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Risk-Averse PDE-Constrained Optimization: Theory, Algorithms, and Statistical Foundations

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Simula Research Laboratory
Oslo, Norway

PGMO Days 2025, November 17, 2025

Figure: Drew P. Kouri, Sandia National Laboratories

D. P. Kouri and T. M. Surowiec.
Risk-averse PDE-constrained optimization using the conditional value-at-risk.
SIAM Journal on Optimization 26, 1 (2016), 365–396.

D. P. Kouri and T. M. Surowiec.
Existence and optimality conditions for risk-averse PDE-constrained optimization
SIAM/ASA J. Uncertainty Quantification 6, 2 (2018), 787–815.

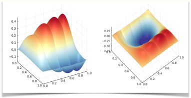
D. P. Kouri and T. M. Surowiec.
Risk-averse optimal control of semilinear elliptic PDEs
ESAIM: Control, Optimisation and Calculus of Variations 26, 53 (2020).

D. P. Kouri and T. M. Surowiec.
Epi-Regularization of Risk Measures
Mathematics of Operations Research 45, 2 (2020), 774–795

D. P. Kouri and T. M. Surowiec.
A primal-dual algorithm for risk minimization
Mathematical Programming 193, 337–363 (2022).

D.P. Kouri, M. Staudigl, and T.M. Surowiec
A Relaxation-based Probabilistic Approach for PDE-constrained Optimization under Uncertainty with Pointwise State Constraints
Comput Optim Appl 85, 441–478 (2023).

From data to decision-making



Data

$\mathcal{D} = \{((A_i, f_i), u_i)\}$

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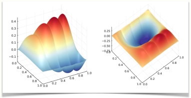

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Statistical inversion
to infer measure $\mu \in \mathcal{P}(\Omega)$

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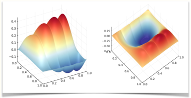

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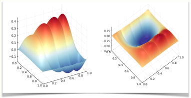

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Risk-Averse Optimization
$\min_{z} \; \mathcal{R}\big(J(u(\cdot), \cdot; z)\big)$

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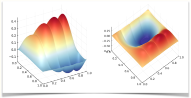

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$\mathcal{R} = \mathbb{E}_\mu, \mathrm{CVaR}_\alpha \ldots$

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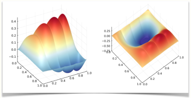

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$z^{k+1} := z^k - \gamma^k \partial \mathcal{J}(z^k, \xi^k)$

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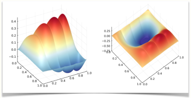

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$z^\star = \mathcal{O}_{\|}(J)$

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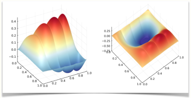

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Asymptotic Statistics
$\frac{1}{N} \sum X_i \to \mathbb{E}[X]$

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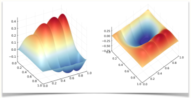

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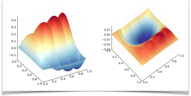
Dealing with random objectives



Using a **random PDE**, means our objective becomes **implicitly random**:

$$\min_{z \in Z_{\mathrm{ad}}} \mathcal{J}(S(z))(\omega) + \rho(z)$$

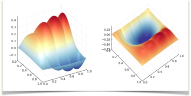
$z \in Z_{\mathrm{ad}}$ decision variables, designs, controls, etc. (**deterministic**)

$z \mapsto S(z)$ solution of the random PDE. (**stochastic**)

$\mathcal{J}$ objective. (either **deterministic** or **stochastic**)

ρ cost or regularization term.

Shaping the Distribution



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$$X_z(\omega) := \mathcal{J}(S(z))(\omega) + \rho(z).$$

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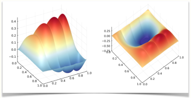

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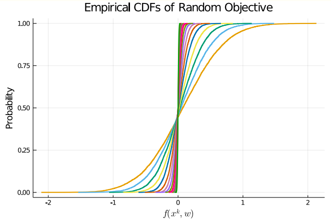
Every z yields a difference distribution for X_z (see fig.)



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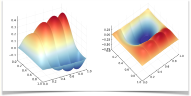

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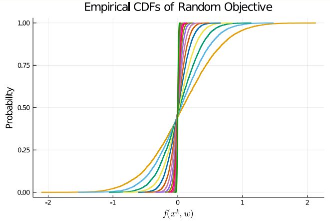
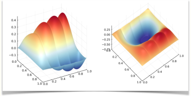


We need to choose a **numerical surrogate for risk** $\mathcal{R}$.

From data to decision-making



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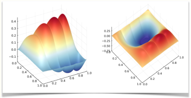
Risk Measures[1]



Risk Neutral $\mathcal{R} = \mathbb{E}$: Optimize to achieve best performance **on average**, ignores outliers, tail events.

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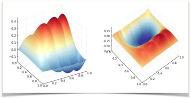

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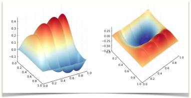

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Value-at-Risk $\mathcal{R}[X] = \inf\{\tau : \mathbb{P}(X \leq \tau) \geq \beta\}$:

Find smallest τ such that with **probability** β, $\mathcal{J}(S(z))$ does not exceed the value τ.

Conceptually very useful, but not subadditive, mathematically difficult to handle numerically.

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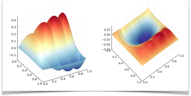

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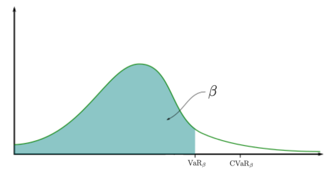

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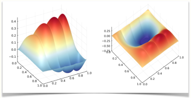

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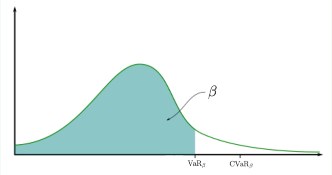

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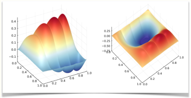

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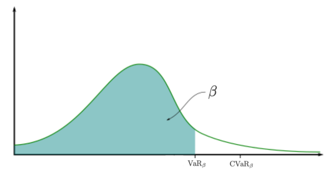

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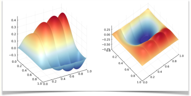

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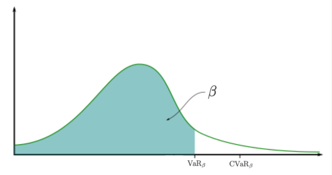

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$$\mathrm{CVaR}_\beta[X] = \inf_{t\in\mathbb{R}}\left\{t + \frac{1}{1-\beta}\mathbb{E}[(X-t)_+]\right\} \quad \text{(Rockafellar, Uryasev 2000)}$$

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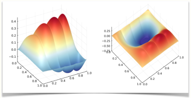
A Contaminant Mitigation Problem[2]



$$
\begin{aligned}
-\nabla \cdot (\epsilon \nabla u) + \mathbb{V} \cdot \nabla u &= f && \text{in } D \\
u &= 0 && \text{on } \Gamma_d = \{0\} \times (0,1) \\
\epsilon \nabla u \cdot n &= 0 && \text{on } \partial D \setminus \Gamma_d
\end{aligned}
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$D = (0,1)^2$ physical domain, u is the advected pollutant.

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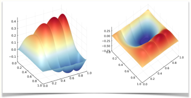

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-\nabla \cdot (\epsilon(\omega)\nabla u) + \mathbb{V}(\omega) \cdot \nabla u &= f(\omega) && \text{in } D, \text{ a.s.} \\
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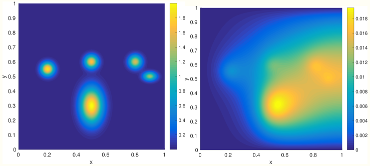
Random inputs: $\epsilon, \mathbb{V}, f$ permeability, wind, sources of contaminant defined over probability space $(\Omega, \mathcal{F}, \mathbb{P})$.



(top left) mean of f

(top right) u with mean values for ϵ, V, f, $z = 0$

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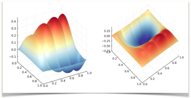

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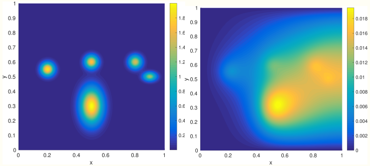
$\mathcal{Z}$ is the control space, e.g., $L^2(D)$ or $\mathbb{R}^n$; $\mathcal{Z}_{\text{ad}} = \{z \in \mathcal{Z} \,|\, 0 \le z \le 1\}$.

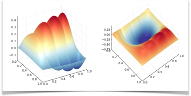


(top left) mean of f

(top right) u with mean values for ϵ, V, f, $z = 0$

[2] D.P. Kouri, T. M. Surowiec, (2018). SIAM/ASA J. Uncertain. Quantif., 6(2), 787-815.

A Contaminant Mitigation Problem[2]



Find optimal placement of mitigating factors z by solving:

$$\min_{z \in \mathcal{Z}_{\text{ad}}} \left\{ \mathcal{R}\left[\frac{1}{2}\int_D S(z)^2 \,\mathrm{d}x\right] + \|z\|_1 \right\}$$

where $S(z) = u : \Omega \to H^1(D)$ solves the weak form of

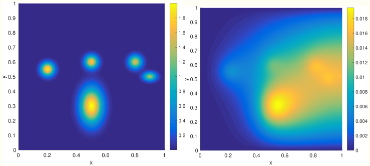
$$\begin{aligned} -\nabla \cdot (\epsilon(\omega)\nabla u) + \mathbb{V}(\omega) \cdot \nabla u &= f(\omega) - Bz && \text{in } D, \text{ a.s.} \\ u &= 0 && \text{on } \Gamma_d = \{0\} \times (0,1), \text{ a.s.} \\ \epsilon(\omega)\nabla u \cdot n &= 0 && \text{on } \partial D \setminus \Gamma_d, \text{ a.s.} \end{aligned}$$

$D = (0,1)^2$ physical domain, u is the advected pollutant.

Random inputs: $\epsilon, \mathbb{V}, f$ permeability, wind, sources of contaminant defined over probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

$\mathcal{Z}$ is the control space, e.g., $L^2(D)$ or $\mathbb{R}^n$; $\mathcal{Z}_{\text{ad}} = \{z \in \mathcal{Z} \,|\, 0 \le z \le 1\}$.

$\mathcal{R} : \mathcal{X} \to \overline{\mathbb{R}}$ is a numerical surrogate for "risk", i.e., a risk measure.



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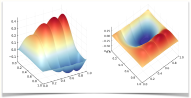

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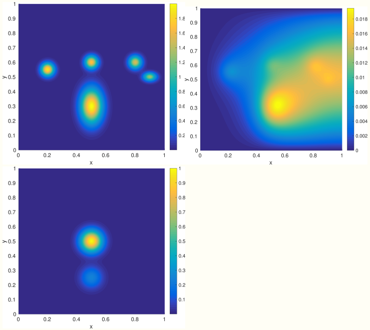

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(bottom left) optimal solution with $\mathcal{R} = \mathbb{E}$

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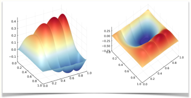
A Contaminant Mitigation Problem[2]



Find optimal placement of mitigating factors z by solving:

$$\min_{z \in \mathcal{Z}_{\text{ad}}} \left\{ \mathrm{CVaR}_\beta \left[\frac{1}{2} \int_D S(z)^2 \, \mathrm{d}x \right] + \|z\|_1 \right\}$$

where $S(z) = u : \Omega \to H^1(D)$ solves the weak form of

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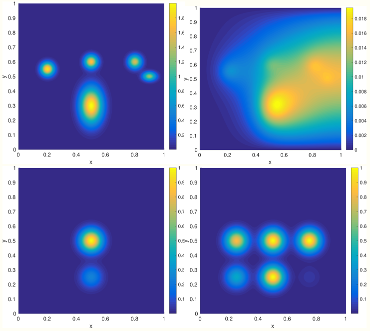

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(bottom right) with $\mathcal{R} = \mathrm{CVaR}\beta$

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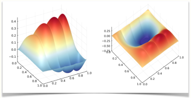
Thermal Compliance Topology Optim.[3]



$$
\begin{aligned}
-\nabla \cdot (\mathbf{E} : \epsilon u) &= F && \text{in } D \\
\epsilon u &= \tfrac{1}{2}(\nabla u + \nabla u^\top) && \text{in } D \\
u &= g && \text{on } \partial D
\end{aligned}
$$

[3] R. Bollapragada, C. Karamanli, B. Keith, B. Lazarov, S. Petrides, & J. Wang (2023). Comput. Math. Appl., 149, 239–258.

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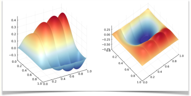

$$-\nabla \cdot (\mathbf{E}(\omega) : \epsilon u) = F(\omega) \qquad \text{in } D, \text{a.s.}$$

$$\epsilon u = \frac{1}{2}(\nabla u + \nabla u^\top) \qquad \text{in } D, \text{a.s.}$$

$$u = g(\omega) \qquad \text{on } \partial D, \text{a.s.}$$

Random inputs: Linear elastic isotropic material with uncertain Lamé coefficients $\mathbf{E}$ traction forces g bulk forces F.

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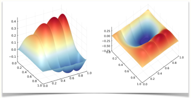

$$-\nabla \cdot (\mathbf{E}(\omega)(z)) : \epsilon u) = F(\omega) \qquad \text{in } D, \text{a.s.}$$

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The material density $z \in \mathcal{Z}_{\text{ad}}$ fulfills

$z : D \to \mathbb{R}$.

$z(x) \in [0, 1]$ a.e. on D ($z = 0$ "no material", $z = 1$ "material").

$\int_D z \, \mathrm{d}x \leq V_0|D|$ (volume fraction).

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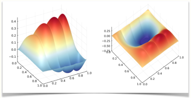

Find optimal material distribution $z^\star$ that minimizes compliance:

$$\min_{z \in \mathcal{Z}_{\text{ad}}} \mathcal{R}\left[\int_D F(\cdot) \cdot S(z)\,\mathrm{d}x\right] + \wp(z)$$

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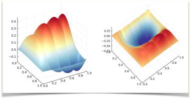

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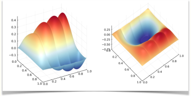
(top) Optimal density field ignoring random inputs

(bottom) optimal density field using $\mathcal{R} = \mathbb{E}$.

[3] R. Bollapragada, C. Karamanli, B. Keith, B. Lazarov, S. Petrides, & J. Wang (2023). Comput. Math. Appl., 149, 239–258.

From data to decision-making



Data
$\mathcal{D} = \{((A_i, f_i), u_i)\}$

Statistical inversion
to infer measure $\mu \in \mathcal{P}(\Omega)$

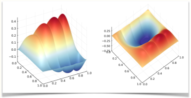
Random PDE
$\nabla \cdot (A(\omega)\, \nabla u) = f(\omega)$

Stoch. Optim.
$z^{k+1} := z^k - \gamma^k \partial \mathcal{J}(z^k, \xi^k)$

Risk-Averse Optimization
$\min_z \; \mathcal{R}\big(J(u(\cdot), \cdot; z)\big)$

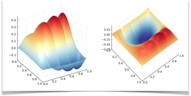
Risk models
$\mathcal{R} = \mathbb{E}_\mu, \mathrm{CVaR}_\alpha \ldots$

A Class of Risk Measures



Using $\Phi(X) = \mathbb{E}[\max\{0, X\}] = \mathbb{E}[(X)_+]$, we can define several useful risk measures:

A Class of Risk Measures

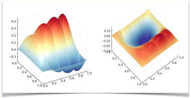


Using $\Phi(X) = \mathbb{E}[\max\{0, X\}] = \mathbb{E}[(X)_+]$, we can define several useful risk measures:

Convex combination of the expected value and CVaR

$$\mathcal{R}(X) = (1-t)\mathbb{E}[X] + t \inf_{a \in \mathbb{R}} \left\{ a + \tfrac{1}{1-\beta}\Phi(X-a) \right\}, \;\; \beta \in (0,1), \;\; t \in (0,1],$$

A Class of Risk Measures



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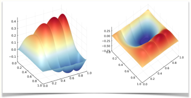
Mean-plus-semideviation-from-target of order 1

$$\mathcal{R}(X) = \mathbb{E}[X] + c\Phi(X - t), \;\; c > 0, \;\; t \in \mathbb{R},$$

A Class of Risk Measures



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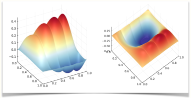
Mean-plus-semideviation of order 1

$$\mathcal{R}(X) = \mathbb{E}[X] + c\Phi(X - \mathbb{E}[X]), \quad c > 0,$$

Abstract Formulation



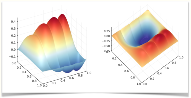
Many problems take the abstract form (with $\mathcal{X} = \mathcal{Z}$ or $\mathcal{Z} \times \mathbb{R}$, $\mathcal{X}_{\text{ad}} \subset \mathcal{X}$):

$$\min_{x \in \mathcal{X}_{\text{ad}}} \{g(x) + \Phi(G(x))\} \tag{P}$$

$G : \mathcal{X} \to \mathcal{Y}$ is a random operator (e.g. $J(S(z)(\omega))$), g is a differentiable function (e.g. $\mathbb{E}[J(S(z))]$), $(\Omega, \mathcal{F}, \mathbb{P})$ complete probability space, $\mathcal{Y} := L^2(\Omega, \mathcal{F}, \mathbb{P})$.

$\Phi : \mathcal{Y} \to \mathbb{R}$ is a functional that maps random variables into $\mathbb{R}$ (e.g. part of a risk measure)

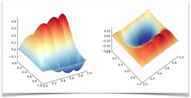
Assumptions and Observations on Φ



Assume $\Phi : \mathcal{Y} \to \mathbb{R}$ convex, positively homogeneous, monotonic wrt partial order on $\mathcal{Y}$

Example: $\Phi(Y) := \mathbb{E}[(Y)_+]$.

Assumptions and Observations on Φ



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Φ finite and convex on all of $\mathcal{Y}$ $\Rightarrow$ Φ is continuous and subdifferentiable.

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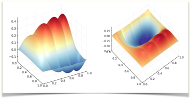

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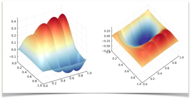
$\Phi(Y) = \Phi^{**}(Y)$ and $\Phi(Y) = \sup_{\lambda \in \partial\Phi(0)} \mathbb{E}[\lambda Y] \quad \forall Y \in \mathcal{Y}$.

Assumptions and Observations on Φ



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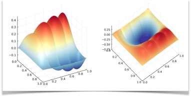
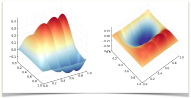
$\partial\Phi(0) := \mathfrak{A} \subseteq \{\lambda \in \mathcal{Y} \,|\, \lambda \geq 0 \text{ a.s.}\}$ is a nonempty, closed, bounded, convex "risk envelope"

Towards a Primal-Dual Algorithm



$$\min_{x\in\mathcal{X}_{\mathsf{ad}}} g(x) + \Phi(G(x)) = \min_{x\in\mathcal{X}_{\mathsf{ad}}} \sup_{\lambda\in\mathfrak{A}} g(x) + \mathbb{E}[\lambda G(x)] \quad \text{set} \quad \ell(x,\lambda) := g(x) + \mathbb{E}[\lambda G(x)].$$

Towards a Primal-Dual Algorithm



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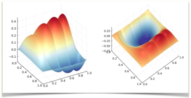
The generalized augmented Lagrangian is then

$$L(x, \lambda, r) = \max_{\theta \in \mathfrak{A}} \left\{ \ell(x, \theta) - \frac{1}{2r} \mathbb{E}[(\lambda - \theta)^2] \right\}$$

Towards a Primal-Dual Algorithm



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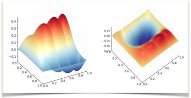
where $\Phi_{r,\lambda}(Y) = \inf_Z\{\Phi(Z) + \Psi_{r,\lambda}(Z-Y)\}$ with $\Psi_{r,\lambda}(Y) = \mathbb{E}[\lambda Y] + \frac{r}{2}\mathbb{E}[Y^2]$.

[a] See D. P. Kouri and T. M. Surowiec, *Epi-Regularization of Risk Measures* Mathematics of Operations Research 45, 2 (2020), 774–795 for more on this technique.

Towards a Primal-Dual Algorithm



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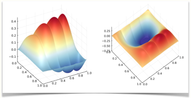
$$L(x,\lambda,r) = \max_{\theta\in\mathfrak{A}}\left\{\ell(x,\theta)-\frac{1}{2r}\mathbb{E}[(\lambda-\theta)^2]\right\} = g(x)+\Phi_{r,\lambda}(G(x)).$$

where $\Phi_{r,\lambda}(Y) = \inf_Z\{\Phi(Z)+\Psi_{r,\lambda}(Z-Y)\}$ with $\Psi_{r,\lambda}(Y) = \mathbb{E}[\lambda Y]+\frac{r}{2}\mathbb{E}[Y^2]$.

Since $\Phi_{r,\lambda}$ is an epi-regularized[a] version of Φ, it is convex, monotonic, $\mathcal{C}^1$ and Γ-converges to $\Phi(\cdot)$.

[a] See D. P. Kouri and T. M. Surowiec, *Epi-Regularization of Risk Measures* Mathematics of Operations Research 45, 2 (2020), 774–795 for more on this technique.

Towards a Primal-Dual Algorithm



$$\min_{x \in \mathcal{X}_{\text{ad}}} g(x) + \Phi(G(x)) = \min_{x \in \mathcal{X}_{\text{ad}}} \sup_{\lambda \in \mathfrak{A}} g(x) + \mathbb{E}[\lambda G(x)] \quad \text{set} \quad \ell(x, \lambda) := g(x) + \mathbb{E}[\lambda G(x)].$$

The generalized augmented Lagrangian is then

$$L(x, \lambda, r) = \max_{\theta \in \mathfrak{A}} \left\{ \ell(x, \theta) - \frac{1}{2r} \mathbb{E}[(\lambda - \theta)^2] \right\} = g(x) + \Phi_{r,\lambda}(G(x)).$$

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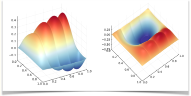
Letting $\mathfrak{P}$ be the L^2-projection onto $\mathfrak{A}$, the maximizer above is given by

$$\Lambda(x, \lambda, r) := \mathfrak{P}_{\mathfrak{A}}(rG(x) + \lambda).$$

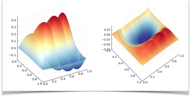
[a] See D. P. Kouri and T. M. Surowiec, *Epi-Regularization of Risk Measures* Mathematics of Operations Research 45, 2 (2020), 774–795 for more on this technique.

Towards a Primal-Dual Algorithm



This viewpoint goes back to the method of multipliers/augmented Lagrangian e.g. Hestenes 1969, Powell 1969, Rockafellar 1976.

Towards a Primal-Dual Algorithm

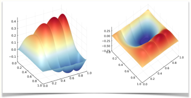


This viewpoint goes back to the method of multipliers/augmented Lagrangian e.g. Hestenes 1969, Powell 1969, Rockafellar 1976.

Solve a sequence of subproblems in x for minimizing $L(x, \lambda, r)$.

Use $\Lambda(x, \lambda, r)$ to update λ.

The PD-Risk Algorithm



Algorithm The Primal-Dual Risk Minimization Algorithm [4]

Given $x_0 \in \mathcal{X}_{\text{ad}}$, $r_0 \in (0,\infty)$, $\lambda_0 \in \mathfrak{A}$, and stationarity error: $\mathfrak{S}(x,\lambda,r) := \|x - \mathfrak{P}_{\mathcal{X}_{\text{ad}}}(x - \nabla_x L(x,\lambda,r))\|_{\mathcal{X}}$
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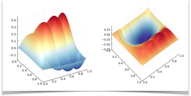

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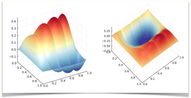

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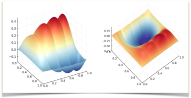

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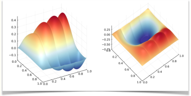

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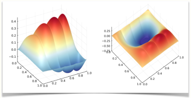

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5. Set $\tau_{x,k+1} = \rho_x \tau_{x,k}$ and $\tau_{\lambda,k+1} = \rho_\lambda \tau_{\lambda,k}$. ▷ Decrease tolerances for increased accuracy (continuation)

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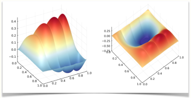
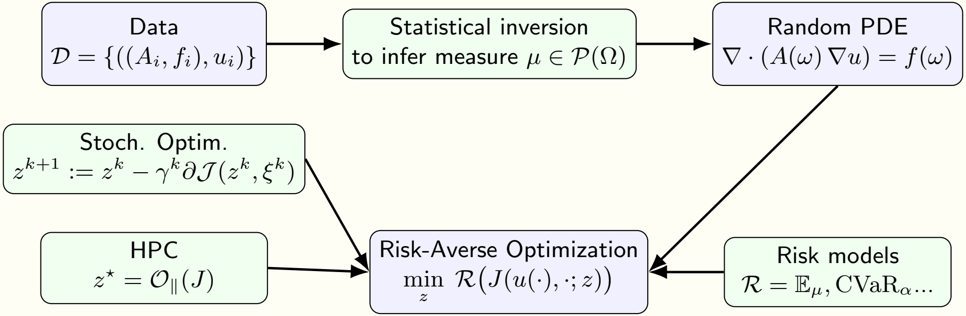
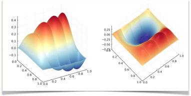
From data to decision-making





∞-Dim. Stochastic Optimization?



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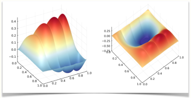

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PD-Risk (and related methods) require:

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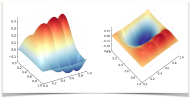

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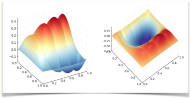

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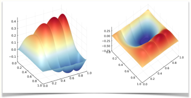

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PDE-Optimization:

Replace Z by a finite-dim. space Z_h.

$S(z)$ requires the solution of a PDE.

PDEs can be very expensive to solve.

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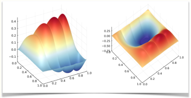

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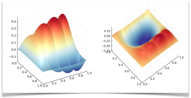
PDEs can be very expensive to solve.

Stochastic Optimization:

Cannot evaluate $S(z)$ or $S(z_h)$.

Replace $\mathbb{E}[\mathcal{J}(z)]$ by $\frac{1}{N}\sum_{i=1}^{N} \mathcal{J}(z, \xi^i)$.

Essential computations



How do we efficiently **compute gradients**?

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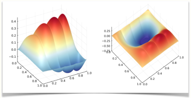
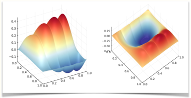


How do we efficiently **compute gradients**?

How do we efficiently **compute Hessian-vector products**?

The true **Hessian** is never available, **Hessian-vector products** suffice for second-order methods.

Computing the Gradient



Given $z \in Z$, calculate **state** $u = S(z)$ by solving $e(u, z) = 0$.

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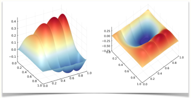

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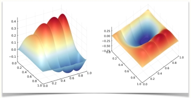

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Given z, u, λ calculate the full gradient

$$\nabla \mathcal{J}(z) = \partial_z e(u, z)^* \lambda + \partial_u J(u, z).$$

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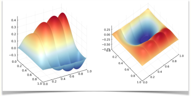

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This might require an additional solve for the true discrete gradient.

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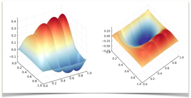

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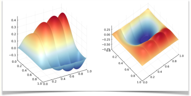

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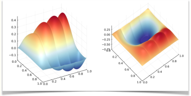
In the linear case, we only require the solution of **sparse structured linear** systems for $\nabla \mathcal{J}$.

Without **adjoints**, $\nabla \mathcal{J}$ would contain large dense matrices due to the solution operators S, P.

Parallelization in Smooth Linear Case



Computing the gradient when using Monte Carlo is largely parallelizable.

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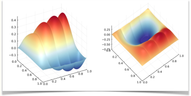

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N **states solves** (in parallel)

$$\boldsymbol{A}(\xi^i)u = \boldsymbol{B}(\xi^i)z + f(\xi^i) \text{ in } H^{-1}(D) \quad i = 1, \ldots, N,$$

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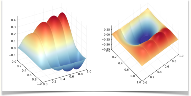

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N **adjoint solves** (in parallel, serial with i^{th} state u^i)

$$\boldsymbol{A}(\xi^i)^* \lambda = -J_u'(u(\xi^i), z) \text{ in } H^{-1}(D) \quad i = 1, \dots, N,$$

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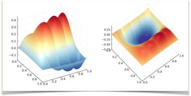

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N **adjoint solves** (in parallel, serial with i^{th} state u^i)

$$\boldsymbol{A}(\xi^i)^*\lambda = -J'_u(u(\xi^i), z) \text{ in } H^{-1}(D) \quad i = 1, \dots, N,$$

N **“matrix-vector” products**

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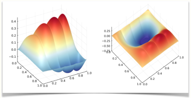
Parallelization in Smooth Linear Case



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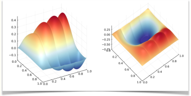

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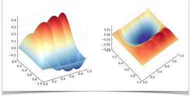
Don't forget the Riesz maps/proper inner products!

Essential computations

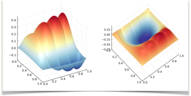


A similar computation can be made for Hessian vector products that requires:

$$N \times \ \textbf{State} \ + N \times \ \textbf{Adjoint} \ + N \times \ \textbf{State Sensitivity} \ + N \times \ \textbf{Adjoint Sensitivity}$$

solves, where each class of N solves can be made in parallel.

CVaR Minimization



We consider the following stochastic optimization problem:

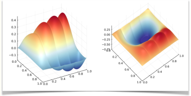
$$\min \left\{ t + \frac{1}{2(1-\beta)} \mathbb{E}\left[\left(\|S(z) - u_d\|_{L^2}^2 - t \right)_+ \right] + \frac{\alpha}{2}\|z\|_Z^2 \text{ over } (z,t) \in \mathcal{Z}_{\text{ad}} \times \mathbb{R} \right\}, \tag{1}$$

where $\mathcal{Z}_{\text{ad}} \subset Z$ is a nonempty, closed, and convex set and $S(z) = u$ is the unique solution to

$$\text{Find } u \in \mathcal{U} : \mathbb{E}\left[\int_D A\nabla u \cdot \nabla v \mathrm{d}x \right] = \mathbb{E}[\langle Bz + f, v\rangle_{U^*,U}], \quad \forall v \in \mathcal{U}.$$

where $U := H_0^1(\Omega)$, $\mathcal{U} := L^2(\Omega, \mathcal{F}, \mathbb{P}; U)$.

CVaR Minimization



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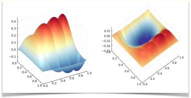
$$\boldsymbol{A}(\omega)u = \boldsymbol{B}(\omega)z + f(\omega) \text{ in } H^{-1}(D), \quad \mathbb{P}\text{-a.s. in } \Omega.$$

Is linear algebra **really** an issue?



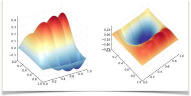
The Robust Stochastic Mirror Descent method yielded the following:

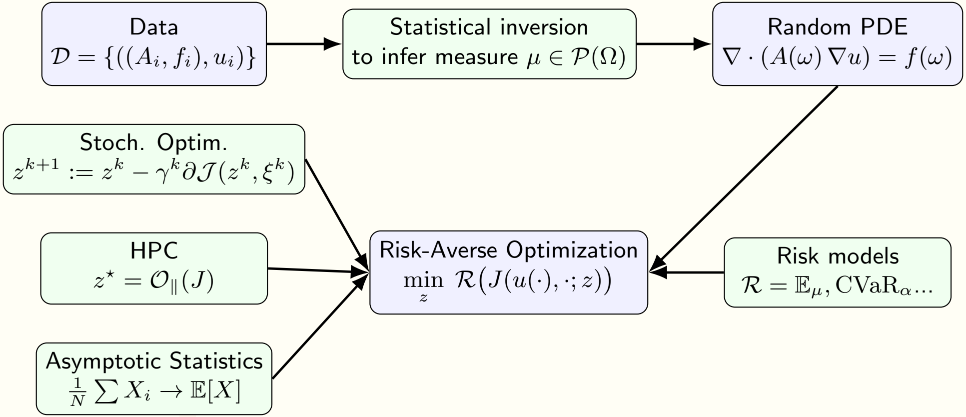
iter	time(s)	fval	abs-err f	rel-err f	abs-err x_k	rel-err x_k
100	1.4	3.1274e-01	1.3403e-01	7.4999e-01	2.7098e+02	8.3928e-01
1000	14.7	2.5017e-01	7.1464e-02	3.9989e-01	2.0949e+02	6.4885e-01
10000	152.0	2.0502e-01	2.6312e-02	1.4723e-01	1.4802e+02	4.5846e-01
100000	2054.2	1.8906e-01	1.0353e-02	5.7933e-02	1.0943e+02	3.3892e-01
1000000	104636.2	1.8411e-01	5.3994e-03	3.0213e-02	8.8822e+01	2.7510e-01

In contrast, using the PD-Risk with Monte Carlo we obtain:

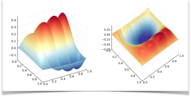
N	time(s)	fval	nstate	nadjoint	nstatesens	nadjointsens	totalsolves
100	680.0	1.7924e-01	27500	7112	36554	36554	107720
1000	2889.3	1.7871e-01	83000	30035	197168	197168	507371
10000	23540.5	1.7871e-01	700000	268612	1594077	1594077	4156766

From data to decision-making





A More *Statistical* Numerical Analysis



Are solutions and optimal values stable with respect to shifts in distribution?

Can this stability be quantified?

What happens asymptotically in the “big data” limit?

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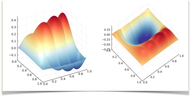

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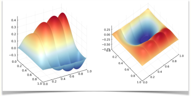

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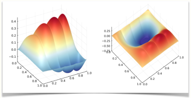
Fundamentally different questions to traditional numerical analysis in PDEs and optimal control:

FEM: finite dimensional spaces replace ∞-dimensional ones, but we always use the Lebesgue measure.

Mesh refinement increases the dimension of these spaces, but refinements are not random.

Defining Stability[5]



Optimal Value: $$\nu(P) = \inf_{z \in \mathcal{Z}_{\mathrm{ad}}} \int_\Omega f(z,\omega)\, \mathrm{d}P(\omega).$$

Optimal Solutions: $$z(P) \in \operatorname*{argmin}_{z \in \mathcal{Z}_{\mathrm{ad}}} \int_\Omega f(z,\omega)\, \mathrm{d}P(\omega)$$

[5] See J. Milz, T.M. Surowiec *Asymptotic consistency for nonconvex risk-averse stochastic optimization with infinite-dimensional decision spaces* Mathematics of Operations Research 49 (3), 1403-1418 (2024) for results on the risk averse case.

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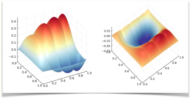

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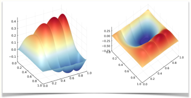

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How does ν or z change, when we replace P by Q?

If $Q := P_N$ such that $P_N \Rightarrow P$, does it hold that

$$\nu(P_N) \to \nu(P) \text{ and } \|z(P_N) - z(P)\|_{\mathcal{Z}} \to 0?$$

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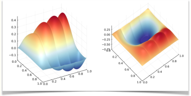
A (Very) Brief History



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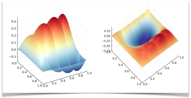

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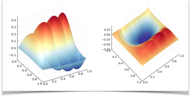

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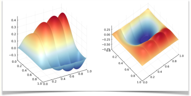

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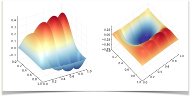
Our Core Framework: Based on the metric approach to studying optimal solutions under varying probability laws, particularly following Rachev & Römisch (2002).

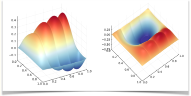
Personal Contributions



- M. Hoffhues, W. Römisch, T.M. Surowiec
  On quantitative stability in infinite-dimensional optimization under uncertainty
  Optim Lett 15, 2733–2756 (2021). https://doi.org/10.1007/s11590-021-01707-2
- W. Römisch, T.M. Surowiec
  Asymptotic properties of Monte Carlo methods in elliptic PDE-constrained optimization under uncertainty
  Numer. Math. 156, 1887–1914 (2024). https://doi.org/10.1007/s00211-024-01436-5

Figure: Werner Römisch, 28.12.1947 - 12.7.2024

Abstract Setting



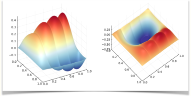
Decision variables are taken from $\mathcal{Z}$, an infinite dimensional Hilbert space.

We assume the randomness arises from a random element $\xi(\omega)$ and work with the law of ξ.

Ξ is a metric space, $P \in \mathcal{P}(\Xi)$ the set of all Borel probability measures.

The probability measures themselves are the exogenous parameters.

A Class of Random Operator Equations



We derive a class of integrands $f : \mathcal{Z} \times \Xi \to \mathbb{R}$ to motivate the general framework.

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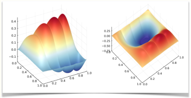

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Our random operator equation will be written in the form

$$\boldsymbol{A}(\xi)u = z + g(\xi) \quad (P\text{-a.e. } \xi \in \Xi).$$

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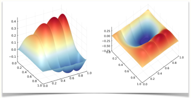

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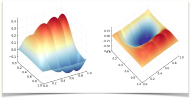

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$$\underbrace{\gamma \|v\|_V^2 \leq \langle A(\xi)v, v\rangle \quad \forall v \in V}_{\text{uniform coercivity}} \quad \text{and} \quad \underbrace{\langle A(\xi)v, w\rangle \leq L\|v\|\|w\| \quad \forall v, w \in V}_{\text{uniform boundedness}}$$

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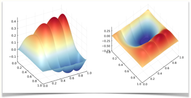

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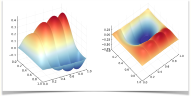
$A : \Xi \to \mathcal{L}(V, V^*)$ is measurable and essentially bounded.

$g : \Xi \to V^*$ is measurable and essentially bounded (or more regular).

Linear Quadratic Risk-Neutral Problems



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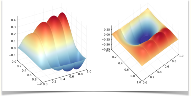

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$\boldsymbol{A}(\xi)^{-1} : V^\star \to V$ exists and is positive definite (with $\frac{1}{L}$) and bounded (with $\frac{1}{\gamma}$).

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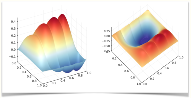

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We consider the integrand inspired by PDE-constrained optimization:

$$f(z,\xi) \quad = \quad \frac{1}{2}\left\|\boldsymbol{A}(\xi)^{-1}(z - g(\xi)) - u_d\right\|_H^2 + \frac{\alpha}{2}\|z\|_{\mathcal{Z}}^2 \quad (\alpha > 0, u_d \in H)$$

for $z \in \mathcal{Z}_{\mathrm{ad}}$ (closed, bounded, convex set) and $\xi \in \Xi$.

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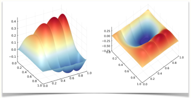

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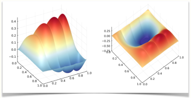
for $z \in \mathcal{Z}_{\mathrm{ad}}$ (closed, bounded, convex set) and $\xi \in \Xi$.

This leads to the risk-neutral optimization problem:

$$\min \left\{ \mathbb{E}_P[f(z)] = \int_\Xi f(z,\xi)\mathrm{d}P(\xi) : z \in \mathcal{Z}_{\mathrm{ad}} \right\}.$$

A Growth Condition



Lemma (Hoffhues, Römisch, Surowiec (2021))

For each $P \in \mathcal{P}(\Xi)$ *and any* $z \in \mathcal{Z}_{\mathrm{ad}}$ *we have*

$$\|z - z(P)\|_{\mathcal{Z}}^2 \leq \frac{8}{\alpha}(\mathbb{E}_P[f(z)] - v(P))$$

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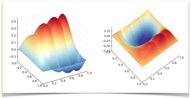

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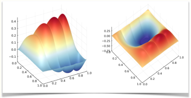
$$\|z - z(P)\|_{\mathcal{Z}}^2 \leq \frac{8}{\alpha}(\mathbb{E}_P[f(z)] - v(P))$$

For any two $P, Q \in \mathcal{P}(\Xi)$ we immediately deduce

$$\begin{aligned}\|z(Q) - z(P)\|_{\mathcal{Z}}^2 \leq \frac{4}{\alpha}\Big[&\mathbb{E}_P[f(z(Q))] - \mathbb{E}_Q[f(z(Q))] \\ &+ \mathbb{E}_Q[f(z(P))] - \mathbb{E}_P[f(z(P))]\Big]\end{aligned}$$

A Growth Condition



Lemma (Hoffhues, Römisch, Surowiec (2021))

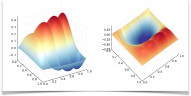
For each $P \in \mathcal{P}(\Xi)$ *and any* $z \in \mathcal{Z}_{\mathrm{ad}}$ *we have*

$$\|z - z(P)\|_{\mathcal{Z}}^2 \leq \frac{8}{\alpha}(\mathbb{E}_P[f(z)] - v(P))$$

For any two $P, Q \in \mathcal{P}(\Xi)$ we immediately deduce

$$\|z(Q) - z(P)\|_{\mathcal{Z}}^2 \leq \frac{4}{\alpha}\Big[\mathbb{E}_P[f(z(Q))] - \mathbb{E}_Q[f(z(Q))] + \mathbb{E}_Q[f(z(P))] - \mathbb{E}_P[f(z(P))]\Big] \Leftarrow \text{Difference of expectations}$$

Zolotarev's ζ-distances

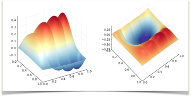


Zolotarev ζ-distance on $\mathcal{P}(\Xi)$ (Zolotarev 83):

$$d_{\mathfrak{F}}(\mathbb{P},\mathbb{Q}) = \sup_{f\in\mathfrak{F}} |\mathbb{E}_P[f] - \mathbb{E}_Q[f]| \Leftarrow \text{A difference of expectations!}$$

where $\mathfrak{F}$ is a family of real-valued Borel measurable functions on Ξ and $\mathbb{P},\mathbb{Q} \in \mathcal{P}(\Xi)$.

Probability Metrics



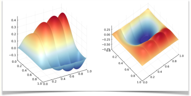
Compared with classical probability metrics we consider a much smaller family $\mathfrak{F}$:

$$\mathfrak{F}_{\mathrm{mi}} = \{f(z,\cdot) : z \in \mathcal{Z}_{\mathrm{ad}}\} \quad \text{(minimal information (m.i.) family).}$$

This leads to a pseudometric[a]: the minimal information (m.i.) distance $d_{\mathfrak{F}_{\mathrm{mi}}}$.

[a] Distance between distinct points can be zero.

Probability Metrics



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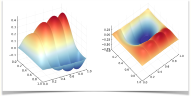
[a] Distance between distinct points can be zero.

Rachev & Römisch: Use the right metric for your problem class!

Wasserstein W_1 is a ζ-distance, but $\mathfrak{F}$[a] is too rich and would give worse convergence rates.

[a] set of all 1-Lipschitz functions on Ξ

Lipschitz-type Estimates for $d_{\mathfrak{F}_{\mathrm{mi}}}$



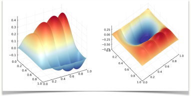
Theorem (Hoffhues, Römisch, Surowiec (2021))

Under the standing assumptions we obtain the estimates

$$
\begin{aligned}
|v(Q) - v(P)| &\leq d_{\mathfrak{F}_{mi}}(P, Q) \\
\|z(Q) - z(P)\|_{\mathcal{Z}} &\leq 2\sqrt{\frac{2}{\alpha} d_{\mathfrak{F}_{mi}}(P, Q)^{\frac{1}{2}}}
\end{aligned}
$$

for P and $Q \in \mathcal{P}(\Xi)$.

A Refined Estimate



Theorem (Römisch, Surowiec (2024))

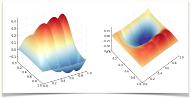
Under the standing assumptions the Lipschitz-type estimate

$$\|z(Q) - z(P)\|_H \leq \frac{8}{\alpha} d_{\mathfrak{F}_{di}}(P, Q)$$

holds for all $P, Q \in \mathcal{P}(\Xi)$, *where* $\mathfrak{F}_{di}$ *denotes the following function class on* Ξ

$$\mathfrak{F}_{di} = \{\langle \partial_z f(z, \cdot), h \rangle_H : z \in Z_{\mathrm{ad}}, \|h\|_H \leq 1\}.$$

A Refined Estimate



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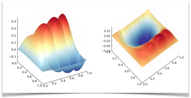
Quantifies changes under shifts in the underlying distribution.

Nonasymptotic, what about when $Q = P_N$ with $N \to +\infty$?

A Key Property for Asymptotics



Need $\mathfrak{F}$ to be a $\mathbb{P}$-uniformity class[a], i.e. $(\mathbb{P}_N)$, $\mathbb{P} \in \mathcal{P}(\Xi)$, weak conv. of $(\mathbb{P}_N)$ to $\mathbb{P}$ implies

$$\lim_{N \to \infty} d_{\mathfrak{F}}(\mathbb{P}_N, \mathbb{P}) = 0.$$

[a] Can be guaranteed by, e.g., Topsøe (1967) under uniform boundedness and equicontinuity conditions

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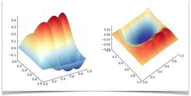

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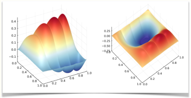
The standing assumptions imply both $\mathfrak{F}_{mi}$ and $\mathfrak{F}_{di}$ are $\mathbb{P}$-uniformity classes.

Monte Carlo Approximation



For an iid random sample $\{\xi^i\}_{i=1}^n$ with law P, the empirical measure $P_n(\cdot)$ is a perturbation of P.

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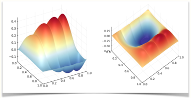

For an iid random sample $\{\xi^i\}_{i=1}^n$ with law P, the empirical measure $P_n(\cdot)$ is a perturbation of P.

By the LLN $(P_n(\cdot)) \Rightarrow P$ $\mathbb{P}$-almost surely.

Since $\mathfrak{F}_{mi}$, $\mathfrak{F}_{di}$ are P-uniformity classes, it holds that

$$v(P_n(\cdot)) \to v(P) \text{ and } z(P_n(\cdot)) \to z(P) \qquad \mathbb{P}\text{-a.s.}$$

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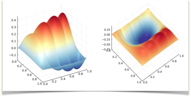

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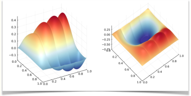
P-uniformity (consistency) and stability implies convergence of the estimators![a]

[a] Cf. Lax-Richtmyer (1956) Equivalence Theorem: Stability + Consistency $\Rightarrow$ Convergence for numerical PDEs!

Rate of Convergence and CLT



Theorem (Römisch, Surowiec (2024))

$\Xi \subset \mathbb{R}^d$ *is bounded, convex set and* $\Xi \subseteq \mathrm{cl\,int}\,\Xi$

$A(\cdot)$, $g(x,\cdot)$ *have continuous partial derivatives up to order* k[a], *with bounded, measurable derivatives,*

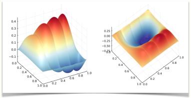
$k \in \mathbb{N}$ *satisfies* $d < 2k$.

Then:

1. $\mathbb{E}_{\mathbb{P}}[|v(P_n(\cdot)) - v(P)|] = O(n^{-\frac{1}{2}})$,
2. $\mathbb{E}_{\mathbb{P}}[\|z(P_n(\cdot)) - z(P)\|_H] = O(n^{-\frac{1}{2}})$,
3. $(\sqrt{n}(v(P_n(\cdot)) - v(P))) \rightsquigarrow \mathcal{N}(0, P(f(z(P)))^2)$.

[a] A is generated by a second-order elliptic operator with coefficient functions $a_{ij}(x,\xi)$ exibiting the requisite smoothness.

A Random Elliptic Operator



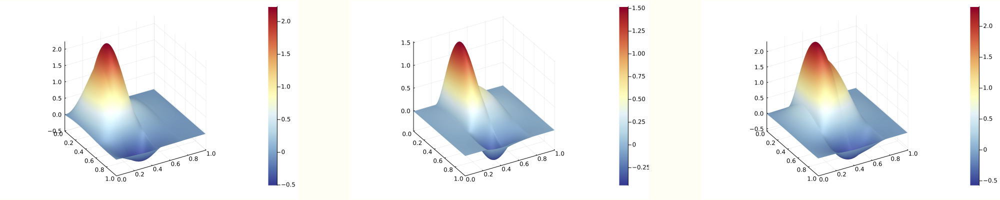
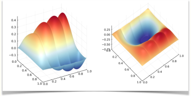


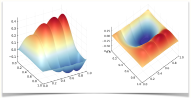
Figure: Uncontrolled, random states: Three realizations of $u(\xi)$ computed by setting $z \equiv 0$.

Convergence Rate Validation



Reference Solution: Compute $v(P_n)$ and $z(P_n)$ with sample size $n = 500$ and FEM for function spaces.

Convergence Rate Validation



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Test Runs: Select a range of smaller sample sizes $m = 1, \ldots, 100$.

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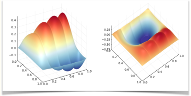

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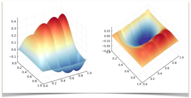

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Metrics: Compute the error against the reference solution P_n:

$$\|z(P_{m,j}) - z(P_n)\|_{L^2} \text{ and } |v(P_{m,j}) - v(P_n)|.$$

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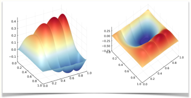

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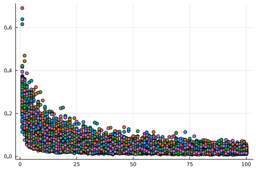
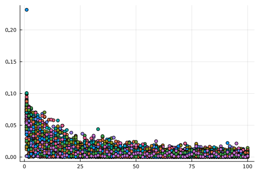
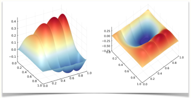


Solution Error (Control)



Value Error

Convergence Rate Validation



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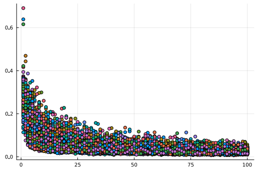

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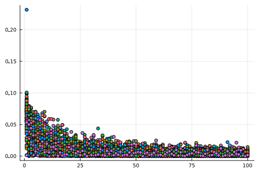
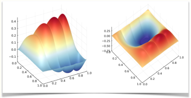


Value Error

Empirical Convergence Rates:

- **Optimal Solution (z)**: $O(m^{-0.536})$
- **Optimal Value (v)**: $O(m^{-0.660})$

Using the CLT

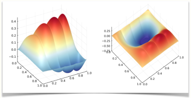


We know the estimated optimal value $v(P_n)$ converges asymptotically to the true value $v(P)$:

$$\sqrt{n}(v(P_n) - v(P)) \xrightarrow{d} \mathcal{N}(0, P(f(z(P)))^2)$$

where $P(f(z(P)))^2$ is the asymptotic variance.

Using the CLT



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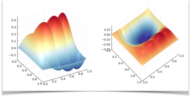
The practical limiations are

- The true optimal value, $v(P)$, is **unknown**.
- The asymptotic variance, $P(f(z(P)))^2$, is **unknown**.
- We cannot directly use the CLT to construct confidence intervals.

Using the CLT



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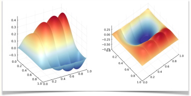
We remedy this with subsampling, which provides an asymptotically equivalent distribution that **mimics the true limiting distribution** $\mathcal{N}(0, P(f(z(P)))^2)$.

The Subsampling Mechanics



The Idea: Right Distribution, Wrong Size

- **Traditional Bootstrap** $\rightarrow$ Uses size n (right size), often inconsistent.
- **Subsampling** $\rightarrow$ Uses size $b \ll n$ (wrong size), achieves the **right limiting distribution**.

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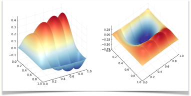

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- **Subsample Size**: Choose b, such that $b \rightarrow \infty$ but $\frac{b}{n} \rightarrow 0$. (sample without replacement)
- **Iteration Count**: m replicates, $m \rightarrow \infty$.
- For $j = 1, \ldots, m$, draw subsample, compute $v(P_n^*(N_j^{n,b}))$, optimal value for subsample.

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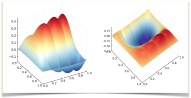

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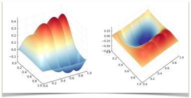
The distribution of the subsampled statistic converges to the desired asymptotic distribution:

$$\sqrt{b}(v(P_n^*(N_j^{n,b})) - v(P_n)) \xrightarrow{d} \mathcal{N}(0, P(f(z(P)))^2)$$

Practical Application



Using $L_{n,b}(\cdot)$, the empirical CDF of the m subsample statistics, $(1-\alpha)$ confidence interval for $v(P)$:

$$\mathsf{CI}_{1-\alpha}(v(P)) = \left[v(P_n) - n^{-1/2}L_{n,b}^{-1}(1-\alpha/2), \quad v(P_n) - n^{-1/2}L_{n,b}^{-1}(\alpha/2)\right]$$

$v(P_n)$ provides the center; $n^{-1/2}$ scales the empirical quantiles $L_{n,b}^{-1}(\cdot)$.

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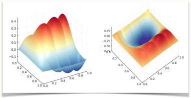

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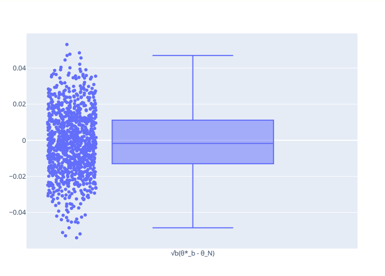
$v(P_n)$ provides the center; $n^{-1/2}$ scales the empirical quantiles $L_{n,b}^{-1}(\cdot)$.

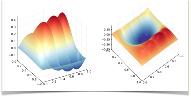
Parameters Used: $n = 2000$, $b = 1000$, $m = 1000$
Confidence Level: 95% $(\alpha = 0.05)$ yields

$$\mathbf{CI}_{95\%} = [0.089148, \quad 0.090720]$$

Validation Check: The 95% CI was found to capture the true optimal value in 84 out of 100 runs.



Conclusion

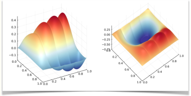


Key Scientific Achievements

Modeling $\longrightarrow$ Algorithms $\longrightarrow$ Implementation $\longrightarrow$ Stability

Open Challenges & Future Directions

Conclusion



Key Scientific Achievements

Modeling $\rightarrow$ Algorithms $\rightarrow$ Implementation $\rightarrow$ Stability

PD-Risk: A robust primal-dual framework for **non-smooth risk-averse optimization** in ∞-dimensions.

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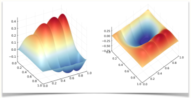

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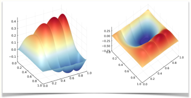

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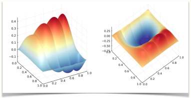

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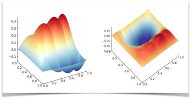

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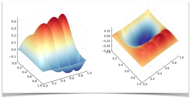

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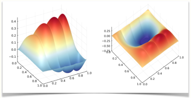

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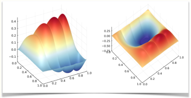

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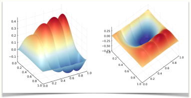

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Thank You!

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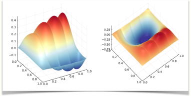

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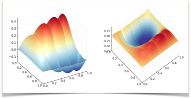

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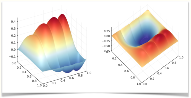

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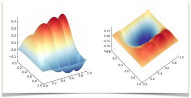

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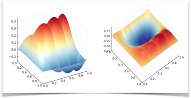

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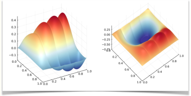

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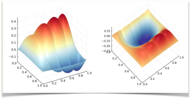

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Over $+$100K runs with random data, SSN converges in less than 10 iterations on average.