\xi, u) d\epsilon + \Phi(\xi(\tau))]$$

where the dynamic is given by

$$d\xi = f(\xi, u)ds + \sigma(\xi, u)dB_s, \quad s \in [0, \tau]$$
$$\xi(0) = x$$

Idempotent algorithm for stochastic control

- Dynamic programming principle

$$V_{t+\tau} = S^\tau[V_t]$$

where the semigroup $(S^t)_{t\geq 0}$ is defined by:

$$S^t[\phi] := \sup_{u\in\mathcal{U}_t} \mathbb{E}[\int_0^t \ell(\xi,u)d\epsilon + \phi(\xi(t))]$$

- Propagation steps [Kaise,McEneaney 10]
  - At time t, $V_t \simeq \sup_{i=1,\dots,p} \phi_i$
  - At time $t+\tau$,

$$V_{t+\tau} \simeq S^\tau[\sup_i \phi_i]$$

$$\simeq \sup_{u\in U} \tau\ell(x,u) + \mathbb{E}[\sup_i \phi_i(x+\tau f(x,u)+\sigma(x,u)\omega)]$$

Idempotent algorithm for stochastic control

- Propagation steps [Kaise,McEneaney 10]
  - At time t, $V_t \simeq \sup_{i=1,\dots,p} \phi_i$
  - At time $t+\tau$,

$$V_{t+\tau} \simeq \sup_{u\in U} \tau\ell(x,u) + \mathbb{E}[\sup_i \phi_i(x+\tau f(x,u)+\sigma(x,u)\omega)]$$
$$= \sup_{u\in U} \sup_{\tilde{i}\in\mathcal{I}} \int_{\mathbb{R}^m} \tau\ell(x,u) + \phi_{\tilde{i}}(x+f(x,u)\tau+\sigma(x,u)\omega)d\mathbb{P}_\tau$$

where $\mathcal{I} = \{\tilde{i} : \mathbb{R}^m \to \{1,\dots,p\}\}$ and $\mathbb{P}_\tau$ is the distribution of a Gaussian r.v. with mean 0 and covariance τI.

Main technical issue: pruning an infinite number of functions.

Main technical issue

Let I be an infinite set. The main technical issue is to approximate the function

$$\sup_{i \in I} \phi_i$$

by

$$\sup(\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_k})$$

Main technical issue

Let I be a finite set. The main technical issue is to approximate the function

$$\sup_{i \in I} \phi_i$$

by

$$\sup(\phi_{i_1}, \phi_{i_2}, \ldots, \phi_{i_k})$$

Infinite horizon switched problem [McEneaney 07]

- Infinite horizon switched optimal control problem:

$$V(x) = \sup_u \sup_\mu \sup_{T>0} \int_0^T \frac{1}{2} x(t)^\top D^{\mu(t)} x(t) + (\ell_1^{\mu(t)})^T x(t) + \alpha^{\mu(t)} - \frac{\gamma^2}{2} |u(t)|^2 dt,$$

where the state dynamics are given by

$$\dot{x}(t) = A^{\mu(t)} x(t) + \ell_2^{\mu(t)} + \sigma^{\mu(t)} u(t), x_0 = x,$$

- Arising from nonlinear robust H_∞ control, nonconvex problem.

McEneaney's COD free problem [McEneaney 07]

- The semigroup associated to the switched problem:

$$S^t[\phi](x) = \sup_u \sup_\mu \int_0^t \frac{1}{2} x(t)^\top D^{\mu(t)} x(t) + (\ell_1^{\mu(t)})^T x(t) + \alpha^{\mu(t)} - \frac{\gamma^2}{2} |u(t)|^2 dt + \phi(x(t)).$$

- The Hamiltonian: $H = \sup_{m \in 1,\dots,M} H^m(x,p)$ where

$$H^m(x,p) = \frac{1}{2} x^\top D^m x + \frac{1}{2} p^\top \Sigma^m p + (A^m x)^\top p + (\ell_1^m)^\top x + (\ell_2^m)^\top p + \alpha^m.$$

McEneaney's COD free problem [McEneaney 07]

- Under some technical assumptions (on the finiteness of the value function of the infinite horizon problem), it is shown that :

$$V(x) = \lim_{T \to +\infty} S^T[0]$$

is the unique viscosity solution of

$$H(x, \nabla V) = \max_{m=1,\dots,M} H^m(x, \nabla V) = 0$$

in a class of bounded functions.

- Finite horizon approximation

$$V(x) \simeq V_T(x) = S^T[0](x).$$

Max-plus basis method [McEneaney 07]

- Finite horizon approximation

$$V(x) \simeq V_T(x) = S^T[0](x).$$

- Maxplus propagation
  - Discretize the time interval $[0, T]$ into $\{0, \tau, 2\tau, \dots, N\tau\}$.
  - At time t,

    $$V_t \simeq \tilde{V}_t = \sup_{i=1,\dots,q_t} \phi_i^t$$

    where $\{\phi_i^t\}_i$ are quadratic affine functions.
  - At time $t + \tau$,

    $$V_{t+\tau} \simeq S^\tau[\tilde{V}_t] = \sup_{i=1,\dots,q_t} S^\tau[\phi_i^t].$$

Approximation of the semigroup [McEneaney 07]

- For each $m = 1, \ldots, M$, define the semigroup associated to H^m

$$S_m^t[\phi](x) = \sup_u \int_0^t \frac{1}{2} x(t)^\top D^m x(t) + (\ell_1^m)^T x(t) + \alpha^m - \frac{\gamma^2}{2} |u(t)|^2 dt + \phi(x(t)).$$

where the dynamics are given by

$$\dot{x}(t) = A^m x(t) + \ell_2^m + \sigma^m u(t), x_0 = x,$$

$S_m^t[\phi]$ is a quadratic affine function if ϕ is. (Riccati)

- Approximation of S^τ

$$S^\tau \simeq \sup_{m=1,\ldots,M} S_m^\tau$$

Maxplus propagation [McEneaney 07]

- Propagation principle
  - At time t,

$$V_t \simeq \tilde{V}_t = \sup_{i=1,\dots,q_t} \phi_i^t$$

  - At time $t+\tau$,

$$V_{t+\tau} \simeq S^\tau[\tilde{V}_t] = \sup_{i=1,\dots,q_t} S^\tau[\phi_i^t]$$

$$\simeq \sup_{i=1,\dots,q_t} \sup_{m=1,\dots,M} S_m^\tau[\phi_i^t]$$

Maxplus propagation [McEneaney 07]

- Propagation principle
  - At time t,

$$V_t \simeq \tilde{V}_t = \sup_{i=1,\dots,q_t} \phi_i^t$$

  - At time $t+\tau$,

$$V_{t+\tau} \simeq S^\tau[\tilde{V}_t] = \sup_{i=1,\dots,q_t} S^\tau[\phi_i^t]$$

$$\simeq \sup_{i=1,\dots,q_t} \sup_{m=1,\dots,M} \underbrace{S_m^\tau[\phi_i^t]}_{\text{Riccati}}$$

Curse of complexity

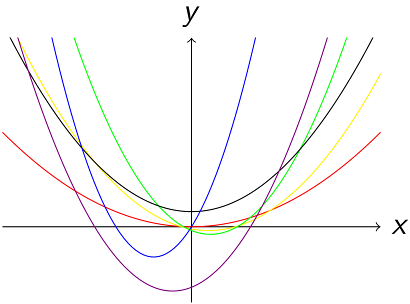
Theorem ([QU 12])

Under certains technical assumptions, the number of algebraic operations of the COD free max-plus method to obtain an error ϵ is of order:

$$O(|M|^{O(-\log(\epsilon)/\epsilon)}d^3), \quad \text{as} \quad \epsilon \to 0$$

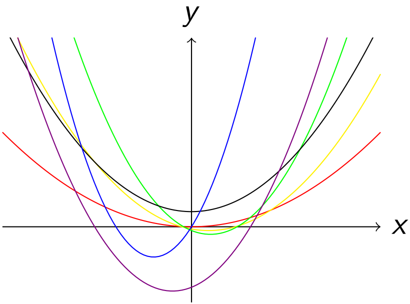
- The number of quadratic functions is multiplied by M at each steps. At the end of N steps, the number of basis functions is M^N.
- Such *curse of complexity* can be reduced by carring on pruning operations.

Pruning operation:

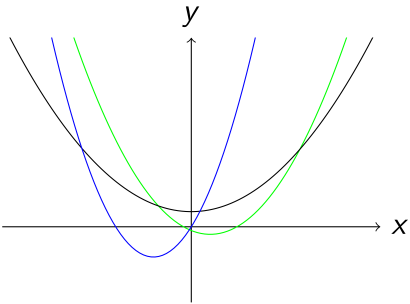


$$\phi = \sup(\phi_{green}, \phi_{red}, \phi_{violet}, \phi_{yellow}, \phi_{black}, \phi_{blue})$$

Pruning operation:



$$\phi = \sup(\phi_{green}, \phi_{red}, \phi_{violet}, \phi_{yellow}, \phi_{black}, \phi_{blue})$$



$$\phi = \sup(\phi_{green}, \phi_{black}, \phi_{blue})$$

Pruning algorithms

Let $Q_1, \dots, Q_n$ be $(d+1) \times (d+1)$ symmetric matrices such that the quadratic functions are given by

$$\phi_i(x) = (x^T \;\; 1) Q_i \begin{pmatrix} x \\ 1 \end{pmatrix}, \quad i = 1, \dots, n.$$

1 Pairwise pruning

$$\phi_i \geq \phi_j \Leftrightarrow Q_i \geq Q_j \Rightarrow \text{ Remove the function } \phi_j$$

2 Global pruning

$$\sup_{i \neq j} \phi_i \geq \phi_j \Rightarrow \text{ Remove the function } \phi_j$$

Pruning algorithms

- Importance metric

$$\sup_{i\neq j}\phi_i \geq \phi_j \Leftrightarrow \nu_j := \sup_x \frac{\phi_j(x) - \sup_{i\neq j}\phi_i(x)}{1+|x|^2} \leq 0$$

ν_j is called the *importance metric* of the function ϕ_j.

- Optimisation form

$$\begin{aligned} \nu_j = \max \quad & \nu \\ & \nu \leq z^\top (Q_j - Q_i) z, \quad \forall i \neq j \\ & z^T z = 1. \end{aligned}$$

Pruning algorithms

Optimisation problem

$$\nu_j = \max \ \nu$$
$$\nu \leq z^\top (Q_j - Q_i) z$$
$$z^T z = 1.$$

SDP relaxation

$$\overline{\nu}_j = \max \ \nu$$
$$\nu \leq \mathrm{trace}((Q_j - Q_i) Z)$$
$$Z \geq 0, \ \ \mathrm{trace}(Z) = 1.$$

- Conservative pruning[McEneaney,Deshpande,Gaubert 08]:

$$\overline{\nu}_j \leq 0 \Rightarrow \nu_j \leq 0 \Rightarrow: \ \text{Remove the function } \phi_j.$$

- Over-pruning[McEneaney,Deshpande,Gaubert 08]:

sort $\overline{\nu}_j$, keep at most k functions and prune the rest.

Pruning algorithms: a combinatorial modelisation

- Discrete points $\tilde{X} = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$. The *lost* at point x_k if we remove the function ϕ_j is:

$$c(j,k) := \sup_i \phi_i(x_k) - \sup_{i \neq j} \phi_i(x_k) \geq 0$$

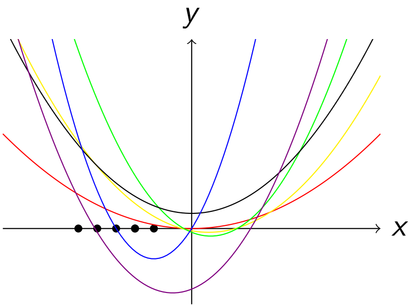
- Combinatorial optimisation problem
  - Minimizing the average lost $\rightarrow$ discrete k-median problem:

$$\min_{|S|=k} \sum_{k=1}^{N} [\min_{j \in S} c(j,k)] \ .$$

  - Minimizing the maximal lost $\rightarrow$ discrete k-center problem:

$$\min_{|S|=k} \max_{k=1,\ldots,N} [\min_{j \in S} c(j,k)] \ .$$

Pruning algorithms: distribution of witness points



Pruning algorithms: generation of witness points

[Gaubert,McEneaney,Qu 11]

Optimisation problem

$$\nu_j = \max \ \nu \quad \nu \leq z^\top (Q_j - Q_i) z \quad z^T z = 1.$$

SDP relaxation

$$\overline{\nu}_j = \max \ \nu \quad \nu \leq \mathrm{trace}((Q_j - Q_i)Z) \quad Z \geq 0, \ \ \mathrm{trace}(Z) = 1.$$

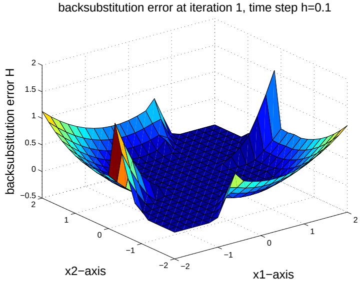
Randomization technique [Aspremont,Boyd 03]: For each function j, let Z_j be the optimal solution of the SDP, generate random points of distribution $\mathcal{N}(0, Z_j)$.

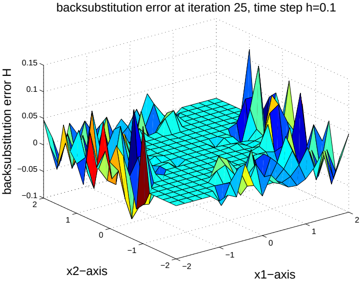
Experimental results

The method has been applied to a quantum optimal control of dimension 15 where the state space is the unitary group $SU(4)$, see [Sridharan,James,McEneaney 10].

Experimental results

Instance : $d = 6$, $M = 6$ Backsubstitution error at point x : $H(x, \nabla V(x))$.

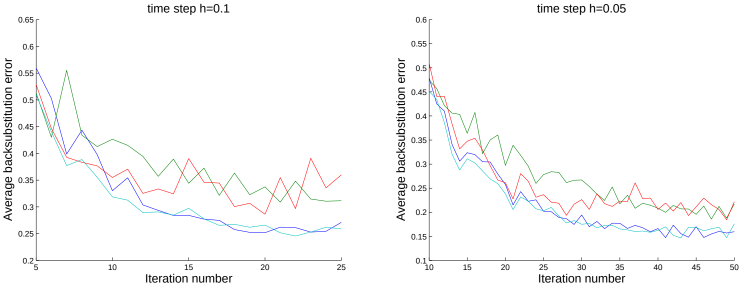




Experimental results

$M = 6$, $d = 6$

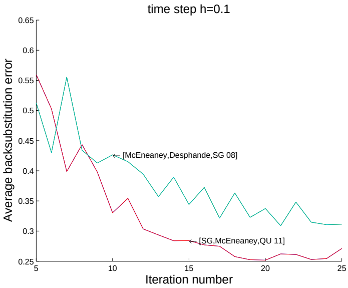
Average backsubstitution error evolution obtained using 4 different pruning algorithms

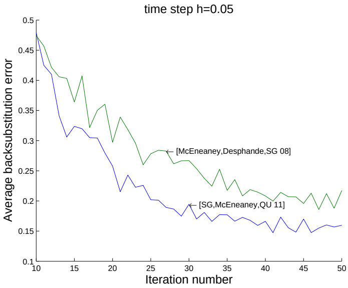


Experimental results

$M = 6$, $d = 6$

Average backsubstitution error evolution





Conclusions and perspectives

- New class of numerical method
  - Analogous max-plus approach of finite element method
    - Curse of dimensionality is inevitable
  - Structured problem
    - Curse of dimensionality converted to curse of complexity
    - Pruning is essentiel and it works
- Current works
  - Apply the COD-free approach to more general Hamiltonian

$$H \simeq \sup_m H^m$$

  - Second order HJB PDE

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