

Random dynamics on complex surfaces.

Romain Dujardin

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Journée d'accueil des post-doctorant(e)s IDF

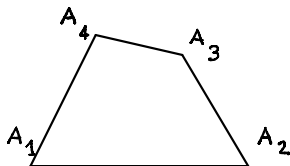
Plan

1. Folding quadrilaterals and pentagons
2. Generalities on random dynamical systems and classical examples
3. Some results

Based on a series of papers in collaboration with **Serge Cantat** (Rennes)

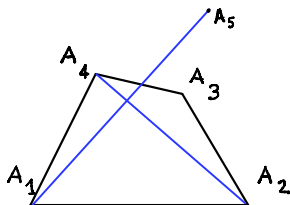
Folding plane quadrilaterals

Start with a plane quadrilateral $A_1A_2A_3A_4$ with generic side lengths.



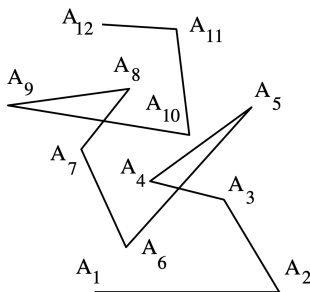
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To analyse this dynamics it is natural to consider the map obtained after 4 iterations (cyclic folding map)

$$\varphi : q_1 = A_1 A_2 A_3 A_4 \longmapsto q_2 = A_5 A_6 A_7 A_8$$

The dynamics of this map was (almost) [completely described by Benoist and Hulin \(2004\)](#).

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Observation

φ preserves side lengths, so it induces a self-map on

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A first step of the study of this iteration is to study the shape of the successive quadrilaterals without taking care of their location in the plane, so we introduce the induced maps

$$\dot{\varphi} : \text{Quad}(\ell_1, \ell_2, \ell_3, \ell_4) / \text{translations} \hookrightarrow$$

$$\ddot{\varphi} : \text{Quad}(\ell_1, \ell_2, \ell_3, \ell_4) / \text{Isom}^+(\mathbb{R}^2) \hookrightarrow .$$

Folding plane quadrilaterals

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$$\text{Quad}^0(\ell_1, \ell_2, \ell_3, \ell_4) = \text{Quad}(\ell_1, \ell_2, \ell_3, \ell_4) / \text{Isom}^+(\mathbb{R}^2).$$

and note that

$$\text{Quad}^0(\ell_1, \ell_2, \ell_3, \ell_4) \simeq \text{quadrilaterals with } A_1 = (0, 0) \text{ and } A_2 = (\ell_1, 0).$$

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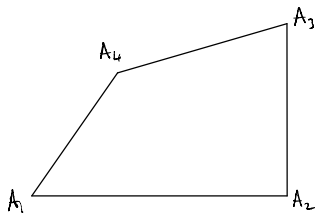
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Consider the map $\psi : A_1 A_2 A_3 A_4 \mapsto A_1 A_2 A'_3 A'_4$ on Quad^0 obtained by folding along two successive diagonals (composition of two involutions). Actually $\tilde{\varphi} = \psi^2$.



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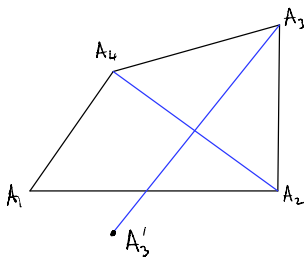
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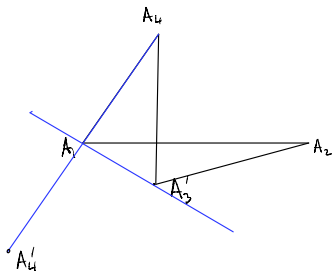
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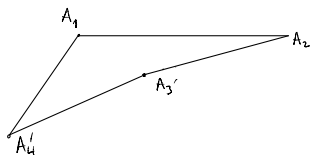
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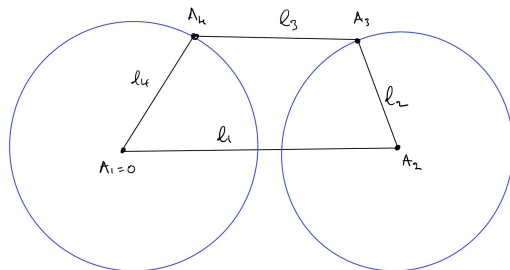
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Recall $\psi : \text{Quad}^0(\ell_1, \ell_2, \ell_3, \ell_4) \hookrightarrow$. What is this space?

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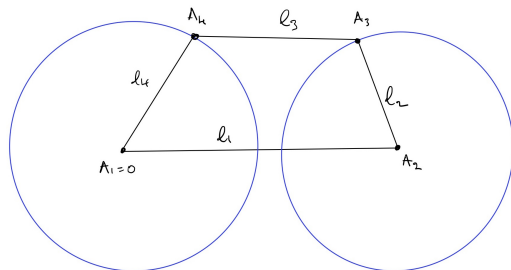
Fact: it is 1-dimensional.



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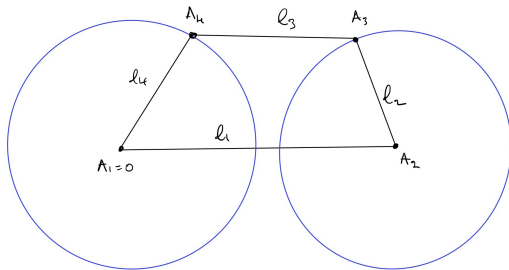


More delicate fact: It can have one or two components.

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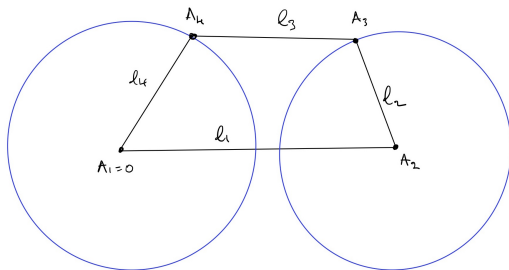


: not possible by continuous deformation.

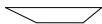
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DE L'EMPLOI DES FONCTIONS ELLIPTIQUES DANS LA THÉORIE DU QUADRILATÈRE PLAN;

PAR M. G. DARBOUX.

La théorie des systèmes articulés, qui doit son origine à la belle découverte du colonel Peaucellier, et à laquelle les travaux récents des géomètres anglais ont donné une réelle importance, repose tout entière sur la considération de polygones dont les angles changent, mais dont les côtés conservent des dimensions invariables. On a donc été conduit à considérer les figures géométriques sous un point de vue nouveau et à étudier les relations auxquelles peut donner naissance la déformation d'une figure polygonale dont les différents côtés conservent toujours leur grandeur et peuvent être assimilés à des tiges solides articulées les unes aux autres. Je me propose de traiter, dans ce travail, le plus simple des polygones articulés, le quadrilatère, et de mettre en évidence les ressources que peut offrir la théorie des fonctions elliptiques dans la recherche des propriétés géométriques et des relations entre les angles, les côtés et les diagonales du quadrilatère.

Considérons un quadrilatère, dont nous désignerons les côtés par a, b, c, d , et supposons qu'un mobile ait parcouru les différents côtés du quadrilatère en marchant toujours dans le même sens. Appelons $\omega_1, \omega_2, \omega_3, \omega_4$ les angles que font les différents côtés, pris dans le sens où ils ont été parcourus par le mobile, avec une droite quelconque du plan. On aura les deux équations

$$(1) \quad \begin{cases} ae^{i\omega_1} + be^{i\omega_2} + ce^{i\omega_3} + de^{i\omega_4} = 0, \\ ae^{-i\omega_1} + be^{-i\omega_2} + ce^{-i\omega_3} + de^{-i\omega_4} = 0, \end{cases}$$

qui contiennent la théorie complète du quadrilatère et qui donnent naissance à toutes les relations existant entre les angles des différents côtés. Or, si l'on pose

Darboux alternative

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Theorem (Darboux 1879)

If $\ell_1 \pm \ell_2 \pm \ell_3 \pm \ell_4 \neq 0$,
 $\text{Quad}^0(\ell_1, \ell_2, \ell_3, \ell_4)$ is a real elliptic curve.

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The involutions induced by the reflexions and ψ are algebraic automorphisms so they respect the group structure.

Darboux alternative

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quadrilatère $(dcab)$, dont les deux diagonales sont (a, b) , (a, c) ; supposons que, prenant l'un des deux triangles qui forment le quadrilatère, par exemple le triangle formé des côtés a, b , on le fasse tourner de 180° autour de la diagonale (a, b) : on aura une nouvelle position du quadrilatère dans laquelle la diagonale (a, b) aura conservé sa grandeur, mais l'autre diagonale (a, c) aura changé. Opérons maintenant de même autour de (a, c) , c'est-à-dire faisons tourner de 180° autour de (a, c) l'un des deux triangles dans lesquels cette diagonale partage le quadrilatère. En répétant indéfiniment ces opérations, on formera une suite illimitée de quadrilatères telle que deux quadrilatères consécutifs aient toujours une diagonale commune. Cela posé, je dis que, si pour une certaine valeur des angles du quadrilatère primitif cette suite illimitée est périodique, c'est-à-dire comprend seulement un nombre limité de formes distinctes, il en sera de même quand on répétera les mêmes opérations sur le quadrilatère initial déformé d'une manière quelconque.

En effet, désignons par $Q_1, Q_2, Q_3, \dots$ les quadrilatères obtenus successivement; appelons v_n, w_n les valeurs de v, w correspondantes au quadrilatère Q_n . On aura

$$(26) \quad v_n - w_n = h_1.$$

Pour les quadrilatères Q_1, Q_2 , et en général pour les quadrilatères Q_{2p+1}, Q_{2p+2} , la diagonale (a, b) a la même valeur. On a donc

$$\sin w_{2p+1} = \sin w_{2p+2},$$

et par conséquent, à un multiple près de $4K$, on a

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Au contraire, pour les quadrilatères Q_2, Q_3 , et en général Q_{2p}, Q_{2p+1} , la diagonale (a, c) a la même valeur. On a donc

$$(28) \quad \sin v_{2p} = \sin v_{2p+1},$$

ce qui donne

$$v_{2p} + v_{2p+1} = 2K.$$

Les égalités (26), (27), (28) déterminent les valeurs successives

Bull. des Sciences mathém., 1^{re} Série, t. III. (Mars 1879.)

Consequence: ψ is a translation.

The dynamics of a translation on $\mathbb{R}/\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is easy to describe: it is either periodic or all its orbits are dense (possibly on one component of the curve).

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The dynamics of a translation on $\mathbb{R}/\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is easy to describe: it is either periodic or all its orbits are dense (possibly on one component of the curve). So we get:

Darboux alternative

Fix ℓ_i s.t. $\ell_1 \pm \ell_2 \pm \ell_3 \pm \ell_4 \neq 0$.

If for some $q_0 \in \text{Quad}^0(\ell_1, \ell_2, \ell_3, \ell_4)$, $\psi^n(q_0)$ is periodic, then $\psi^n(q)$ is periodic for all q . Otherwise $\psi^n(q)$ is dense for all $q \in \text{Quad}^0$

This completely describes the dynamics of ψ .

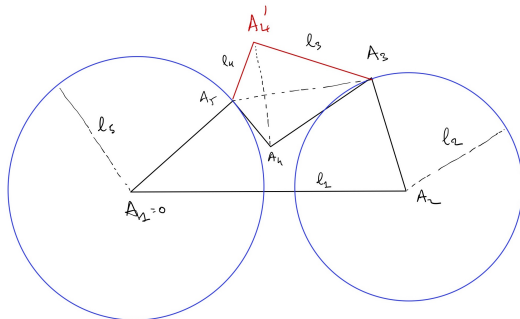
Folding plane pentagons

- Given $(\ell_i)_{i=1}^5$ such that $\sum \pm \ell_i \neq 0$ we now define:

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$$\text{and } \text{Pent}^0(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5) = \text{Pent}(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5) / \text{Isom}^+(\mathbb{R}^2).$$

- We have several involutions σ_{ij} folding the pentagon along one of its diagonals.



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Problem

Describe the dynamics of the group generated by these involutions.

Folding plane pentagons

- ▶ $X = \text{Pent}^0(\ell_i)$ is now a real algebraic **surface**. (e.g. it is defined by equations like $\|A_1 - A_2\|^2 = \ell_1^2$, etc.)
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- ▶ The involutions σ_{ij} are algebraic (birational) mappings on X .
- ▶ Imitating Darboux' calculations show that X embeds as a smooth quartic surface in $\mathbb{P}^3$, so it is a **K3 surface**.
- ▶ As a consequence the σ_{ij} are algebraic diffeomorphisms (automorphisms) on X and preserve a smooth volume form. (This follows from the K3 structure)

Conclusion of first part

This will be the **general dynamical setting** of this talk: the dynamical study of a group Γ of algebraic diffeomorphisms of a real or complex projective surface X , preserving a smooth volume form.

Basic problem

Describe the dynamics of Γ on X .

We will approach this problem by using tools from **random dynamics**, **ergodic theory** and **holomorphic dynamics**.

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- ▶ Classification of orbit closures $\overline{\Gamma \cdot x}$ for $x \in X$.
- ▶ Description of finite orbits:

$$\text{Per}(\Gamma) = \{x \in X, \Gamma \cdot x \text{ is finite}\} \quad \text{“}\Gamma\text{-periodic points”}.$$

Outline

What do we want to know?

- ▶ Classification of orbit closures $\overline{\Gamma \cdot x}$ for $x \in X$.
- ▶ Description of finite orbits:

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- ▶ Asymptotic distribution of orbits.

Probabilistic approach:

Introduce a random walk $\omega = (\omega_n)$ on Γ and study the asymptotic behavior of $(\omega_n \cdot x)$ for generic ω and x .

Random dynamical systems

- ▶ Setting of (iid) random dynamics is the following: let X be some (say) compact metric space and ν be a probability measure on $\mathcal{C}(X, X)$. Consider a iid sequence of mappings $(f_n)_{n \geq 0}$ with distribution ν .

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- ▶ The general purpose of random dynamics is to describe the sample paths

$$f_n \circ \cdots \circ f_0(x) \text{ for } x \in X \text{ and } (f_n) \in \text{Supp}(\nu)^{\mathbb{N}}.$$

(= trajectories of the random walk induced by ν on X).

We will consider this problem from the point of view of ergodic theory.

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- ▶ An important idea is that many properties of the RDS (X, ν) often only depend on those of the **closed group generated by $\text{Supp}(\nu)$** :

$$\Gamma_\nu = \langle \text{Supp}(\nu) \rangle \leq \text{Diff}(X).$$

Invariant and stationary (probability) measures

- ▶ $\mu \in \text{Proba}(X)$ is ν -invariant if for every $f \in \text{Supp}(\nu)$, $f_*\mu = \mu$.

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Fact

Stationary measures always exist (fixed point of $\mu \mapsto \int f_*\mu d\nu(f)$).
Invariant measures are more rigid and typically don't.

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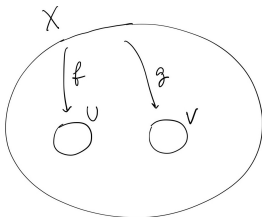
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Fact

Stationary measures always exist. Invariant measures are more rigid and typically don't.

Typical obstruction: contraction property.



Random orbits and stationary measures

- ▶ For every $x \in X$, every cluster value of

$$\nu^{(n)} * \delta_x := \int \delta_{f_n \circ \dots \circ f_1(x)} d\nu(f_1) \cdots d\nu(f_n)$$

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For every $x \in X$, for $\nu^{\mathbb{N}}$ -a.e. (f_n) , every cluster value of

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Random orbits and stationary measures

General philosophy

understanding stationary measures $\Rightarrow$ understanding random orbits.

(not completely correct)

$\times 2 \times 3$ on the circle (Furstenberg 1966)

Example 1:

View the circle as $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Consider $f_2, f_3 : \mathbb{T} \rightarrow \mathbb{T}$ defined by

$$f_2 : x \mapsto 2x \bmod 1 \text{ and } f_3 : x \mapsto 3x \bmod 1$$

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Theorem (Furstenberg)

If x is irrational, its orbit is dense in $\mathbb{T}$.

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- ▶ Finite orbits carry invariant measures, and $\text{Leb}_{\mathbb{T}}$ is invariant.

Conjecture (Furstenberg's $\times 2 \times 3$ problem)

The only diffuse invariant measure is Lebesgue.

(still open!)

Products of 2×2 random matrices

Example 2

View the circle as the space $\mathbb{P}^1(\mathbb{R})$ of lines in $\mathbb{R}^2$. Let ν be a finitely supported probability measure on $\mathrm{SL}(2, \mathbb{R})$. This defines a RDS by Möbius transformations on the circle.

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If Γ_ν is a non-elementary subgroup of $\mathrm{SL}(2, \mathbb{R})$ then there exists a **unique** ν -stationary measure μ on $\mathbb{P}^1(\mathbb{R})$. Moreover μ is **not** invariant.

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► We see in this example that stationary measures can satisfy some **rigidity properties** (cf. uniqueness).

Linear actions on tori

Example 3

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- ▶ $\mathrm{Leb}_{\mathbb{T}^2}$
- ▶ $\forall x \in \mathbb{Q}^2 / \mathbb{Z}^2$, its orbit $\mathrm{SL}(2, \mathbb{Z}) \cdot x$ is a finite set. So there is a ν -invariant measure on $\mathrm{SL}(2, \mathbb{Z}) \cdot x$.

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Theorem (Bourgain, Furman, Lindenstrauss, Mozes (2008))

If Γ_ν is not elementary then all ergodic invariant measures are of this form. Furthermore every ν -stationary measure is ν -invariant.

This last property is called **ν -stiffness** and was conjectured by Furstenberg (1996). (cf. also Furstenberg $\times 2 \times 3$ problem). The proof is difficult and based on Fourier analysis and additive combinatorics.

Further developments

- ▶ Benoist and Quint later gave an ergodic theoretic proof of the BFLM theorem, which proves stiffness for some “linear” RDS on homogeneous spaces like $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$.

It also implies:

Theorem (Benoist, Quint (2011))

For every $x \notin \mathbb{Q}^2/\mathbb{Z}^2$, a.s. $\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f_n \dots f_1(x)} \rightarrow \mathrm{Leb}_{\mathbb{T}^2}$.

Further developments

- ▶ Brown and Rodriguez-Hertz built on the techniques of Benoist-Quint and Eskin-Mirzakhani to extend this theory to some non-linear actions by C^2 diffeos on compact surfaces.
- ▶ More recent work by Brown, Eskin, Filip, Rodriguez-Hertz (2025) partially extends this to higher dimension.

This implies for instance:

Theorem (Brown, Rodriguez-Hertz (2015))

Assume $\nu_0 \in \text{Proba}(\text{SL}(2, \mathbb{Z}))$ is finitely supported and Γ_{ν_0} is non-elementary. Let ν be a small perturbation of ν_0 in $\text{Diff}_{\text{Leb}}^2(\mathbb{T}^2)$. Then every ν -stationary measure on $\mathbb{T}^2$ is ν -invariant. Furthermore if ν is ergodic it is either finitely supported or Lebesgue

Our setting

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- ▶ This assumption was later removed by Megan Roda (2024): can work directly over X .
- ▶ We also often assume that there is a smooth Γ -invariant volume on X (resp. $X(\mathbb{R})$).

Examples

- ▶ Examples of such X with a large group of automorphisms are rare and rigid: need a recipe. ($\text{Aut}(X)$ is generally **discrete**). Many of them live on K3 surfaces, in which case the volume preserving assumption is automatic.
- ▶ An example is given by $\text{SL}(2, \mathbb{Z})$ acting on $\mathbb{R}^2/\mathbb{Z}^2 \hookrightarrow \mathbb{C}^2/\mathbb{Z}[i]^2$ (cf. BFLM).
- ▶ Another example is given by pentagon folding!

Stiffness

Theorem (Cantat-D, 2020)

Let X be a smooth projective surface and $Y \subset X$ a totally real surface (Typically $Y = X(\mathbb{R})$). Let ν be a finitely supported probability measure on $\text{Aut}(X)$. Assume that Y is Γ_ν -invariant and that every $f \in \Gamma_\nu$ preserves a smooth volume form on Y .

Then every ergodic ν -stationary measure on Y is:

- ▶ either supported on a Γ_ν -invariant curve;
- ▶ or Γ_ν -invariant.

In particular if there is no Γ_ν -invariant curve then ν -stiffness holds.

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Theorem (Roda, 2024)

Let X be a smooth projective surface and $Y \not\subset X$ a totally real surface (Typically $Y \not\subset X(\mathbb{R})$). Let ν be a finitely supported probability measure on $\text{Aut}(X)$. Assume that Y is Γ -invariant and that every $f \in \Gamma_\nu$ preserves a smooth volume form.

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- ▶ If there is an invariant curve $C \subset X$ stiffness should not be expected because $\Gamma_\nu|_C$ could restrict to a Möbius action (random product of 2×2 matrices) on $\mathbb{P}^1(\mathbb{R})$. This is essentially the only obstruction.

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- ▶ This contains the BFLM theorem since $Y = \mathbb{R}^2/\mathbb{Z}^2$ is the “real part” of the complex torus $X = \mathbb{C}^2/\mathbb{Z}[i]^2$.
- ▶ The theorem is the union of two quite different results: **elementary** and **non-elementary actions**. Stiffness in the elementary case follows from general arguments (Choquet-Deny, compactness), while the non-elementary case requires a fair amount of holomorphic dynamics and ergodic theory.

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Goal: prove a rigidity statement like: “ if Γ is a non-elementary group of automorphisms, then any ergodic Γ -invariant measure is:

- ▶ either finitely supported,
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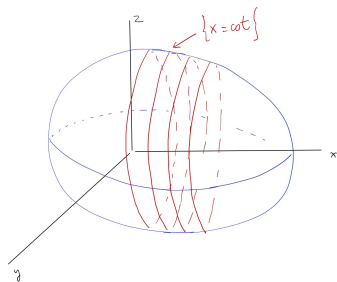
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This is known for linear automorphisms of $\mathbb{T}^2$ by BFLM. The analogous statement on $\mathbb{T}^1$ is Furstenberg’s $\times 2 \times 3$ conjecture.

Parabolic automorphisms

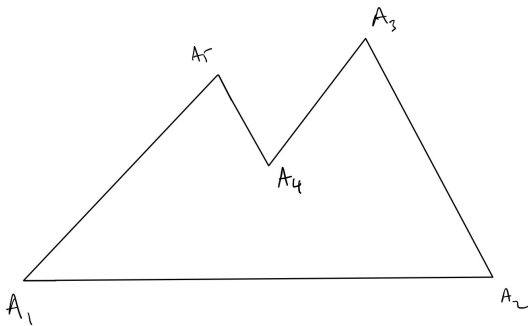
A **parabolic automorphism** of X is an automorphism preserving a fibration by elliptic curves, and acting as a translation along the fibers.

It is an example of **integrable system**.



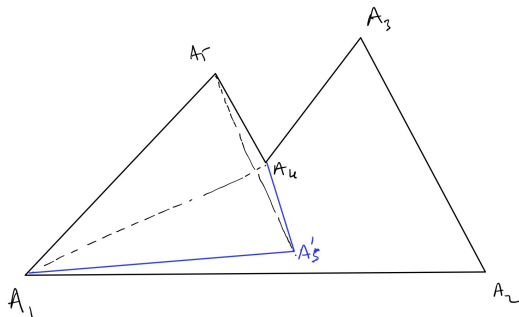
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An example is given by folding along 2 successive diagonals in a pentagon, $f = \sigma_{25} \circ \sigma_{14}$ since it preserves the quadrilateral obtained by forgetting the vertex A_2 .



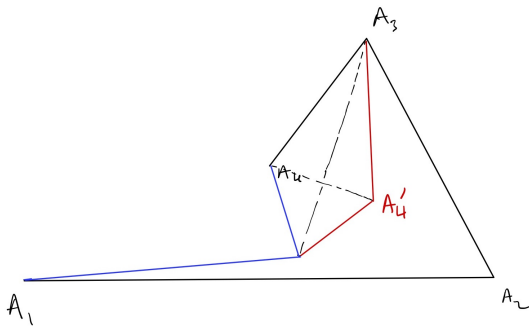
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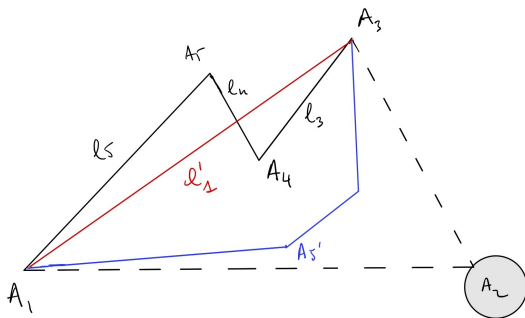
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Classification of invariant measures

Theorem (Cantat-D, 2022)

Let X be a smooth projective surface and Γ be a non-elementary group of automorphisms preserving Y . Assume that Γ contains a parabolic element.

Then any ν -invariant ergodic stationary measure is:

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Remark

We have a classification of orbit closures along the same lines.

Classification: pentagons

For spaces of pentagons this gives :

Corollary

The ergodic invariant measures for the pentagon folding group are

- ▶ measures supported on finite Γ -orbits (i.e. periodic pentagons);
- ▶ or the natural Lebesgue measure induced by the K3 structure.

Furthermore, every infinite Γ -orbit is dense.

In addition there are only **finitely many finite orbits**.

Classification II: finite orbits

The last assertion follows from

Theorem (Cantat-D, 2021)

Let X be a smooth projective surface **which is not a complex torus**, and Γ be a non-elementary group of automorphisms on X , containing parabolic transformations. If Γ has no invariant curve then Γ admits finitely many finite orbits.

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► (Elementary?) Open question: for a general pentagon, is the set of periodic orbits empty?

Conclusion

So did we solve our problem for pentagons?

- ▶ We know from Breiman's theorem that for every $x \in X$ and a.e. (f_n) the limit points of $\frac{1}{n} \sum_{k=1}^n \delta_{f_k \dots f_1(x)}$ are stationary measures.
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If x is not Γ -periodic, a.e. sample path equidistributes towards the invariant volume.

- ▶ For this we need to show that periodic orbits are **repelling on average**. There are many situations where we can prove this, but **not** for generic pentagons.

Bonus

- ▶ A conjecture for probabilists!

Thank you for your attention!