Mean-Field Games

Second lecture: Potential case, Common Noise, Master Equation

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Based on joint works with R. Carmona, P. Cardaliaguet, A. Cecchin, D. Crisan, J.F. Chassagneux, R. Foguen, D. Lacker, J.M. Lasry, P.L. Lions, K. Ramaman

Part IV. Potential Games

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a. More on Pontryagin principle

Pontryagin in $\mathbb{R}^d$

- Go back to MFG but $\sigma \equiv 0$
  - stochastic optimal control problem in the environment $(\mu_t)_{0 \le t \le T}$

$$dX_t = b(X_t, \mu_t, \alpha_t)dt$$

  - cost functional (randomness inside the initial condition)

$$J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t)dt\Big]$$

- First order condition of optimality

$$X_t = X_0 + \int_0^t b\Big(X_s, \mu_s, \alpha^\star(X_s, \mu_s, Y_s)\Big)\,ds$$

$$Y_t = \partial_x g(X_T, \mu_T) + \int_t^T \partial_x H\Big(X_s, \mu_s, \alpha^\star(X_s, \mu_s, Y_s), Y_s\Big)\,ds$$

- $H(x, \mu, \alpha, z) = b(x, \mu, \alpha) \cdot z + f(x, \mu, \alpha)$
  - $\alpha^\star(x, \mu, z) = \text{argmin}_{\alpha \in A} H(x, \mu, \alpha, z)$

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$$Y_t = \partial_x g(X_T, \mu_T) + \int_t^T \partial_x H\Big(X_s, \mu_s, \alpha^\star(X_s, \mu_s, Y_s), Y_s\Big)\, ds$$

- Unique minimizer and sufficient condition for each $(\mu_t)_{0 \le t \le T}$ if
  - $b(x, \mu, \alpha) = b_0(\mu) + b_1 x + b_2 \alpha$ and $\partial_x f$, $\partial_\alpha f$, $\partial_x g$ Lip. in (x, α)
  - g and f convex in (x, α) with f strict convex in α ( )

Pontryagin in $\mathbb{R}^d$

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  - stochastic optimal control problem in the environment $(\mu_t)_{0 \le t \le T}$

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- First order condition of optimality

$$X_t = X_0 + \int_0^t b\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s)\Big)\, ds$$

$$Y_t = \partial_x g(X_T, \mathcal{L}(X_T)) + \int_t^T \partial_x H\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s), Y_s\Big)\, ds$$

- obtain MFG by replacing μ_s by $\mathcal{L}(X_s)$
  - similar principle when $\sigma \not\equiv 0$ using backward SDEs (●)

Linear-quadratic in $d = 1$, σ constant

- Take
  - $b(t,x,\mu,\alpha) = a_t x + a'_t \mathbb{E}(\mu) + b_t \alpha_t$
  - $g(x,\mu) = \frac{1}{2}[qx + q'\mathbb{E}(\mu)]^2$
  - $f(t,x,\mu,\alpha) = \frac{1}{2}[\alpha^2 + (m_t x + m'_t \mathbb{E}(\mu))^2]$
- Pontryagin

$$dX_t = [a_t X_t + a'_t \mathbb{E}(X_t) - b_t^2 Y_t]dt + \sigma dW_t$$
$$dY_t = -[a_t Y_t + m_t(m_t X_t + m'_t \mathbb{E}(X_t))]dt + Z_t dW_t$$
$$Y_T = q[qX_T + q'\mathbb{E}(X_T)]$$

  - take the mean

$$d\mathbb{E}(X_t) = [(a_t + a'_t)\mathbb{E}(X_t) - b_t^2 \mathbb{E}(Y_t)]dt$$
$$d\mathbb{E}(Y_t) = -[a_t \mathbb{E}(Y_t) + m_t(m_t + m'_t)\mathbb{E}(X_t)]dt$$
$$\mathbb{E}(Y_T) = q(q + q')\mathbb{E}(X_T)$$

- existence and uniqueness if $q(q + q') \geq 0$, $m_t(m_t + m'_t) \geq 0$ ( )

Part IV. Potential Games

b. MFG as a first order condition

Optimization problem over the whole population

- Same dynamics as before! rewrite the dynamics of the particles

$$dX_t^i = b(X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt + \sigma dW_t^i$$

- Same cost functional! to player $i \in \{1, \dots, N\}$

$$J^i(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2, \dots, \boldsymbol{\alpha}^N) = \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^N) + \int_0^T f(X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt\Big]$$

- Reduce to Markov feedback policies $\alpha_t^i = \alpha^i(t, X_t^1, \dots, X_t^N)$
- Central planner! $\Rightarrow$ Forces all the players to use the same $\alpha^i = \alpha$!

  ◦ exchangeability (symmetry in law) $\Rightarrow J^1 = \cdots = J^N$ is the cost to the society

  ◦ minimize any J^i with respect to α!

Asymptotic Social Optimization

- Recall the finite problem

$$dX_t^i = b(X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt + \sigma dW_t^i$$

  - with Markov feedback policies $\alpha_t^i = \alpha^i(t, X_t^1, \ldots, X_t^N)$
  - minimize

$$J(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2, \ldots, \boldsymbol{\alpha}^N) = \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^N) + \int_0^T f(X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt\Big]$$

- Asymptotic problem should be to minimize

$$J(\boldsymbol{\alpha}) = \mathbb{E}\Big[g(X_T, \mathcal{L}(X_T)) + \int_0^T f(X_t, \mathcal{L}(X_t), \alpha_t)dt\Big]$$

over $dX_t = b(X_t, \mathcal{L}(X_t), \alpha_t)dt + \sigma dW_t$

  - or, written for Fokker-Planck equations

$$J(\boldsymbol{\alpha}) = \int_{\mathbb{R}^d} g(x, \mu_T)d\mu_T(x) + \int_0^T \int_{\mathbb{R}^d} f(x, \mu_t, \alpha_t(x))d\mu_t(x)dt$$

over $\partial_t \mu_t = -\mathrm{div}_x(b(x, \mu_t, \alpha(t,x))\mu_t) + \frac{1}{2}\sigma^2 \Delta_x \mu_t$

Formal application of Pontryagin

- Choose $b(\alpha) = \alpha \in \mathbb{R}^d$ and

$$g(x,\mu) = \frac{1}{2} \int_{\mathbb{R}^d} G(x-y)d\mu(y)$$

$$f(x,\mu,\alpha) = \frac{1}{2} \int_{\mathbb{R}^d} F(x-y)d\mu(y) + \frac{1}{2}|\alpha|^2, \quad F \text{ and } G \text{ even}$$

- variable $\mu \in \mathcal{P}(\mathbb{R}^d) \Rightarrow$ adjoint is a continuous function u on $\mathbb{R}^d$
  - formal Hamiltonian

$$H(\mu,\alpha,u) = \int_{\mathbb{R}^d} \alpha(x)\partial_x u(x)d\mu(x) + \frac{1}{2}\int_{\mathbb{R}^d} \sigma^2 \Delta_x u(x)d\mu(x) + \frac{1}{2}\int_{\mathbb{R}^d} |\alpha(x)|^2 d\mu(x) + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)d\mu(x)d\mu(y)$$

  - optimizer $\alpha^\star(x) = -\partial_x u(x)$
  - linearizing (differential calculus???) w.r.t. μ (●)

$$\partial_t u_t(x) = -\frac{1}{2}\sigma^2 \Delta_x u_t(x) + \frac{1}{2}|\partial_x u_t(x)|^2 - \int_{\mathbb{R}^d} F(x-y)d\mu_t(x)$$

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$$\partial_t u_t(x) = -\frac{1}{2}\sigma^2 \Delta_x u_t(x) + \frac{1}{2}|\partial_x u_t(x)|^2 - \int_{\mathbb{R}^d} F(x-y)d\mu_t(x)$$

  - terminal condition $u_T(x) = \int_{\mathbb{R}^d} G(x-y)d\mu_T(x)$
  - combine with $\partial_t \mu_t = \mathrm{div}_x(\partial_x u(t,x)\mu_t) + \frac{1}{2}\sigma^2 \Delta_x \mu_t$ and

get an MFG over $dX_t = \alpha_t dt + \sigma dW_t$ with cost functional

$$J(\alpha) = \mathbb{E}\Big[\int_{\mathbb{R}^d} G(X_T - y)d\mu_T(y) + \int_0^T \Big(\frac{1}{2}|\alpha_t|^2 + \int_{\mathbb{R}^d} F(X_t - y)d\mu_t(y)\Big)dt\Big]$$

Formal application of Pontryagin

- Choose $b(\alpha) = \alpha \in \mathbb{R}^d$ and

$$g(x,\mu) = \frac{1}{2}\int_{\mathbb{R}^d} G(x-y)d\mu(y)$$

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- variable $\mu \in \mathcal{P}(\mathbb{R}^d) \Rightarrow$ adjoint is a continuous function u on $\mathbb{R}^d$

  ◦ linearizing (differential calculus???) w.r.t. μ ( )

$$\partial_t u_t(x) = -\frac{1}{2}\sigma^2 \Delta_x u_t(x) + \frac{1}{2}|\partial_x u_t(x)|^2 - \int_{\mathbb{R}^d} F(x-y)d\mu_t(x)$$

  ◦ terminal condition $u_T(x) = \int_{\mathbb{R}^d} G(x-y)d\mu_T(x)$

  ◦ MFG is a first order condition for optimal problem on space of probability measures ( ) ( )

  ◦ MFG and optimal problem on space of probabilities do not have same coefficients but share same solutions $\Rightarrow$ Mechanism design

Part V. Solving MFG with a Common Noise

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a. Formulation

MFG with a common noise

- Mean field game with common noise B
  - asymptotic formulation for a finite player game with

$$dX_t^i = b(X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt + \sigma(X_t^i, \bar{\mu}_t^N)dW_t^i + \sigma^0(X_t^i, \bar{\mu}_t^N)dB_t$$

  - uncontrolled version $\rightsquigarrow$ asymptotic SDE with $\bar{\mu}_t^N$ replaced by $\mathcal{L}(X_t|(B_s)_{0\le s\le T}) = \mathcal{L}(X_t|(B_s)_{0\le s\le t})$ (●)
  - particles become independent conditional on B and converge to the solution

$$dX_t = b(X_t, \mathcal{L}(X|B))dt + \sigma(X_t, \mathcal{L}(X|B))dW_t + \sigma^0(X_t, \mathcal{L}(X|B))dB_t$$

MFG with a common noise

- Mean field game with common noise B
  - asymptotic formulation for a finite player game with $A = \mathbb{R}^k$ and
$$dX_t^i = \Big(b(X_t^i, \bar{\mu}_t^N) + \alpha_t^i\Big)dt + \sigma dW_t^i + \eta dB_t$$
  - uncontrolled version $\rightsquigarrow$ $\bar{\mu}_t^N$ replaced by $\mathcal{L}(X_t|B)$
- Equilibrium as a fixed point $\rightsquigarrow$ time $[0, T]$, state in $\mathbb{R}^d$
  - candidate $\rightsquigarrow$ $(\mu_t)_{t\in[0,T]}$ $\mathbb{F}^B$ prog-meas with values in space of probability measures with a finite second moment $\mathcal{P}_2(\mathbb{R}^d)$
  - representative player with control α
$$dX_t = (b(X_t, \mu_t) + \alpha_t)dt + \sigma dW_t + \eta dB_t$$
$\rightsquigarrow$ $X_0 \sim \mu_0$, $\sigma, \eta \in \{0, 1\}$, W and B $\mathbb{R}^d$-valued $\perp\!\!\!\perp$ B.M.
  - cost functional $J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T \Big(f(X_t, \mu_t) + \frac{1}{2}|\alpha_t|^2\Big)dt\Big]$
  - find $(\mu_t)_{t\in[0,T]}$ such that $\mu_t = \mathcal{L}(X_t^\star | (B_s)_{0\le s\le T})$

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- Mean field game with common noise B
  - asymptotic formulation for a finite player game with

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$$dX_t = (b(X_t, \mu_t) + \alpha_t)dt + \sigma dW_t + \eta dB_t$$

$\leadsto$ $X_0 \sim \mu_0$, $\sigma, \eta \in \{0,1\}$, W and B $\mathbb{R}^d$-valued $\perp\!\!\!\perp$ B.M.

  - cost functional $J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T \big(f(X_t, \mu_t) + \frac{1}{2}|\alpha_t|^2\big)dt\Big]$
  - find $(\mu_t)_{t\in[0,T]}$ such that $\mu_t = \mathcal{L}(X_t^\star|(B_s)_{0\le s\le t})$

Forward-backward formulation

- Forward-backward formulation must account for $(\mu_t)_{0 \leq t \leq T}$ random

    ◦ systems of two forward-backward SPDEs [Carmona D, Cardaliaguet D Lasry Lions, Cardaliaguet Souganidis]

Forward-backward formulation

- Forward-backward formulation must account for $(\mu_t)_{0\le t\le T}$ random
  - systems of two forward-backward SPDEs

$\rightsquigarrow$ one backward stochastic HJB equation [Peng]

$$d_t u(t,x) + \Big(\underbrace{b(x,\mu_t)\cdot\partial_x u(t,x) + \frac{\sigma^2+\eta^2}{2}\Delta_x u(t,x)}_{Laplace\ generator} + \underbrace{f(x,\mu_t) - \frac{1}{2}|\partial_x u(t,x)|^2}_{standard\ Hamiltonian\ in\ HJB}$$

$$+ \underbrace{\eta \operatorname{div}[v(t,x)]}_{Ito\ Wentzell\ cross\ term\ (\ )} \Big) dt - \underbrace{1_{\{\eta\neq 0\}} v(t,x)\cdot dB_t}_{backward\ term} = 0$$

with boundary condition: $u(T,\cdot) = g(\cdot,\mu_T)$

$\rightsquigarrow$ one forward stochastic Fokker-Planck equation

$$d_t\mu_t = \Big(-div(\mu_t[b(x,\mu_t) - \partial_x u(t,x)])dt + \frac{\sigma^2+\eta^2}{2}\Delta_x^2 \mu_t \Big) dt$$

$$- \eta div(\mu_t dB_t)$$

Part V. MFG with Common Noise

b. Strong solutions

Continuation method

(Cardaliaguet-D.-Lasry-Lions)

- Standard method for handling nonlinear equations
- Stochastic Fokker Planck equation

$$d_t\mu_t = \{\tfrac{1}{2}(1+\eta^2)\Delta\mu_t + div(\mu_t\partial_x u(t,x))\}dt - \eta\, div(\mu_t dB_t)$$

- Stochastic HJB equation

$$\begin{aligned} d_t u(t,x) = \{&-\tfrac{1}{2}(1+\eta^2)\Delta u(t,x) + \tfrac{1}{2}|\partial_x u(t,x)|^2 \\ &- f(x,\mu_t) - \eta\, div(v(t,x))\}dt \\ &+ v(t,x)\cdot dB_t \\ u(T,x) = g(x,\mu_T)& \end{aligned}$$

Continuation method

(Cardaliaguet-D.-Lasry-Lions)

- Standard method for handling nonlinear equations
- Stochastic Fokker Planck equation

$$d_t\mu_t = \{\tfrac{1}{2}(1+\eta^2)\Delta\mu_t + div(\mu_t\partial_x u(t,x))\}dt - \eta\, div(\mu_t dB_t)$$

- Stochastic HJB equation

$$\begin{aligned} d_t u(t,x) = \{&-\tfrac{1}{2}(1+\eta^2)\Delta u(t,x) + \tfrac{1}{2}|\partial_x u(t,x)|^2 \\ &- \beta f(x,\mu_t) - \varphi_t(x) - \eta\, div(v(t,x))\}dt \\ &+ v(t,x)\cdot dB_t \\ u(T,x) = {}& \beta g(x,\mu_T) + \gamma(x) \end{aligned}$$

- Continuation method
  - increase step by step the coupling parameter β
  - $\beta = 0 \Rightarrow$ stochastic HJB is decoupled!

Decoupled case $\beta = 0$

- Conditional on $\mathcal{F}_T^B$, action of $(B_t)_t$ reduced to a transport

$$d(X_t - \eta B_t) = \alpha_t dt + dW_t$$

  - $\tilde{u}(t,x) = u(t, x + \eta B_t)$ and $\tilde{\mu}_t = \mu_t \circ (x \mapsto x - \eta B_t)^{-1}$
  - reduced Stoc. HJB / Stoc. FP system

$$d_t\tilde{\mu}_t = \{\tfrac{1}{2}\Delta\tilde{\mu}_t + div(\tilde{\mu}_t\partial_x\tilde{u}(t,x))\}dt$$
$$d_t\tilde{u}(t,x) = \{-\tfrac{1}{2}\Delta\tilde{u}(t,x) + \tfrac{1}{2}|\partial_x\tilde{u}(t,x)|^2 - \tilde{\varphi}_t(x)\}dt - \tilde{v}(t,x)\cdot dB_t$$
$$\tilde{u}(T,x) = \tilde{\gamma}(x)$$

- If $p_t(x)$ is the heat kernel $\Rightarrow$ express $\tilde{u}(t,x)$ as

$$\tilde{u}(t,x) = \mathbb{E}\Big[\int_{\mathbb{R}^d}\tilde{\gamma}(x-y)p_{T-t}(y)dy + \int_t^T\int_{\mathbb{R}^d}(\tilde{\varphi}(s,\cdot) - \tfrac{1}{2}|\partial_x\tilde{u}(s,\cdot)|^2)(x-y)p_{s-t}(y)dy \,|\, \mathcal{F}_t^B\Big].$$

Small coupling $\beta \ll 1$

- Picard fixed point theorem for solving the system when $\beta \ll 1$

$$d_t\tilde{\mu}_t = \{\tfrac{1}{2}\Delta\tilde{\mu}_t + div(\tilde{\mu}_t\partial_x\tilde{u}(t,x))\}dt$$
$$d_t\tilde{u}(t,x) = \{-\tfrac{1}{2}\Delta\tilde{u}(t,x) + \tfrac{1}{2}|\partial_x\tilde{u}(t,x)|^2 - \beta\tilde{f}(x,\tilde{\mu}_t)\}dt - \tilde{v}(t,x)\cdot dB_t$$
$$\tilde{u}(T,x) = \beta\tilde{g}(x,\tilde{\mu}_T)$$

- Contraction with

$$d_t\tilde{\mu}_t = \{\tfrac{1}{2}\Delta\tilde{\mu}_t + div(\tilde{\mu}_t\partial_x\tilde{u}(t,x))\}dt$$
$$d_t\tilde{u}(t,x) = \{-\tfrac{1}{2}\Delta\tilde{u}(t,x) + \tfrac{1}{2}|\partial_x\tilde{u}(t,x)|^2 - \tilde{\varphi}_t(x)\}dt - \tilde{v}(t,x)\cdot dB_t$$
$$\tilde{u}(T,x) = \tilde{\gamma}(x)$$

  - $\tilde{\varphi}_t(x) = \beta\tilde{f}(x,\tilde{\mu}_t^{input})$, $\quad \tilde{\gamma}(x) = \beta\tilde{g}(x,\tilde{\mu}_T^{input})$
  - $\tilde{\varphi}'_t(x) = \beta\tilde{f}(x,\tilde{\mu}_t^{input,\prime})$, $\quad \tilde{\gamma}'(x) = \beta\tilde{g}(x,\tilde{\mu}_T^{input,\prime})$

- Stability if f and g and their derivatives are Lipschitz in μ

$$\operatorname{essup}_{\omega\in\Omega}\sup_{t\in[0,T]} W_1(\mu_t,\mu'_t) \le C\Big[\operatorname{essup}_{\omega\in\Omega}\Big(\|\tilde{\gamma}-\tilde{\gamma}'\|_{1+\alpha} + \sup_{t\in[0,T]}\|\tilde{\varphi}_t-\tilde{\varphi}'_t\|_{\alpha}\Big)\Big]$$

Method of continuation

- Increase the value of β progressively in

$$d_t\tilde{\mu}_t = \{\tfrac{1}{2}\Delta\tilde{\mu}_t + div(\tilde{\mu}_t\partial_x\tilde{u}(t,x))\}dt$$

$$d_t\tilde{u}(t,x) = \{-\tfrac{1}{2}\Delta\tilde{u}(t,x) + \tfrac{1}{2}|\partial_x\tilde{u}(t,x)|^2 - \beta\tilde{f}(x,\tilde{\mu}_t) - \tilde{\varphi}_t(x)\}dt - \tilde{v}(t,x)\cdot dB_t$$

$$\tilde{u}(T,x) = \beta\tilde{g}(x,\tilde{\mu}_T) + \tilde{\gamma}(x)$$

- Show $\exists\epsilon > 0$ s.t. $\exists!$ for $\beta \in [0,1) \Rightarrow \exists!$ for $\beta + \epsilon$

- Same principle as above

  - $\tilde{\varphi}_t(x) = \epsilon\tilde{f}(x,\tilde{\mu}_t^{input}), \quad \tilde{\gamma}(x) = \epsilon\tilde{g}(x,\tilde{\mu}_T^{input})$
  - $\tilde{\varphi}'_t(x) = \epsilon\tilde{f}(x,\tilde{\mu}_t^{input,\prime}), \quad \tilde{\gamma}'(x) = \epsilon\tilde{g}(x,\tilde{\mu}_T^{input,\prime})$

- Need stability for $\beta \in (0,1)$!

$$\mathrm{essup}_{\omega\in\Omega} \sup_{t\in[0,T]} W_1(\mu_t,\mu'_t)$$

$$\leq C\,\mathrm{essup}_{\omega\in\Omega}\Big(\|\tilde{\gamma} - \tilde{\gamma}'\|_{1+\alpha} + \sup_{t\in[0,T]} \|\tilde{\varphi}_t - \tilde{\varphi}'_t\|_{\alpha}\Big)$$

Method of continuation

- Increase the value of β progressively in

$$d_t\tilde{\mu}_t = \{\tfrac{1}{2}\Delta\tilde{\mu}_t + div(\tilde{\mu}_t\partial_x\tilde{u}(t,x))\}dt$$

$$d_t\tilde{u}(t,x) = \{-\tfrac{1}{2}\Delta\tilde{u}(t,x) + \tfrac{1}{2}|\partial_x\tilde{u}(t,x)|^2 - \beta\tilde{f}(x,\tilde{\mu}_t) - \tilde{\varphi}_t(x)\}dt - \tilde{v}(t,x)\cdot dB_t$$

$$\tilde{u}(T,x) = \beta\tilde{g}(x,\tilde{\mu}_T) + \tilde{\gamma}(x)$$

- Show $\exists\epsilon > 0$ s.t. $\exists!$ for $\beta \in [0,1) \Rightarrow \exists!$ for $\beta + \epsilon$

- Same principle as above

  - $\tilde{\varphi}_t(x) = \epsilon\tilde{f}(x,\tilde{\mu}_t^{input})$, $\quad \tilde{\gamma}(x) = \epsilon\tilde{g}(x,\tilde{\mu}_T^{input})$
  - $\tilde{\varphi}'_t(x) = \epsilon\tilde{f}(x,\tilde{\mu}_t^{input,\prime})$, $\quad \tilde{\gamma}'(x) = \epsilon\tilde{g}(x,\tilde{\mu}_T^{input,\prime})$

- Need stability for $\beta \in (0,1)$! Consequence of monotonicity

$$\text{essup}_{\omega\in\Omega} \sup_{t\in[0,T]} W_1(\mu_t,\mu'_t)$$
$$\leq C\text{essup}_{\omega\in\Omega}\Big(\|\tilde{\gamma} - \tilde{\gamma}'\|_{1+\alpha} + \sup_{t\in[0,T]} \|\tilde{\varphi}_t - \tilde{\varphi}'_t\|_\alpha\Big)$$

Part VI. Master Equation

Part VI. Master Equation

a. Derivation of the master equation

Generalized value function

- Throughout this section $\rightsquigarrow$ existence and uniqueness of equilibria
  - for instance $\rightsquigarrow$ smooth coefficients and monotonicity
  - definition on $\mathbb{R}^d$ first, and analysis on $\mathbb{T}^d$
- Initial condition of the population μ^0 at time t_0
  - uniqueness $\rightsquigarrow$ flow $(\mu_t)_{t_0 \le t \le T}$ describing the equilibrium
  - solution of optimal control starting from x_0 under $\boldsymbol{\mu} = (\mu_t)_{t_0 \le t \le T}$

$$dX_t = -\partial_x u^{\boldsymbol{\mu}}(t, X_t)dt + dW_t + \eta dB_t \quad t \in [t_0, T],$$

with $X_{t_0} = x$ and

$$\begin{aligned} d_t u^{\boldsymbol{\mu}}(t,x) &= \{-\tfrac{1}{2}(1+\eta^2)\Delta u^{\boldsymbol{\mu}}(t,x) + \tfrac{1}{2}|\partial_x u^{\boldsymbol{\mu}}(t,x)|^2 - f(x,\mu_t) \\ &\qquad - \eta div(v^{\boldsymbol{\mu}}(t,x))\}dt + v^{\boldsymbol{\mu}}(t,x) \cdot dB_t \\ u^{\boldsymbol{\mu}}(T,x) &= g(x,\mu_T) \end{aligned}$$

- Generalized value function : $\mathcal{U}(t_0, x_0, \mu^0) = u^{\boldsymbol{\mu}:\mu_{t_0}=\mu^0}(t_0, x_0)$

Dynamic Programming

• $(X^\star)_{t_0 \le t \le T} \rightsquigarrow$ optimal trajectory starting from x_0 at t_0 under equilibrium $\boldsymbol{\mu}$ starting from μ^0 at t_0

$$\mathcal{U}(t_0, x_0, \mu^0) = \mathbb{E}\Big[\int_{t_0}^{T} \Big[f(X_s^\star, \mu_s) + \tfrac{1}{2}|\alpha_s^\star|^2\Big] ds + g(X_T^\star, \mu_T)\Big]$$

• Flow property at the equilibrium

$$\mathcal{U}(t_0, x_0, \mu^0) = \mathbb{E}\Big[\int_{t_0}^{t_0+\epsilon} \Big[f(X_s^\star, \mu_s) + \tfrac{1}{2}|\alpha_s^\star|^2\Big] ds + \mathcal{U}(t_0 + \epsilon, X_{t_{0+\epsilon}}^\star, \mu_{t_0+\epsilon})\Big]$$

• If $\mathcal{U}$ is smooth w.r.t. three arguments $\Rightarrow$ solution of a PDE on $[0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$

- ◦ needs differential calculus and chain rule
- ◦ use Lions' approach to differential calculus on Wasserstein space

Differential calculus on Wasserstein space

- Approach of the differentiation on $\mathcal{P}_2(\mathbb{R}^d)$ due to Lions
- Given $\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$
- Lifting of $\mathcal{U}$

$$\hat{\mathcal{U}} : L^2(\Omega, \mathbb{P}) \ni X \mapsto \mathcal{U}(\mathcal{L}(X) = \text{Law}(X))$$

  - $\mathcal{U}$ differentiable if $\hat{\mathcal{U}}$ Fréchet differentiable
- Differential of $\mathcal{U}$
  - Fréchet derivative of $\hat{\mathcal{U}}$

$$D\hat{\mathcal{U}}(X) = \partial_\mu \mathcal{U}(\mu)(X), \quad \partial_\mu \mathcal{U}(\mu) : \mathbb{R}^d \ni x \mapsto \partial_\mu \mathcal{U}(\mu)(x) \quad \mu = \mathcal{L}(X)$$

  - derivative of $\mathcal{U}$ in $\mu \rightsquigarrow \partial_\mu \mathcal{U}(\mu) \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$
- Finite-dimensional projection ( )

$$\partial_{x_i}\Big[\mathcal{U}\Big(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)\Big] = \frac{1}{N}\partial_\mu \mathcal{U}\Big(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)(x_i), \quad x_1, \ldots, x_N \in \mathbb{R}^d$$

First-order differentiability

- Example: $\mathcal{U}(\mu) = \int_{\mathbb{R}^d} h(y)d\mu(y)$
  - h C^1 and ∇h at most of linear growth

$$\hat{\mathcal{U}}(X+Y) = \mathbb{E}[h(X+Y)] = \mathbb{E}[h(X)] + \mathbb{E}[\nabla h(X)\cdot Y] + o(\|Y\|_2)$$

$$\Rightarrow D\hat{\mathcal{U}}(X) = \nabla h(X) \Rightarrow \partial_\mu \mathcal{U}(\mu)(v) = \nabla h(v)$$

- Equivalent form (close to geometric approach, Tudorascu (17))
  - action of $\mathcal{U}$ along measure transported by a vector field

$b : \mathbb{R}^d \to \mathbb{R}^d$

$$dX_t = b(X_t)dt, \quad X_0 \sim \mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$$

  - action of $\mathcal{U}$ along $(\mu_t = \mathcal{L}(X_t))_t$?

$$\frac{d}{dt}_{|t=0}\mathcal{U}(\mu_t) = \frac{d}{dt}_{|t=0}\mathbb{E}[\hat{\mathcal{U}}(X_t)] = \mathbb{E}[\partial_\mu \mathcal{U}(\mu)(X_0)\cdot b(X_0)]$$

$$= \int_{\mathbb{R}^d} \partial_\mu \mathcal{U}(\mu)(v)\cdot b(v)d\mu_0(v)$$

Second-order differentiability

- Need for existence of second-order derivatives
  - asking the lift to be twice Fréchet is too strong
  - only discuss the existence of second-order partial derivatives
- Requires
  - $\partial_\mu \mathcal{U}(\mu)(v)$ is differentiable in v and μ

$$\partial_v \partial_\mu \mathcal{U}(\mu)(v) \qquad \partial_\mu^2 \mathcal{U}(\mu)(v, v')$$

  - $\partial_v \partial_\mu \mathcal{U}(\mu)(v)$ and $\partial_\mu^2 \mathcal{U}(\mu)(v, v')$ continuous in (μ, v, v') (for W_2 in μ) with suitable growth
- Finite-dimensional projection

$$\partial^2_{x_i\ x_j}\left[\mathcal{U}\left(\frac{1}{N}\sum_{k=1}^N \delta_{x_k}\right)\right] = \frac{1}{N}\partial_v\partial_\mu \mathcal{U}\left(\frac{1}{N}\sum_{k=1}^N \delta_{x_k}\right)(x_i)\ \delta_{i,j} + \frac{1}{N^2}\partial_\mu^2 \mathcal{U}\left(\frac{1}{N}\sum_{k=1}^N \delta_{x_k}\right)(x_i, x_j)$$

Itô's formula on $\mathcal{P}_2(\mathbb{R}^d)$

- Process $dX_t = b_t dt + dW_t + dB_t$ with $\mathbb{E}\int_0^T |b_t|^2 dt < \infty$
  - μ_t = conditional law of X_t given B
- $\mathcal{U}$ Fréchet differentiable with $\mathbb{R}^d \ni v \mapsto \partial_\mu \mathcal{U}(\mu)(v)$ differentiable in v and μ
  - Itô's formula for $(\mathcal{U}(\mu_t))_{t\geq 0}$?
- Space discretization: Approximation of μ_t by a particle system

$$\mu_t \sim \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j} \quad \text{with} \quad (X_t^i)_t \text{ conditionally i.i.d. given } B$$

- Limit on standard Itô's formula for $d_t\Big[\mathcal{U}\Big(\frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}\Big)\Big]$

$$d\mathcal{U}(\mu_t) = \mathbb{E}[b_t \cdot \partial_\mu \mathcal{U}(\mu_t)(X_t^1)\,|\,B] + \mathbb{E}[\mathrm{Trace}(\partial_v \partial_\mu \mathcal{U}(\mu_t)(X_t^1))\,|\,B]dt$$
$$+ \frac{1}{2}\mathbb{E}[\mathrm{Trace}(\partial_\mu^2 \mathcal{U}(\mu_t)(X_t^1, X_t^2))\,|\,B]dt + \mathbb{E}[\partial_\mu \mathcal{U}(\mu_t)(X_t^1)\,|\,B] \cdot dB_t$$

Form of the master equation

- Formal identification in the dynamic programming expansion
- Master equation at order 2

$$
\begin{aligned}
&\partial_t \mathcal{U}(t,x,\mu) - \int_{\mathbb{R}^d} \partial_x \mathcal{U}(t,v,\mu) \cdot \partial_\mu \mathcal{U}(t,x,\mu,v) d\mu(v) \\
&\quad - \tfrac{1}{2}|\partial_x \mathcal{U}(t,x,\mu)|^2 + f(x,\mu) + \tfrac{1}{2}(1+\eta^2)\mathrm{Trace}\Big(\partial_x^2 \mathcal{U}(t,x,\mu)\Big) \\
&\quad + \tfrac{1}{2}(1+\eta^2) \int_{\mathbb{R}^d} \mathrm{Trace}\Big(\partial_v \partial_\mu \mathcal{U}(t,x,\mu)(v)\Big) d\mu(v) \\
&\quad + \eta^2 \int_{\mathbb{R}^d} \mathrm{Trace}\Big(\partial_x \partial_\mu \mathcal{U}(t,x,\mu)(v)\Big) d\mu(v) \\
&\quad + \tfrac{1}{2}\eta^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{Trace}\Big(\partial_\mu^2 \mathcal{U}(t,x,\mu)(v,v')\Big) d\mu(v) d\mu(v') = 0
\end{aligned}
$$

- ◦ Not a HJB! (MFG $\neq$ optimization) (●)

Typical statement

- Lions, Chassagneux-Crisan-D., Cardaliaguet-D.-Lasry-Lions, Gangbo Swiech ($T \ll 1$)
- Require monotonicity and bounded coefficients
- Require first-order smoothness of the coefficients (same for g)
  - $\partial_x f(x,\mu)$ bounded and Lipschitz in (x,μ)
  - $\partial_\mu f(x,\mu)(v)$ bounded and Lipschitz
- Require second-order smoothness of the coefficients (same for g)
  - $\partial_x^2 f(x,\mu)$ bounded and Lipschitz in (x,μ)
  - $\partial_\mu f(x,\mu)(v)$ is differentiable in x, v and μ
  - $\partial_x \partial_\mu f(x,\mu)(v)$, $\partial_v \partial_\mu f(x,\mu)(v)$ are bounded and Lipschitz
  - $\partial_\mu^2 f(x,\mu)(v,v')$ is bounded and Lipschitz
- Then existence and uniqueness of a classical solution with
  - $\mathcal{U}(t,\cdot,\cdot)$ having the same smoothness as f and g and continuously differentiable in time

Extensions

-

Part VI. Master Equation

b. Linearization ($\eta = 0$)

Road map to regularity of $\mathcal{U}$

- To proceed with the $\boxed{\text{analysis} \rightsquigarrow \text{torus}}$
- Look at $\mathcal{U}$ as

$$\mathcal{U} : [0,T] \times \mathcal{P}(\mathbb{T}^d) \ni (t,\mu) \mapsto \underbrace{\left(\mathbb{T}^d \ni x \mapsto \mathcal{U}(t_0,x,\mu)\right)}_{\mathcal{U}(t_0,\cdot,\mu)}$$

  - typical example $\rightsquigarrow \mathcal{U}(t_0,\cdot,\mu) \in C^{n+\alpha}(\mathbb{T}^d)$
  - n, α depending on the smoothness of f and g
- Objective is to $\boxed{\text{understand smoothness w.r.t. } \mu}$
  - recall $\rightsquigarrow \mathcal{U}(t_0,\cdot,\mu) = \underbrace{u^{\mu:\mu_{t_0}=\mu}(t_0,\cdot)}_{\text{HJB with FP initialized at } (t_0,\mu)}$
  - differentiability w.r.t. $\mu^0 \rightsquigarrow$ use $\boxed{\text{convex perturbation}}$

$$\frac{d}{d\varepsilon}_{|\varepsilon=0+} u^{(1-\varepsilon)\mu+\varepsilon\mu'}(t_0,\cdot)$$

$$= \frac{d}{d\varepsilon}_{|\varepsilon=0+} \mathcal{U}(t_0,\cdot,(1-\varepsilon)\mu+\varepsilon\mu') \quad \mu,\ \mu' \in \mathcal{P}(\mathbb{T}^d)$$

Other approach of differentiation on $\mathcal{P}(\mathbb{T}^d)$

- We say that $\mathcal{V} : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is C^1 if

$$\frac{d}{d\varepsilon_{|\varepsilon=0+}} \mathcal{V}((1-\varepsilon)\mu + \varepsilon\mu') = \underbrace{\int_{\mathbb{T}^d} \frac{\delta \mathcal{V}}{\delta m}(\mu)(v) d(\mu' - \mu)(v)}_{\frac{\delta \mathcal{V}}{\delta m}(\mu)(\cdot) \cdot (\mu' - \mu)}$$

for a continuous map $\frac{\delta \mathcal{V}}{\delta m} : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$ ( ) ( ) ( )

  - unique up to an additive constant $\rightsquigarrow$ impose zero mean under μ_0

- Connection with Wasserstein derivative ( )

$$\partial_\mu \mathcal{V}(\mu)(v) = \partial_v \frac{\delta \mathcal{V}}{\delta m}(\mu)(v)$$

  - $\exists$ conditions under which equality holds true

- $\mathcal{V}$ is C^2 if
  - for all $v \in \mathbb{T}^d$ $\quad \mathcal{P}(\mathbb{T}^d) \ni \mu \mapsto \frac{\delta \mathcal{V}}{\delta m}(\mu)(v)$ is C^1

Linearized MFG system

- Assume that f and g are C^1 w.r.t. m with

$$\frac{\delta f}{\delta m}, \ \frac{\delta g}{\delta m} : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \ni (x, \mu, v) \mapsto \frac{\delta f}{\delta m}(x, \mu)(v), \ \frac{\delta g}{\delta m}(x, \mu)(v)$$

smooth enough in x and v

- Formal differentiation of the MFG system
  - perturbation of μ along a direction $\mu' - \mu$
  - we let $z_t = \underbrace{\frac{d}{d\varepsilon}_{|\varepsilon=0+} u^{(1-\varepsilon)\mu+\varepsilon\mu'}(t, \cdot)}_{\text{function}}, \quad m_t = \underbrace{\frac{d}{d\varepsilon}_{|\varepsilon=0+} \mu_t^{(1-\varepsilon)\mu+\varepsilon\mu'}}_{\text{distribution}}$

- should solve

$$\partial_t m_t - \tfrac{1}{2}\Delta m_t - div\Big(m_t \partial_x u(t, x) + \mu_t \partial_x z(t, x)\Big) = 0$$

$$\partial_t z(t, x) + \tfrac{1}{2}\Delta z(t, x) - \partial_x u(t, x) \cdot \partial_x z(t, x) + \frac{\delta f}{\delta m}(x, \mu_t)(\cdot) \cdot m_t(\cdot) = 0$$

$$z_T(x) = \frac{\delta g}{\delta m}(x, \mu_T)(\cdot) \cdot m_T(\cdot)$$

Linearized MFG system

- Assume that f and g are C^1 w.r.t. m with

$$\frac{\delta f}{\delta m}, \frac{\delta g}{\delta m} : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \ni (x, \mu, v) \mapsto \frac{\delta f}{\delta m}(x, \mu)(v), \frac{\delta g}{\delta m}(x, \mu)(v)$$

  smooth enough in x and v

- Formal differentiation of the MFG system
  - perturbation of μ along a direction $\mu' - \mu$
  - we let $z_t = \underbrace{\frac{d}{d\varepsilon}_{|\varepsilon=0+} u^{(1-\varepsilon)\mu+\varepsilon\mu'}(t, \cdot)}_{\text{function}}, \quad m_t = \underbrace{\frac{d}{d\varepsilon}_{|\varepsilon=0+} \mu_t^{(1-\varepsilon)\mu+\varepsilon\mu'}}_{\text{distribution}}$

- should solve

$$\partial_t m_t - \tfrac{1}{2}\Delta m_t - div\Big(m_t \partial_x u(t,x) + \mu_t \partial_x z(t,x)\Big) = 0$$

$$\partial_t z(t,x) + \tfrac{1}{2}\Delta z(t,x) - \partial_x u(t,x) \cdot \partial_x z(t,x) + \underbrace{\frac{\delta f}{\delta m}(x, \mu_t)(\cdot) \cdot m_t(\cdot)}_{\text{balance reg in } v \text{ / singularity } m} = 0$$

Initialization of the linearized system

- Assume $\frac{\delta f}{\delta m}$ $\frac{\delta g}{\delta m}$ $C^{n+2+\alpha}$ in (x, y), $\quad n \geq 0, \alpha \in (0, 1)$
- Fix initial condition of linearized system $m_{t_0}(\cdot) \in C^{-(n+1+\alpha)}(\mathbb{T}^d)$
  - $\rightsquigarrow$ $\exists!$ solution to linearized system with

$$(z(t, \cdot), \mu_t(\cdot))_{t_0 \leq t \leq T} \in C([0, T], C^{n+2+\alpha}(\mathbb{T}^d) \times C^{-(n+1+\alpha)}(\mathbb{T}^d))$$

  - more than uniqueness $\rightsquigarrow$ stability
- Example: $m_{t_0} = \delta_v \rightsquigarrow z(t_0, x) = \mathcal{V}^0(t_0, x, \mu_0)(v)$
- if $m_{t_0}(\cdot)$ is finite signed measure $\rightsquigarrow$ linearity

$$z(t_0, x) = \int_{\mathbb{T}^d} \mathcal{V}^0(t_0, x, \mu_0)(v) dm_{t_0}(v)$$

Initialization of the linearized system

- Assume $\frac{\delta f}{\delta m}$ $\frac{\delta g}{\delta m}$ $C^{n+2+\alpha}$ in (x,y), $\quad n \geq 0, \alpha \in (0,1)$
- Fix initial condition of linearized system $m_{t_0}(\cdot) \in C^{-(n+1+\alpha)}(\mathbb{T}^d)$
  - $\rightsquigarrow$ $\exists!$ solution to linearized system with

$$(z(t,\cdot), \mu_t(\cdot))_{t_0 \leq t \leq T} \in C([0,T], C^{n+2+\alpha}(\mathbb{T}^d) \times C^{-(n+1+\alpha)}(\mathbb{T}^d))$$

  - more than uniqueness $\rightsquigarrow$ stability

- Example: $m_{t_0} = (-1)^\ell \frac{d^\ell}{dv^\ell}\delta_v$, $\ell \leq n+1 \rightsquigarrow$

$$z(t_0,x) = \underbrace{\mathcal{V}^\ell(t_0,x,\mu_0)(v)}_{\partial_v^\ell \mathcal{V}^0(t_0,x,\mu_0)(v)}$$

- if $m_{t_0}(\cdot)$ is finite signed measure $\rightsquigarrow$ linearity

$$z(t_0,x) = \int_{\mathbb{T}^d} \mathcal{V}^0(t_0,x,\mu_0)(v)dm_{t_0}(v)$$

- Distributions in $C^{-(n+1+\alpha)}(\mathbb{T}^d) \rightsquigarrow \mathcal{V}^0$ is $C^{n+1+\alpha}(\mathbb{T}^d)$ in v

General strategy

- Aim at solving

$$\partial_t m_t - \tfrac{1}{2}\Delta m_t - div\Big(m_t \partial_x u(t,x) + \mu_t \partial_x z(t,x)\Big) = 0$$

$$\partial_t z(t,x) + \tfrac{1}{2}\Delta z(t,x) - \partial_x u(t,x) \cdot \partial_x z(t,x) + \frac{\delta f}{\delta m}(x,\mu_t) \cdot m_t(\cdot) = 0$$

$$z(T,x) = \frac{\delta g}{\delta m}(x,\mu_T) \cdot m_T(\cdot)$$

∘ deterministic case $\rightsquigarrow$ Schauder's theorem for ∃ and monotonicity for !

∘ common noise $\rightsquigarrow$ continuation method

∘ progressive augmentation of coupling parameter

General strategy

- Aim at solving

$$\partial_t m_t - \tfrac{1}{2}\Delta m_t - div\Big(m_t \partial_x u(t,x) + \mu_t \partial_x z(t,x)\Big) = 0$$

$$\partial_t z(t,x) + \tfrac{1}{2}\Delta z(t,x) - \partial_x u(t,x) \cdot \partial_x z(t,x) + \beta \frac{\delta f}{\delta m}(x,\mu_t) \cdot m_t(\cdot) = 0$$

$$z(T,x) = \beta \frac{\delta g}{\delta m}(x,\mu_T) \cdot m_T(\cdot)$$

◦ deterministic case ⇝ Schauder's theorem for ∃ and monotonicity for !

◦ common noise ⇝ continuation method

◦ progressive augmentation of coupling parameter β

◦ $\beta = 0 \Rightarrow z \equiv 0$ and $(m_t)_t$ solved separately

◦ proof of ∃! by induction $\beta = 0, \epsilon, 2\epsilon, \dots, 1$, ϵ small enough

□ First order condition of optimality with noise

$$dX_t = b(X_t, \mu_t, \alpha_t)dt + \sigma dW_t$$

⇝ Pontryagin system (Peng)

$$X_t = X_0 + \int_0^t b\Big(X_s, \mu_s, \alpha^\star(X_s, \mu_s, Y_s)\Big)\, ds + \sigma W_t$$

$$Y_t = \partial_x g(X_T, \mu_T) + \int_t^T \partial_x H\Big(X_s, \mu_s, \alpha^\star(X_s, \mu_s, Y_s), Y_s\Big)\, ds - \int_t^T Z_s dW_s$$

□ First order condition of optimality with noise

$$dX_t = b(X_t, \mu_t, \alpha_t)dt + \sigma dW_t$$

⇝ Pontryagin system (Peng)

$$X_t = X_0 + \int_0^t b\Big(X_s, \mu_s, \alpha^\star(X_s, \mu_s, Y_s)\Big)\,ds + \sigma W_t$$

$$Y_t = \partial_x g(X_T, \mu_T) + \int_t^T \partial_x H\Big(X_s, \mu_s, \alpha^\star(X_s, \mu_s, Y_s), Y_s\Big)\,ds - \int_t^T Z_s dW_s$$

□ First order condition of optimality with noise

$$dX_t = b(X_t, \mu_t, \alpha_t)dt + \sigma dW_t$$

$\rightsquigarrow$ Pontryagin system (Peng)

$$X_t = X_0 + \int_0^t b\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s)\Big)\, ds + \sigma W_t$$

$$Y_t = \partial_x g(X_T, \mathcal{L}(X_T)) + \int_t^T \partial_x H\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s), Y_s\Big)\, ds - \int_t^T Z_s dW_s$$

- Summary: Forward-Backward systems may be ill-posed! But:
  - $\rightsquigarrow$ Noise restores uniqueness!
  - $\rightsquigarrow$ Monotonicity ($\leftrightarrow$ convexity) restores uniqueness!

□ Hint: Either use monotonicity or interpret the FB system as the Pontryagin system of a standard optimal control problem with linear–convex coefficients

$\rightsquigarrow b(t,x,\alpha) = (a_t x + a'_t)x + b_t \alpha_t$

$\rightsquigarrow g(x) = \frac{1}{2} q(q + q')x^2$

$\rightsquigarrow f(t,x,\alpha) = \frac{1}{2}[\alpha^2 + m_t(m_t + m'_t)x^2]$

☐ Exercise: What does monotonicity for the MFG mean for the control problem?

☐ Hint: Write monotonicity as

$$\int_{\mathbb{R}^d}\Big[\int_{\mathbb{R}^d} F(x-y)dm(y) - \int_{\mathbb{R}^d} F(x-y)dm'(y)\Big]d(m-m')(x) \geq 0$$

$$\Leftrightarrow \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)d(m-m')(y)d(m-m')(x) \geq 0$$

⇝ second-order term is positive in linearization ⇔ convexity!

☐ Examples:

⇝ $F(z) = -|z|^2$

⇝ $F(z) = \int_{\mathbb{R}^d} \exp(iz \cdot s)d\lambda(s)$, where λ is symmetric positive finite measure

(take λ a Gaussian, take λ a Cauchy, take λ a combination of two Dirac masses...)

□ Make a convex perturbation of $\mu \in \mathcal{P}(\mathbb{R}^d)$

⇝ take $\nu \in \mathcal{P}(\mathbb{R}^d)$ and expand

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)d((1-\varepsilon)\mu(x)+\varepsilon\nu(x))d((1-\varepsilon)\mu(x)+\varepsilon\nu(x)) \\
&= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)d\mu(x)d\mu(y) \\
&\quad + \varepsilon \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)d\mu(x)d(\nu-\mu)(y) \\
&\quad + \varepsilon^2\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)d(\nu-\mu)(x)d(\nu-\mu)(y)
\end{aligned}
$$

⇝ regard $\nu - \mu$ as direction of linearization

□ Think of

$$X_t = F_t\Big((B_s)_{0\le s\le t}, (W_s^1, \cdots, W_s^N)_{0\le s\le t}\Big)$$

$\rightsquigarrow$ B constructed on Ω^0 and $W^1, \cdots, W^N$ constructed on Ω^1 and equip $\Omega^0 \times \Omega^1$ with product measures $\mathbb{P}^0 \otimes \mathbb{P}^1$

$\rightsquigarrow$ take $\omega^0 \in \Omega^0 \Rightarrow \mathcal{L}(X_t|(B_s)_{0\le s\le t})$ at ω^0 is the law on Ω^1 of

$$F_t\Big((B_s(\omega^0))_{0\le s\le t}, (W_s^1, \cdots, W_s^N)_{0\le s\le t}\Big)$$

□ Take sequence $(X_n)_{n\ge1}$ of r.v. on $\Omega^0 \times \Omega^1$ with values in $\mathbb{R}^d$

$\rightsquigarrow$ assume, $\mathbb{P}^0$ a.s., $(X_n(\omega^0, \cdot))_{n\ge1}$ are under $\mathbb{P}^1$

$\rightsquigarrow$ take $\varphi : \mathbb{R}^d \to \mathbb{R}$ bounded continuous

$$\mathbb{P}^0 \otimes \mathbb{P}^1\Big(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \varphi(X_n) = \mathbb{E}^1[\varphi(X_1)]\Big) = 1$$

□ Optimality says that

$$\left(\int_{\mathbb{R}^d} u(t,x)d\mu_t(x) + \int_0^t \int_{\mathbb{R}^d} \left(f(x,\mu_s) + \frac{1}{2}|\partial_x u(s,x)|^2\right)d\mu_s(x)ds\right)_{0\le t\le T}$$

should be a martingale

⇝ but bracket in the product $\int_{\mathbb{R}^d} u(t,x)d\mu_t(x)$!

$$-\eta \int_{\mathbb{R}^d} \sum_{i=1}^d v^i(t,x)\partial_{x_i}(d\mu_t(x)) = \eta \int_{\mathbb{R}^d} \sum_{i=1}^d \partial_{x_i} v^i(t,x)d\mu_t(x)$$

$$= \eta \int_{\mathbb{R}^d} \mathrm{div} v(t,x)d\mu_t(x)$$

⇝ cancels out with Itô-Wentzell!

□ Prove

$$\partial_{x_i}\left[\mathcal{U}\left(\frac{1}{N}\sum_{j=1}^{N}\delta_{x_j}\right)\right] = \frac{1}{N}\partial_\mu \mathcal{U}\left(\frac{1}{N}\sum_{j=1}^{N}\delta_{x_j}\right)(x_i), \quad x_1,\ldots,x_N \in \mathbb{R}^d$$

□ Choose θ r.v. with values in $\{1,\cdots,N\}$$ equipped with uniform probability

$\rightsquigarrow$ for $x=(x_1,\cdots,x_N), y=(y_1,\cdots,y_N)\in\mathbb{R}^d$, expand

$$\begin{aligned}\hat{\mathcal{U}}(x_\theta+y_\theta) &= \hat{\mathcal{U}}(x_\theta) + \mathbb{E}[D\hat{\mathcal{U}}(x_\theta)\cdot y_\theta] + o(\|y_\theta\|_2)\\ &= \hat{\mathcal{U}}(x_\theta) + \mathbb{E}[\partial_\mu \mathcal{U}(\mathcal{L}(x_\theta))(x_\theta)\cdot y_\theta] + o(\|y_\theta\|_2)\\ &= \hat{\mathcal{U}}(x_\theta) + \frac{1}{N}\sum_{i=1\cdots N}\partial_\mu \mathcal{U}(\bar{\mu}_x^N)(x_i)y_i + o(\|y_\theta\|_2)\end{aligned}$$

with $\bar{\mu}_x^N = \frac{1}{N}\sum_{j=1}^{N}\delta_{x_j}$

□ Exercise: Assume that

$$\partial_v \frac{\delta \mathcal{V}}{\delta m}(\mu)(v)$$

is smooth and expand

$$\begin{aligned}(\mathcal{L}(Y)) - (\mathcal{L}(X)) \\ = \int_0^1 \mathbb{E}\Big[&\frac{\delta}{\delta m}(\lambda \mathcal{L}(Y) + (1-\lambda)\mathcal{L}(X), Y) \\ &- \frac{\delta}{\delta m}(\lambda \mathcal{L}(Y) + (1-\lambda)\mathcal{L}(X), X)\Big] d\lambda\end{aligned}$$

Deduce that

$$\partial_\mu \mathcal{V}(\mu)(v) = \partial_v \frac{\delta \mathcal{V}}{\delta m}(\mu)(v)$$

□ Exercise: Choose $\eta = 0$ and take a potential game

$\rightsquigarrow$ write the HJB equation on the space of probability measures for the social optimization problem

$\rightsquigarrow$ derive formally the value function w.r.t. m

$\rightsquigarrow$ show that this coincides with the master equation for the MFG

$\rightsquigarrow$ see [Gangbo and Swiech], see [C D L L]

□ Exercise: Adapt the notion of derivative to $\mathbb{R}^d$ and check that it is consistant with the linearization procedure used for potential games!

$\rightsquigarrow$ take $\mathcal{V}(\mu) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)d\mu(x)d\mu(y)$

$\rightsquigarrow$ take $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ and expand

$$\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)d((1-\varepsilon)\mu(x)+\varepsilon\nu(x))d((1-\varepsilon)\mu(x)+\varepsilon\nu(x))$$

$$= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)d\mu(x)d\mu(y)$$

$$+\varepsilon\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)d\mu(x)d(\nu-\mu)(y)$$

$$+\varepsilon^2\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)d(\nu-\mu)(x)d(\nu-\mu)(y)$$

$\rightsquigarrow$ deduce $\frac{\delta\mathcal{V}}{\delta m}(y) = \int_{\mathbb{R}^d} F(y-x)d\mu(x)$

☐ Exercise : Consider a more general social optimization problem

$$J(\alpha) = G(\mathcal{L}(X_T)) + \int_0^T F(\mathcal{L}(X_t)) + \tfrac{1}{2}\mathbb{E}\int_0^T |\alpha_t|^2 dt$$

over $dX_t = b(X_t, \mathcal{L}(X_t), \alpha_t)dt + \sigma dW_t$

$\rightsquigarrow$ show the first order condition is given by the MFG system

$$\partial_t u_t(x) = -\tfrac{1}{2}\sigma^2 \Delta_x u_t(x) + \tfrac{1}{2}|\partial_x u_t(x)|^2 - \frac{\delta F}{\delta m}(\mu_t)(x)$$

with terminal condition $u_T(x) = \dfrac{\delta G}{\delta m}(\mu_T)(x)$ and with

$$\partial_t \mu_t = \mathrm{div}_x(\partial_x u(t,x)\mu_t) + \tfrac{1}{2}\sigma^2 \Delta_x \mu_t$$