# Dynamic Risk-Averse Optimization

Andrzej Ruszczyński

University Paris Sud
September 2014

# PART I: Introduction to Risk–Averse Optimization

- Utility models
- Mean–risk models
- Measures of risk
- Optimization of measures of risk
- Stochastic dominance constraints
- Introduction to risk–averse dynamic optimization

# Uncertain Outcomes and Risk

## Why Probabilistic Models?

- Wealth of results of probability theory
- Connection to real data via statistics
- Universal language (engineering, economics, medicine, ...)

- Probability space $(\Omega, \mathcal{F}, P)$
- Decision space $\mathcal{X}$
- Random outcome (*e.g.*, cost) $Z_x(\omega)$, $\quad Z : \mathcal{X} \times \Omega \to \mathbb{R}$

## Expected Value Model

$$\min_x \ \mathbb{E}[Z_x] = \int_\Omega Z_x(\omega)\, P(d\omega)$$

It optimizes the outcome on average (Law of Large Numbers?)

## What is Risk?

Existence of unlikely and undesirable outcomes - high $Z_x(\omega)$ for some $\omega$

# Classical Utility Models

**Expected Utility Models (von Neumann and Morgenstern, 1944)**

$$\min_{x\in X} \mathbb{E}\big[u(Z_x)\big] \qquad \Big( = \int_\Omega u\big(Z_x(\omega)\big)\, dP(\omega)\Big)$$

$u : \mathbb{R} \to \mathbb{R}$ is a nondecreasing disutility function

**Rank Dependent Utility (Distortion) Models (Quiggin, 1982; Yaari, 1987)**

$$\min_{x\in X} \int_0^1 F_{Z_x}^{-1}(p)\, dw(p) \qquad F_{Z_x}^{-1}(\cdot) \text{ - quantile function}$$

$w : [0,1] \to \mathbb{R}$ is a nondecreasing rank dependent utility function

Existence of utility functions is derived from systems of axioms, but in practice they are difficult to elicit

# Axioms of Expected Utility Theory (von Neumann 1944)

$W$ is a lottery of $Z$ and $V$ with probabilities $\alpha \in (0,1)$ and $(1-\alpha)$, if the probability measure $\mu_W$ induced by $W$ on $\mathbb{R}$ is the corresponding convex combination of the probability measures $\mu_Z$ and $\mu_V$ of $Z$ and $V$:

$$\mu_W = \alpha\mu_Z + (1-\alpha)\mu_V.$$

We write the lottery symbolically as

$$W = \alpha Z \oplus (1-\alpha)V.$$

For law invariant preferences on the space of real random variables, von Neumann introduced the axioms:

*Independence Axiom:* For all $Z, V, W \in \mathcal{Z}$ one has

$$Z \lhd V \implies \alpha Z \oplus (1-\alpha)W \lhd \alpha V \oplus (1-\alpha)W, \quad \forall\, \alpha \in (0,1)$$

*Archimedean Axiom:* If $Z \lhd V \lhd W$, then $\alpha, \beta \in (0,1)$ exist such that

$$\alpha Z \oplus (1-\alpha)W \lhd V \lhd \beta Z \oplus (1-\beta)W$$

# Expected Utility

## Integral Representation

Suppose the total preorder $\trianglelefteq$ on $\mathcal{Z}$ is law invariant, and satisfies the independence and Archimedean axioms. Then it has an "affine" numerical representation $U : \mathcal{Z} \to \mathbb{R}$:

$$U(\alpha Z \oplus (1-\alpha)V) = \alpha U(Z) + (1-\alpha)U(V).$$

If $\trianglelefteq$ is weakly continuous, then a continuous and bounded function $u : \mathbb{R} \to \mathbb{R}$ exists, such that

$$U(Z) = \mathbb{E}\big[u(Z)\big] = \int_{\Omega} u\big(Z(\omega)\big)\, P(d\omega).$$

New proof by separation theorem - Dentcheva & R. 2012

In a more general setting, we may consider only r.v. with finite moments, and then the boundedness condition on $u(\cdot)$ can be relaxed.

# Risk-Averse Utility

$$U(Z) = \mathbb{E}\big[u(Z)\big] = \int_{\Omega} u\big(Z(\omega)\big)\, P(d\omega)$$

## Monotonicity

The total preorder $\trianglelefteq$ is monotonic with respect to the partial order $\leq$, if $Z \leq V \implies Z \trianglelefteq V$.

We focus on $\mathcal{Z}$ containing integrable random vectors.

## Risk Aversion

A preference relation $\trianglelefteq$ on $\mathcal{Z}$ is *risk-averse*, if $\mathbb{E}[Z|\mathcal{G}] \trianglelefteq Z$, for every $Z \in \mathcal{Z}$ and every $\sigma$-subalgebra $\mathcal{G}$ of $\mathcal{F}$.

## Nondecreasing Convex Disutility

Suppose a total preorder $\trianglelefteq$ on $\mathcal{Z}$ is weakly continuous, monotonic, risk-averse, and satisfies the independence axiom. Then the utility function $u : \mathbb{R} \to \mathbb{R}$ is nondecreasing and convex.

# Axioms of Dual Utility Theory (Yaari 1987)

Real random variables $Z_i$, $i = 1, \ldots, n$, are comonotonic, if

$$\big(Z_i(\omega) - Z_i(\omega')\big)\big(Z_j(\omega) - Z_j(\omega')\big) \geq 0$$

for all $\omega, \omega' \in \Omega$ and all $i, j = 1, \ldots, n$.

*Dual Independence Axiom:* For all comonotonic random variables $Z$, $V$, and $W$ in $\mathcal{Z}$ one has

$$Z \lhd V \implies \alpha Z + (1-\alpha)W \lhd \alpha V + (1-\alpha)W, \quad \forall\, \alpha \in (0,1)$$

*Dual Archimedean Axiom:* For all comonotonic random variables $Z$, $V$, and $W$ in $\mathcal{Z}$, satisfying the relations

$$Z \lhd V \lhd W,$$

there exist $\alpha, \beta \in (0,1)$ such that

$$\alpha Z + (1-\alpha)W \lhd V \lhd \beta Z + (1-\beta)W$$

# Dual Utility

## Affine Representation

If the total preorder $\trianglelefteq$ on $\mathcal{Z}$ is law invariant, and satisfies the dual independence and Archimedean axioms, then a numerical representation $U : \mathcal{Z} \to \mathbb{R}$ of $\trianglelefteq$ exists, which satisfies for all comonotonic $Z, V \in \mathcal{Z}$ and all $\alpha, \beta \in \mathbb{R}_+$ the equation

$$U(\alpha Z + \beta V) = \alpha U(Z) + \beta U(V).$$

## Integral Representation

Suppose $\mathcal{Z}$ is the set of bounded random variables. If, additionally, $\trianglelefteq$ is continuous in $\mathcal{L}_1$ and monotonic, then a bounded, nondecreasing, and continuous function $w : [0,1] \to \mathbb{R}_+$ exists, such that

$$U(Z) = \int_0^1 F_Z^{-1}(p)\, dw(p), \quad Z \in \mathcal{Z}.$$

Proof by separation - Dentcheva & R. 2012

$$U(Z) = \int_0^1 F_Z^{-1}(p)\, dw(p), \quad Z \in \mathcal{Z} \qquad (*)$$

### Risk Aversion

A preference relation $\trianglelefteq$ on $\mathcal{Z}$ is *risk-averse*, if $\mathbb{E}[Z|\mathcal{G}] \trianglelefteq Z$, for every $Z \in \mathcal{Z}$ and every $\sigma$-subalgebra $\mathcal{G}$ of $\mathcal{F}$.

### Convex Rank-Dependent Utility

Suppose a total preorder $\trianglelefteq$ on $\mathcal{Z}$ is continuous, monotonic, and satisfies the dual independence axiom. Then it is risk-averse if and only if it has the integral representation (*) with a nondecreasing and convex function $w : [0, 1] \to [0, 1]$ such that $w(0) = 0$ and $w(1) = 1$.

## Two Objectives

- Minimize the expected outcome, the mean $\mathbb{E}[Z_x]$
- Minimize a scalar measure of uncertainty of $Z_x$, the risk $r[Z_x]$

$$r[Z] = \mathbb{V}\mathrm{ar}[Z] \qquad \text{(Markowitz' model)}$$

$$\sigma_p^+[Z] = \left(\mathbb{E}[(Z - \mathbb{E}Z)_+^p]\right)^{1/p} \qquad \text{(semideviation)}$$

$$\delta_\alpha^+[Z] = \min_\eta \mathbb{E}\left[\max\left(\eta - Z, \frac{\alpha}{1-\alpha}(Z-\eta)\right)\right] \qquad \text{(deviation from quantile)}$$

## Mean–Risk Optimization

$$\min_{x \in X} \rho\big[Z_x\big] = \mathbb{E}\big[Z_x\big] + \kappa\, r\big[Z_x\big], \qquad 0 \le \kappa \le \kappa_{\max}$$

$r\big[Z_x\big]$ is nonlinear w.r.t. probability and possibly nonconvex in $x$

# Example: Portfolio Optimization

$R_1, R_2, \ldots, R_n$ - random return rates of securities
$x_1, x_2, \ldots, x_n$ - fractions of the capital invested in the securities

Return rate of the portfolio (negative of)

$$Z_x = -\big(R_1 x_1 + R_2 x_2 + \cdots + R_n x_n\big)$$

## Risk Optimization with Fixed Mean

$$\begin{aligned} \min_x \quad & r\big[Z_x\big] \\ \text{s.t.} \quad & \mathbb{E}\big[Z_x\big] = \mu \qquad \text{(parameter)} \\ & x \in X_0. \end{aligned}$$

## Combined Mean–Risk Optimization

$$\min_{x \in X_0} \rho\big[Z_x\big] = \mathbb{E}\big[Z_x\big] + \kappa\, r\big[Z_x\big], \qquad 0 \le \kappa \le \kappa_{\max}$$

Interesting applications of parametric optimization

# Nonlinear Programming Formulations for Discrete Distributions

Suppose $Z$ has finitely many realizations $z_1, z_2, \ldots, z_S$ with probabilities $p_1, p_2, \ldots, p_S$

$$\rho(Z) = \mathbb{E}[Z] + \kappa\sigma_m^+[Z] = \mathbb{E}[Z] + \kappa\Big(\mathbb{E}[(Z - \mathbb{E}Z)_+^m]\Big)^{1/m}$$
$$= \sum_{s=1}^S p_s z_s + \kappa\bigg(\sum_{s=1}^S p_s\Big(z_s - \sum_{j=1}^S p_j z_j\Big)_+^m\bigg)^{1/m}$$

**Equivalent Problem (for $m = 1$ - linear programming)**

$$\begin{aligned} \rho(Z) = \min_{v,\mu} \quad & \mu + \kappa\bigg(\sum_{s=1}^S p_s v_s^m\bigg)^{1/m} \\ \text{s.t.} \quad & \mu = \sum_{s=1}^S p_s z_s \\ & v_s \geq z_s - \mu, \quad s = 1, \ldots, S \\ & v_s \geq 0, \qquad\quad s = 1, \ldots, S \end{aligned}$$

# Application to Portfolios

Suppose the vector of return rates has $S$ realizations with probabilities $p_1, p_2, \dots, p_S$
$R_{js}$ - return rate of asset $j = 1, \dots, n$ in scenario $s = 1, \dots, S$

## Equivalent Problem (for $m = 1$ - linear programming)

$$\min_{x,z,v,\mu} \quad \mu + \kappa\left(\sum_{s=1}^{S} v_s^m\right)^{1/m}$$

$$\text{s.t.} \quad \mu = \sum_{s=1}^{S} p_s z_s$$

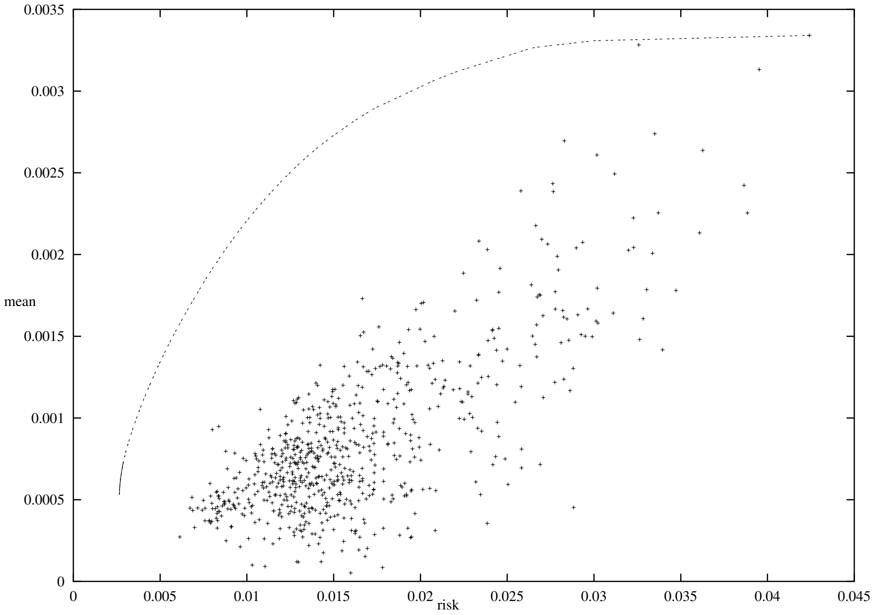
$$z_s = -\sum_{j=1}^{n} R_{sj} x_j, \quad s = 1, \dots, S$$

$$v_s \geq z_s - \mu, \quad s = 1, \dots, S$$

$$v_s \geq 0, \quad s = 1, \dots, S$$

$$x \in X_0$$

# Mean–Semideviation Model (719 stocks); (with R. Vanderbei)



# Key Requirement: Monotonicity

$$\rho(Z) = \mathbb{E}[Z] + \kappa\, r[Z]$$

**Consistency with Stochastic Dominance (Ogryczak–R., 1997)**

$$\mathbb{E}[u(Z)] \le \mathbb{E}[u(W)],\ \forall \text{ nondecreasing and convex } u(\cdot) \Rightarrow\ \rho[Z] \le \rho[W]$$

**Consistency with Pointwise Order (Artzner *et. al.*, 1999)**

$$Z\ \le\ W \text{ a.s.}\ \Rightarrow\ \rho[Z] \le \rho[W]$$

Mean–semideviation and mean–deviation from quantile models are consistent for $0 \le \kappa \le 1$, but not mean–variance.

**Unique optimal solutions of consistent optimization models**

$$\min_{x \in X}\ \rho(Z_x)$$

cannot be strictly dominated (in the corresponding sense)

Space of uncertain outcomes $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty]$

A functional $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$ is a coherent risk measure if it satisfies the following axioms

- Convexity: $\rho(\lambda Z + (1-\lambda)W) \le \lambda\rho(Z) + (1-\lambda)\rho(W)$
  $\forall\ \lambda \in (0,1),\ Z, W \in \mathcal{Z}$
- Monotonicity: If $Z \le W$ then $\rho(Z) \le \rho(W)$, $\quad \forall\ Z, W \in \mathcal{Z}$
- Translation Equivariance: $\rho(Z + a) = \rho(Z) + a$, $\quad \forall\ Z \in \mathcal{Z}$, $a \in \mathbb{R}$
- Positive Homogeneity: $\rho(\tau Z) = \tau\rho(Z)$, $\quad \forall\ Z \in \mathcal{Z}$, $\tau \ge 0$

Kijima-Ohnishi (1993) – no monotonicity
Artzner-Delbaen-Eber-Heath (1999–) - space $\mathcal{L}_\infty$

R.-Shapiro (2005) – spaces $\mathcal{L}_p$, . . .

Good news: $\mathbb{E}[Z]$ is coherent

# Coherent Risk Measures

Space of uncertain outcomes $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty]$

A functional $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$ is a coherent risk measure if it satisfies the following axioms

- Convexity: $\rho(\lambda Z + (1-\lambda)W) \le \lambda\rho(Z) + (1-\lambda)\rho(W)$
  $\forall\ \lambda \in (0,1),\ Z, W \in \mathcal{Z}$
- Monotonicity: If $Z \le W$ then $\rho(Z) \le \rho(W)$, $\quad \forall\ Z, W \in \mathcal{Z}$
- Translation Equivariance: $\rho(Z + a) = \rho(Z) + a$, $\quad \forall\ Z \in \mathcal{Z}$, $a \in \mathbb{R}$
- Positive Homogeneity: $\rho(\tau Z) = \tau\rho(Z)$, $\quad \forall\ Z \in \mathcal{Z}$, $\tau \ge 0$

Kijima-Ohnishi (1993) – no monotonicity
Artzner-Delbaen-Eber-Heath (1999–) - space $\mathcal{L}_\infty$

R.-Shapiro (2005) – spaces $\mathcal{L}_p$, ...

Good news: $\mathbb{E}[Z]$ is coherent

# Coherent Risk Measures

Space of uncertain outcomes $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty]$

A functional $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$ is a coherent risk measure if it satisfies the following axioms

- Convexity: $\rho(\lambda Z + (1 - \lambda)W) \le \lambda\rho(Z) + (1 - \lambda)\rho(W)$
  $\forall\ \lambda \in (0, 1),\ Z, W \in \mathcal{Z}$
- Monotonicity: If $Z \le W$ then $\rho(Z) \le \rho(W), \quad \forall\ Z, W \in \mathcal{Z}$
- Translation Equivariance: $\rho(Z + a) = \rho(Z) + a, \quad \forall\ Z \in \mathcal{Z},\ a \in \mathbb{R}$
- Positive Homogeneity: $\rho(\tau Z) = \tau\rho(Z), \quad \forall\ Z \in \mathcal{Z},\ \tau \ge 0$

Kijima-Ohnishi (1993) – no monotonicity
Artzner-Delbaen-Eber-Heath (1999–) - space $\mathcal{L}_\infty$

R.-Shapiro (2005) – spaces $\mathcal{L}_p, \dots$

Good news: $\mathbb{E}[Z]$ is coherent

# Coherent Risk Measures

Space of uncertain outcomes $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty]$

A functional $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$ is a coherent risk measure if it satisfies the following axioms

- Convexity: $\rho(\lambda Z + (1-\lambda)W) \leq \lambda\rho(Z) + (1-\lambda)\rho(W)$
  $\forall\ \lambda \in (0,1),\ Z, W \in \mathcal{Z}$
- Monotonicity: If $Z \leq W$ then $\rho(Z) \leq \rho(W)$, $\quad \forall\ Z, W \in \mathcal{Z}$
- Translation Equivariance: $\rho(Z + a) = \rho(Z) + a$, $\quad \forall\ Z \in \mathcal{Z}$, $a \in \mathbb{R}$
- Positive Homogeneity: $\rho(\tau Z) = \tau\rho(Z)$, $\quad \forall\ Z \in \mathcal{Z}$, $\tau \geq 0$

Kijima-Ohnishi (1993) – no monotonicity
Artzner-Delbaen-Eber-Heath (1999–) - space $\mathcal{L}_\infty$

R.-Shapiro (2005) – spaces $\mathcal{L}_p$, ...

Good news: $\mathbb{E}[Z]$ is coherent

# Coherent Risk Measures

Space of uncertain outcomes $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty]$

A functional $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$ is a coherent risk measure if it satisfies the following axioms

- Convexity: $\rho(\lambda Z + (1-\lambda)W) \leq \lambda\rho(Z) + (1-\lambda)\rho(W)$
  $\forall\, \lambda \in (0,1),\ Z, W \in \mathcal{Z}$
- Monotonicity: If $Z \leq W$ then $\rho(Z) \leq \rho(W)$, $\quad \forall\, Z, W \in \mathcal{Z}$
- Translation Equivariance: $\rho(Z + a) = \rho(Z) + a$, $\quad \forall\, Z \in \mathcal{Z}, a \in \mathbb{R}$
- Positive Homogeneity: $\rho(\tau Z) = \tau\rho(Z)$, $\quad \forall\, Z \in \mathcal{Z},\ \tau \geq 0$

Kijima-Ohnishi (1993) – no monotonicity
Artzner-Delbaen-Eber-Heath (1999–) - space $\mathcal{L}_\infty$

R.-Shapiro (2005) – spaces $\mathcal{L}_p, \dots$

Good news: $\mathbb{E}[Z]$ is coherent

# Coherent Risk Measures

Space of uncertain outcomes $\mathcal{Z} = \mathcal{L}_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty]$

A functional $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$ is a coherent risk measure if it satisfies the following axioms

- Convexity: $\rho(\lambda Z + (1-\lambda)W) \le \lambda\rho(Z) + (1-\lambda)\rho(W)$
  $\forall\ \lambda \in (0,1),\ Z, W \in \mathcal{Z}$
- Monotonicity: If $Z \le W$ then $\rho(Z) \le \rho(W), \quad \forall\ Z, W \in \mathcal{Z}$
- Translation Equivariance: $\rho(Z + a) = \rho(Z) + a, \quad \forall\ Z \in \mathcal{Z},\ a \in \mathbb{R}$
- Positive Homogeneity: $\rho(\tau Z) = \tau\rho(Z), \quad \forall\ Z \in \mathcal{Z},\ \tau \ge 0$

Kijima-Ohnishi (1993) – no monotonicity
Artzner-Delbaen-Eber-Heath (1999–) - space $\mathcal{L}_\infty$

R.-Shapiro (2005) – spaces $\mathcal{L}_p, \dots$

Good news: $\mathbb{E}[Z]$ is coherent

# Value at Risk

The Value at Risk at level $\alpha \in (0,1)$ of a random cost $Z \in \mathcal{Z}$:

$$\mathrm{V@R}_\alpha^+(Z) \triangleq \inf\{\eta : F_Z(\eta) \geq 1-\alpha\} = F_Z^{-1}(1-\alpha)$$

Monotonicity: $Z \leq V \Longrightarrow \mathrm{V@R}_\alpha^+(Z) \leq \mathrm{V@R}_\alpha^+(V)$

Translation: $\mathrm{V@R}_\alpha^+(Z+c) = \mathrm{V@R}_\alpha^+(Z) + c$, for all $c \in \mathbb{R}$

Positive Homogeneity: $\mathrm{V@R}_\alpha^+(\gamma Z) = \gamma\, \mathrm{V@R}_\alpha^+(Z)$, for all $\gamma \geq 0$

However, it is not convex

## Counterexample: Two independent variables

$$Z = \begin{cases} 0 & \text{with probability } 1-p \\ 1 & \text{with probability } p \end{cases} \qquad V = \begin{cases} 0 & \text{with probability } 1-p \\ 1 & \text{with probability } p \end{cases}$$

For $p < \alpha < 1$ we have $\mathrm{V@R}_\alpha^+(Z) = \mathrm{V@R}_\alpha^+(V) = 0$

If $p < \alpha < 1-(1-p)^2$, we have non-convexity

$$\mathrm{V@R}_\alpha^+\big(\lambda Z + (1-\lambda)V\big) > 0 = \lambda\, \mathrm{V@R}_\alpha^+(Z) + (1-\lambda)\, \mathrm{V@R}_\alpha^+(V)$$

# Average Value at Risk

$$\text{AV@R}_\alpha^+(Z) \triangleq \frac{1}{\alpha}\int_0^\alpha \text{V@R}_\beta^+(Z)\,\mathrm{d}\beta$$

If the $(1-\alpha)$-quantile of $Z$ is unique

$$\text{AV@R}_\alpha^+(Z) = \frac{1}{\alpha}\int_{\text{V@R}_\alpha^+(Z)}^{\infty} z\,\mathrm{d}F_Z(z) = \mathbb{E}\big[Z \,|\, Z \geq \text{V@R}_\alpha^+(Z)\big]$$

## Extremal representation

$$\text{AV@R}_\alpha^+(Z) = \inf_{\eta\in\mathbb{R}} \left\{\eta + \frac{1}{\alpha}\mathbb{E}\big[(Z-\eta)_+\big]\right\}$$

The minimizer $\eta = \text{V@R}_\alpha(Z)$

Connection to weighted deviation from $\alpha$-quantile:

$$\delta_\alpha^+(Z) = \text{AV@R}_\alpha^+(Z) - \mathbb{E}[Z], \quad \alpha \in [0,1].$$

# Linear Programming Representation of AV@R

Suppose $Z$ has finitely many realizations $z_1, z_2, \ldots, z_S$ with probabilities $p_1, p_2, \ldots, p_S$

$$\begin{aligned}
\min_{v,\eta} \quad & \eta + \frac{1}{\alpha} \sum_{s=1}^{S} p_s v_s \\
\text{s.t.} \quad & v_s \geq z_s - \eta, \quad s = 1, \ldots, S \\
& v_s \geq 0, \quad s = 1, \ldots, S
\end{aligned}$$

For portfolios we have to add the constraints

$$z_s = -\sum_{j=1}^{n} R_{sj} x_j, \quad s = 1, \ldots, S$$

$$x \in X_0$$

and include $z$ and $x$ into the decision variables

# Conjugate Duality of Risk Measures

Pairing of a linear topological space $\mathcal{Z}$ with a linear topological space $\mathcal{Y}$ of regular signed measures on $\Omega$ with the bilinear form

$$\langle \mu, Z \rangle = \mathbb{E}_\mu[Z] = \int_\Omega Z(\omega)\,\mu(d\omega)$$

We assume standard conditions on pairing and the polarity: $(\mathcal{Z}_+)^\circ = \mathcal{Y}_-$

## Dual Representation Theorem

If $\rho : \mathcal{Z} \to \overline{\mathbb{R}}$ is a lower semicontinuous* coherent risk measure, then

$$\rho(Z) = \max_{\mu \in \mathcal{A}} \int_\Omega Z(\omega)\,\mu(d\omega), \quad \forall Z \in \mathcal{Z}$$

with a convex closed $\mathcal{A} \subset \mathcal{P}$ (set of probability measures in $\mathcal{Y}$).

Delbaen (2001), Föllmer–Schied (2002), R.–Shapiro (2005), Rockafellar–Uryasev–Zabarankin (2006), . . .

* *Lower semicontinuity is automatic if $\rho$ is finite and $\mathcal{Z}$ is a Banach lattice*

$Z \sim V$ means that $Z$ and $V$ have the same distribution, $\mu_Z = \mu_V$.
$\rho : \mathcal{Z} \to \mathbb{R}$ is law invariant if $Z \sim V \Longrightarrow \rho(Z) = \rho(V)$

**Kusuoka Theorem**

If $(\Omega, \mathcal{F}, P)$ is atomless and $\rho : \mathcal{L}_1(\Omega, \mathcal{F}, P) \to \mathbb{R}$ is law invariant and coherent, then

$$\rho(Z) = \sup_{m \in \mathcal{M}} \int_0^1 \text{AV@R}_\alpha^+(Z)\, m(d\alpha)$$

where $\mathcal{M}$ is a convex set of probability measures on $(0, 1]$.

**Spectral measure**

$$\rho(V) = \int_0^1 \text{AV@R}_\alpha^+(Z)\, m(d\alpha)$$

Spectral measures have dual utility form:

$$\rho(Z) = \int_0^1 F_Z^{-1}(\beta)\, dw(\beta)$$

"Minimize" over $x \in X$ a random outcome $Z_x(\omega) = f(x,\omega)$, $\omega \in \Omega$

### Composite Optimization Problem

$$\min_{x \in X} \rho(Z_x) \qquad \text{(P)}$$

### Theorem

Let $x \mapsto Z_x(\omega)$ be convex and $\rho(\cdot)$ be coherent. Suppose $\hat{x} \in X$ is an optimal solution of (P) and $\rho(\cdot)$ is continuous at $Z_{\hat{x}}$. Then there exists a probability measure $\hat{\mu} \in \partial\rho(Z_{\hat{x}}) \subseteq \mathcal{A}$ such that $\hat{x}$ solves

$$\min_{x \in X} \mathbb{E}_{\hat{\mu}}[Z_x] = \min_{x \in X} \max_{\mu \in \mathcal{A}} \mathbb{E}_{\mu}[Z_x]$$

We also have the duality relation:

$$\min_{x \in X} \rho(Z_x) = \max_{\mu \in \mathcal{A}} \inf_{x \in X} \mathbb{E}_{\mu}[Z_x]$$

# Duality in Portfolio Optimization - Game Model

Suppose the vector of return rates of assets has $S$ realizations

$R_{js}$ - return rate of asset $j = 1, \dots, n$ in scenario $s = 1, \dots, S$

Portfolio return (negative) in scenario $s$

$$Z_s(x) = -\sum_{j=1}^{n} R_{js} x_j$$

## Risk-Averse Portfolio Problem

$$\min_{x \in X} \; \rho\big(Z(x)\big)$$

By homogeneity, we may assume that $\sum_{j=1}^{n} x_j = 1$

## Equivalent Matrix Game

$$\max_{x \in X} \min_{\mu \in \mathcal{A}} \sum_{j=1}^{n} \sum_{s=1}^{S} x_j R_{js} \mu_s$$

- $x$ - mixed strategy of the investor
- $\mu$ - mixed strategy of the opponent (market)

# Stochastic Dominance Constraints (with Darinka Dentcheva, 2003–)

$Z_x$ - random outcome (*e.g.*, cost)
$Y$ - benchmark random outcome, e.g. $Y(\omega) = Z_{\bar{x}}(\omega)$ for some $\bar{x} \in X$

**New Model**

$$\begin{aligned} &\min \mathbb{E}[Z_x] && \text{(or some other objective)} \\ &\text{subject to } Z_x \preceq_{\mathcal{U}} Y && \text{(stochastic ordering constraint)} \\ &x \in X && \end{aligned}$$

$Z_x$ is preferred over $Y$ by all decision makers having disutility functions in the generator $\mathcal{U}$:

$$\mathbb{E}[u(Z_x)] \leq \mathbb{E}[u(Y)] \quad \forall\, u \in \mathcal{U}$$

All nondecreasing $u(\cdot)$ - first order stochastic dominance $\preceq_{\text{st}}$
All nondecreasing convex $u(\cdot)$ - increasing convex order $\preceq_{\text{icx}}$

$$\min \mathbb{E}[Z_x]$$
$$\text{subject to } Z_x \preceq_{\text{icx}} Y$$
$$x \in X$$

$X$ - convex set in $\mathcal{X}$ (separable locally convex Hausdorff vector space)
$x \mapsto Z_x$ is a continuous operator from $\mathcal{X}$ to $\mathcal{L}_1(\Omega, \mathcal{F}, P)$
$x \mapsto Z_x(\omega)$ is convex for $P$-almost all $\omega \in \Omega$

Primal: $\mathbb{E}[u(Z_x)] \le \mathbb{E}[u(Y)]$ for all convex nondecreasing $u : \mathbb{R} \to \mathbb{R}$
Inverse: $\int_0^1 F_{Z_x}^{-1}(p)\, dw(p) \le \int_0^1 F_Y^{-1}(p)\, dw(p)$ for all convex nondecreasing $w : [0,1] \to \mathbb{R}$

## Main Results

- Utility functions $u : \mathbb{R} \to \mathbb{R}$ and rank dependent utility functions $w : [0,1] \to \mathbb{R}$ play the roles of Lagrange multipliers
- Expected utility models and rank dependent utility models are Lagrangian relaxations of the problem

## Lagrangian in Direct Form

$$L(x, u) = \mathbb{E}\Big[Z_x + u(Z_x) - u(Y)\Big]$$

$u(\cdot)$ - convex function on $\mathbb{R}$

## Theorem

Assume Uniform Dominance Condition (a form of Slater constraint qualification). If $\hat{x}$ is an optimal solution of the problem then there exists a function $\hat{u} \in \mathcal{U}$ such that

$$L(\hat{x}, \hat{u}) = \min_{x \in X} L(x, \hat{u}) \tag{1}$$

$$\mathbb{E}\Big[\hat{u}(Z_{\hat{x}})\Big] = \mathbb{E}\Big[\hat{u}(Y)\Big] \tag{2}$$

Conversely, if for some function $\hat{u} \in \mathcal{U}$ an optimal solution $\hat{x}$ of (1) satisfies the dominance constraint and (2), then $\hat{x}$ is optimal

# Implied Rank Utility (Distortion) Function (with Darinka Dentcheva)

## Lagrangian in Inverse Form

$$\Phi(x, w) = \int_0^1 F_{Z_x}^{-1}(p)\, d(p + w(p)) - \int_0^1 F_Y^{-1}(p)\, dw(p)$$

$w(\cdot)$ - convex function on $[0, 1]$

## Theorem

Assume Uniform Dominance Condition (a form of Slater constraint qualification). If $\hat{x}$ is an optimal solution of the problem, then there exists a function $\hat{w} \in \mathcal{W}$ such that

$$\Phi(\hat{x}, \hat{w}) = \min_{x \in X} \Phi(x, \hat{w}) \tag{3}$$

$$\int_0^1 F_{Z_{\hat{x}}}^{-1}(p)\, d\hat{w}(p) = \int_0^1 F_Y^{-1}(p)\, d\hat{w}(p) \tag{4}$$

If for some $\hat{w} \in \mathcal{W}$ an optimal solution $\hat{x}$ of (3) satisfies the inverse dominance constraint and (4), then $\hat{x}$ is optimal

Probability space $(\Omega, \mathcal{F}, P)$ with filtration $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T \subset \mathcal{F}$

Adapted sequence of random variables (costs) $Z_1, Z_2, \ldots, Z_T$

Spaces: $\mathcal{Z}_t = \mathcal{L}_{\bar{s}}(\Omega, \mathcal{F}_t, P)$, $\bar{s} \in [1, \infty)$, and $\mathcal{Z}_{t,T} = \mathcal{Z}_t \times \cdots \times \mathcal{Z}_T$

## Conditional Risk Measure

A mapping $\rho_{t,T} : \mathcal{Z}_{t,T} \to \mathcal{Z}_t$ satisfying the monotonicity condition:

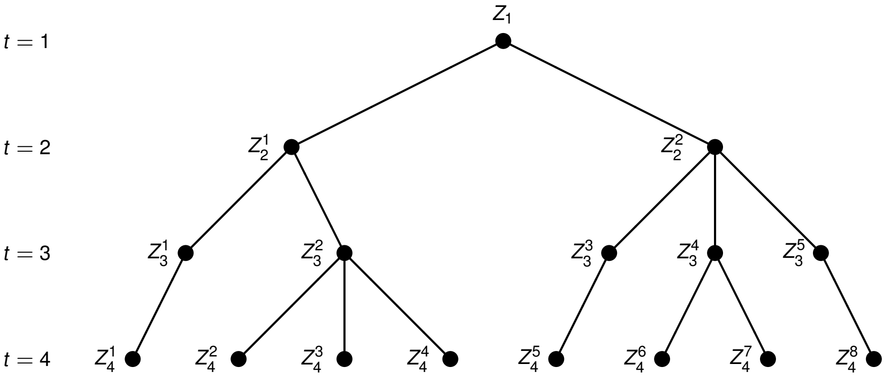
$$\rho_{t,T}(Z) \le \rho_{t,T}(W) \text{ for all } Z, W \in \mathcal{Z}_{t,T} \text{ such that } Z \le W$$

## Dynamic Risk Measure

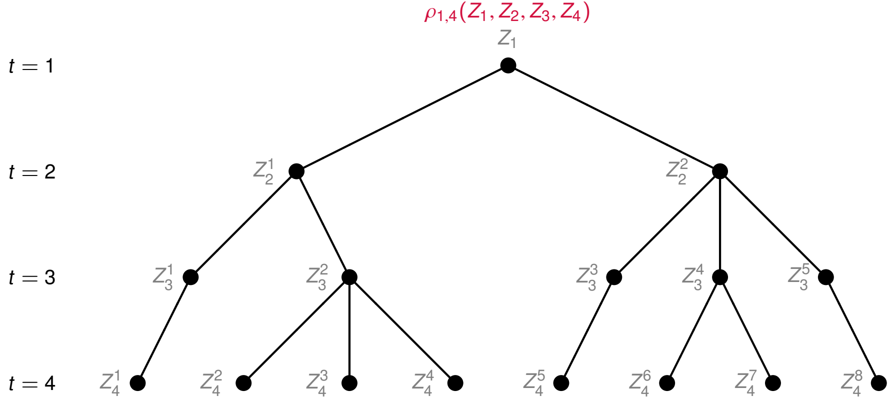
A sequence of conditional risk measures $\rho_{t,T} : \mathcal{Z}_{t,T} \to \mathcal{Z}_t$, $t = 1, \ldots, T$

$$\rho_{1,T}(Z_1, Z_2, Z_3, \ldots, Z_T) \in \mathcal{Z}_1 = \mathbb{R}$$
$$\rho_{2,T}(Z_2, Z_3, \ldots, Z_T) \in \mathcal{Z}_2$$
$$\rho_{3,T}(Z_3, \ldots, Z_T) \in \mathcal{Z}_3$$
$$\vdots$$

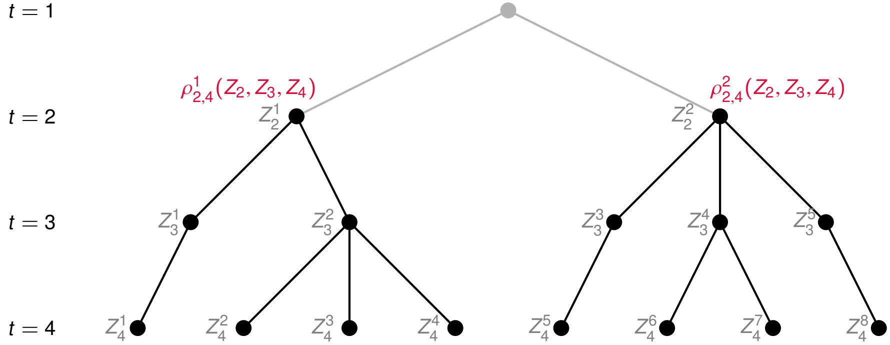
# Evaluating Risk on a Scenario Tree



# Evaluating Risk on a Scenario Tree



# Evaluating Risk on a Scenario Tree



# References for Part I


- D. Dentcheva, A. Ruszczyński, Common mathematical foundations of expected utility and dual utility theories, *SIAM Jornal on Optimization* 23 (2013), No. 1, 381–405.
- W. Ogryczak and A. Ruszczyński, Dual stochastic dominance and related mean–risk models, *SIAM Journal on Optimization* 13 (2002) 60–78.
- W. Ogryczak and A. Ruszczyński, From stochastic dominance to mean–risk models: Semideviations as risk measures, *European Journal of Operational Research* 116 (1999) 33–50
- A. Ruszczyński and R.J. Vanderbei, Frontiers of stochastically non-dominated portfolios, *Econometrica* 71 (2003) 1287–1297.
- A. Ruszczyński and A. Shapiro, Optimization of convex risk functions, *Mathematics of Operations Research* 31 (2006) 433–452.
- D. Dentcheva and A. Ruszczyński, Optimization with stochastic dominance constraints, *SIAM Journal on Optimization* 14 (2003) 548–566.
- D. Dentcheva and A. Ruszczyński, Inverse stochastic dominance constraints and rank dependent expected utility theory, *Mathematical Programming* 108 (2006) 297–311.

# PART II: Optimization of Dynamic Risk Measures

- Dynamic measures of risk
- Time consistency and local property
- Interchangeability
- Risk optimization on a tree
- Application to Markov models
- Stochastic conditional time-consistency
- Markov risk measures
- Dynamic programming
- Solution methods
- Examples

# How to Measure Risk of Sequences?

Probability space $(\Omega, \mathcal{F}, P)$ with filtration $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T \subset \mathcal{F}$
Adapted sequence of random variables (costs) $Z_1, Z_2, \ldots, Z_T$
Spaces: $\mathcal{Z}_t = \mathcal{L}_{\bar{s}}(\Omega, \mathcal{F}_t, P)$, $\bar{s} \in [1, \infty)$, and $\mathcal{Z}_{t,T} = \mathcal{Z}_t \times \cdots \times \mathcal{Z}_T$

## Conditional Risk Measure

A mapping $\rho_{t,T} : \mathcal{Z}_{t,T} \to \mathcal{Z}_t$ satisfying the monotonicity condition:

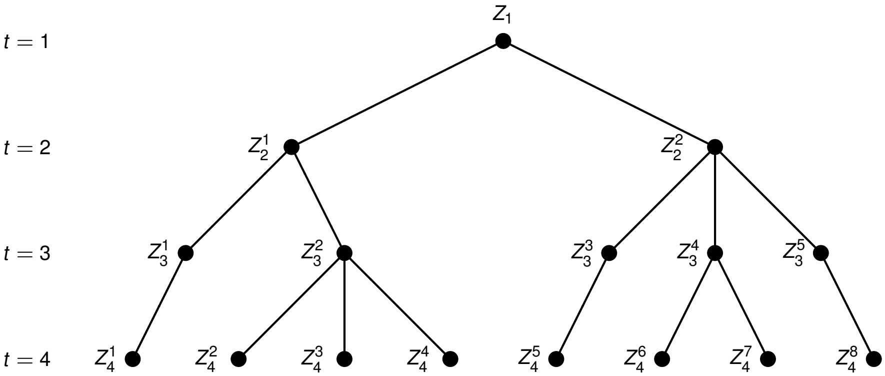
$$\rho_{t,T}(Z) \leq \rho_{t,T}(W) \text{ for all } Z, W \in \mathcal{Z}_{t,T} \text{ such that } Z \leq W$$

## Dynamic Risk Measure

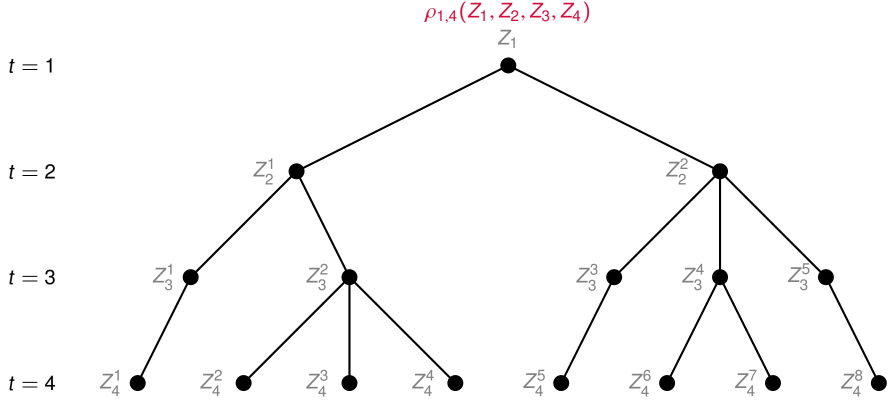
A sequence of conditional risk measures $\rho_{t,T} : \mathcal{Z}_{t,T} \to \mathcal{Z}_t$, $t = 1, \ldots, T$

$$\rho_{1,T}(Z_1, Z_2, Z_3, \ldots, Z_T) \in \mathcal{Z}_1 = \mathbb{R}$$
$$\rho_{2,T}(Z_2, Z_3, \ldots, Z_T) \in \mathcal{Z}_2$$
$$\rho_{3,T}(Z_3, \ldots, Z_T) \in \mathcal{Z}_3$$
$$\vdots$$

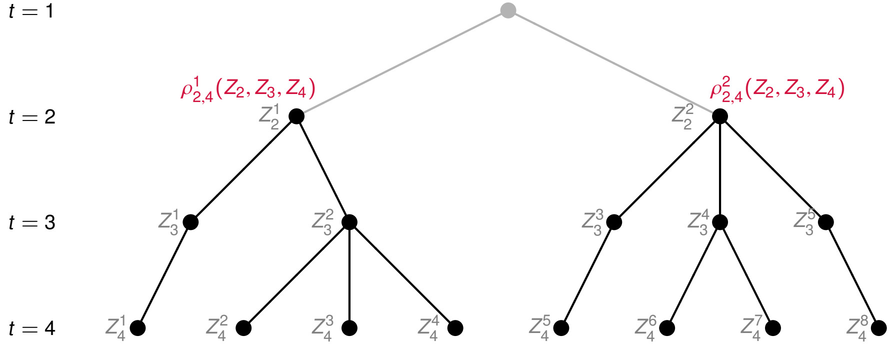
# Evaluating Risk on a Scenario Tree



# Evaluating Risk on a Scenario Tree



# Evaluating Risk on a Scenario Tree



# Time Consistency of Dynamic Risk Measures

A dynamic risk measure $\left\{\rho_{t,T}\right\}_{t=1}^{T}$ is time-consistent if for all $\tau < \theta$

$$Z_k = W_k,\ k = \tau, \dots, \theta - 1 \quad \text{and} \quad \rho_{\theta,T}(Z_\theta, \dots, Z_T) \le \rho_{\theta,T}(W_\theta, \dots, W_T)$$

imply that $\quad \rho_{\tau,T}(Z_\tau, \dots, Z_T) \le \rho_{\tau,T}(W_\tau, \dots, W_T)$

Define $\quad \rho_t(Z_{t+1}) = \rho_{t,T}(0, Z_{t+1}, 0, \dots, 0)$

## Nested Decomposition Theorem

Suppose a dynamic risk measure $\left\{\rho_{t,T}\right\}_{t=1}^{T}$ is time-consistent and

$$\rho_{t,T}(Z_t, Z_{t+1}, \dots, Z_T) = Z_t + \rho_{t,T}(0, Z_{t+1}, \dots, Z_T)$$

Then for all $t$ we have the representation

$$\rho_{t,T}(Z_t, \dots, Z_T) = Z_t + \rho_t\Big(Z_{t+1} + \rho_{t+1}\Big(Z_{t+2} + \cdots + \rho_{T-1}(Z_T)\Big)\cdots\Big)$$

# Coherent One-Step Conditional Risk Measures

Stronger assumptions about one-step measures $\rho_t : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$:

- Convexity: $\rho_t(\lambda Z + (1-\lambda)W) \le \lambda\rho_t(Z) + (1-\lambda)\rho_t(W)$
  $\forall\ \lambda \in (0,1),\ Z, W \in \mathcal{Z}_{t+1}$
- Monotonicity: If $Z \le W$ then $\rho_t(Z) \le \rho_t(W),\ \forall\ Z, W \in \mathcal{Z}_{t+1}$
- Predictable Translation Equivariance:
  $\rho_t(Z + W) = Z + \rho_t(W),\ \forall\ Z \in \mathcal{Z}_t,\ W \in \mathcal{Z}_{t+1}$
- Positive Homogeneity: $\rho_t(\tau Z) = \tau\rho_t(Z),\ \forall\ Z \in \mathcal{Z}_{t+1},\ \tau \ge 0$

Scandolo ('03), Riedel ('04), R.-Shapiro ('06), Cheridito-Delbaen-Kupper ('06), Föllmer-Penner ('06), Artzner-Delbaen-Eber-Heath-Ku ('07), Pflug-Römisch ('07)

## Example: Conditional Mean–Semideviation

$$\rho_t(Z_{t+1}) = \mathbb{E}[Z_{t+1}|\mathcal{F}_t] + \kappa\mathbb{E}\Big[\big(Z_{t+1} - \mathbb{E}[Z_{t+1}|\mathcal{F}_t]\big)_+^s \Big| \mathcal{F}_t\Big]^{\frac{1}{s}}$$

Here $s \in [1, \bar{s}]$ and $\kappa \in [0,1]$ may be $\mathcal{F}_t$-measurable

# Coherent One-Step Conditional Risk Measures

Stronger assumptions about one-step measures $\rho_t : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$:

- Convexity: $\rho_t(\lambda Z + (1-\lambda)W) \leq \lambda \rho_t(Z) + (1-\lambda)\rho_t(W)$
  $\forall\ \lambda \in (0,1),\ Z, W \in \mathcal{Z}_{t+1}$
- Monotonicity: If $Z \leq W$ then $\rho_t(Z) \leq \rho_t(W),\ \ \forall\ Z, W \in \mathcal{Z}_{t+1}$
- Predictable Translation Equivariance:
  $\rho_t(Z + W) = Z + \rho_t(W),\ \forall\ Z \in \mathcal{Z}_t,\ \ W \in \mathcal{Z}_{t+1}$
- Positive Homogeneity: $\rho_t(\tau Z) = \tau \rho_t(Z),\ \forall\ Z \in \mathcal{Z}_{t+1},\ \ \tau \geq 0$

Scandolo ('03), Riedel ('04), R.-Shapiro ('06), Cheridito-Delbaen-Kupper ('06), Föllmer-Penner ('06), Artzner-Delbaen-Eber-Heath-Ku ('07), Pflug-Römisch ('07)

## Example: Conditional Mean–Semideviation

$$\rho_t(Z_{t+1}) = \mathbb{E}[Z_{t+1}|\mathcal{F}_t] + \kappa \mathbb{E}\Big[\Big(Z_{t+1} - \mathbb{E}[Z_{t+1}|\mathcal{F}_t]\Big)_+^s \Big|\mathcal{F}_t\Big]^{\frac{1}{s}}$$

Here $s \in [1, \bar{s}]$ and $\kappa \in [0,1]$ may be $\mathcal{F}_t$-measurable

# Coherent One-Step Conditional Risk Measures

Stronger assumptions about one-step measures $\rho_t : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$:

- Convexity: $\rho_t(\lambda Z + (1-\lambda)W) \leq \lambda \rho_t(Z) + (1-\lambda)\rho_t(W)$
  $\forall\ \lambda \in (0,1),\ Z, W \in \mathcal{Z}_{t+1}$
- Monotonicity: If $Z \leq W$ then $\rho_t(Z) \leq \rho_t(W)$, $\ \forall\ Z, W \in \mathcal{Z}_{t+1}$
- Predictable Translation Equivariance:
  $\rho_t(Z + W) = Z + \rho_t(W),\ \forall\ Z \in \mathcal{Z}_t,\ W \in \mathcal{Z}_{t+1}$
- Positive Homogeneity: $\rho_t(\tau Z) = \tau \rho_t(Z),\ \forall\ Z \in \mathcal{Z}_{t+1},\ \tau \geq 0$

Scandolo ('03), Riedel ('04), R.-Shapiro ('06), Cheridito-Delbaen-Kupper ('06), Föllmer-Penner ('06), Artzner-Delbaen-Eber-Heath-Ku ('07), Pflug-Römisch ('07)

## Example: Conditional Mean–Semideviation

$$\rho_t(Z_{t+1}) = \mathbb{E}[Z_{t+1}|\mathcal{F}_t] + \kappa \mathbb{E}\Big[\Big(Z_{t+1} - \mathbb{E}[Z_{t+1}|\mathcal{F}_t]\Big)_+^s \Big| \mathcal{F}_t\Big]^{\frac{1}{s}}$$

Here $s \in [1, \bar{s}]$ and $\kappa \in [0,1]$ may be $\mathcal{F}_t$-measurable

# Coherent One-Step Conditional Risk Measures

Stronger assumptions about one-step measures $\rho_t : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$:

- Convexity: $\rho_t(\lambda Z + (1-\lambda)W) \le \lambda\rho_t(Z) + (1-\lambda)\rho_t(W)$
  $\forall\ \lambda \in (0,1),\ Z, W \in \mathcal{Z}_{t+1}$
- Monotonicity: If $Z \le W$ then $\rho_t(Z) \le \rho_t(W),\ \forall\ Z, W \in \mathcal{Z}_{t+1}$
- Predictable Translation Equivariance:
  $\rho_t(Z + W) = Z + \rho_t(W),\ \forall\ Z \in \mathcal{Z}_t,\ W \in \mathcal{Z}_{t+1}$
- Positive Homogeneity: $\rho_t(\tau Z) = \tau\rho_t(Z), \forall\ Z \in \mathcal{Z}_{t+1},\ \tau \ge 0$

Scandolo ('03), Riedel ('04), R.-Shapiro ('06), Cheridito-Delbaen-Kupper ('06), Föllmer-Penner ('06), Artzner-Delbaen-Eber-Heath-Ku ('07), Pflug-Römisch ('07)

## Example: Conditional Mean–Semideviation

$$\rho_t(Z_{t+1}) = \mathbb{E}[Z_{t+1}|\mathcal{F}_t] + \kappa\mathbb{E}\Big[\Big(Z_{t+1} - \mathbb{E}[Z_{t+1}|\mathcal{F}_t]\Big)_+^s \Big|\mathcal{F}_t\Big]^{\frac{1}{s}}$$

Here $s \in [1, \bar{s}]$ and $\kappa \in [0,1]$ may be $\mathcal{F}_t$-measurable

# Coherent One-Step Conditional Risk Measures

Stronger assumptions about one-step measures $\rho_t : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$:

- Convexity: $\rho_t(\lambda Z + (1-\lambda)W) \le \lambda\rho_t(Z) + (1-\lambda)\rho_t(W)$
  $\forall\ \lambda \in (0,1),\ Z, W \in \mathcal{Z}_{t+1}$
- Monotonicity: If $Z \le W$ then $\rho_t(Z) \le \rho_t(W)$, $\forall\ Z, W \in \mathcal{Z}_{t+1}$
- Predictable Translation Equivariance:
  $\rho_t(Z+W) = Z + \rho_t(W)$, $\forall\ Z \in \mathcal{Z}_t,\ W \in \mathcal{Z}_{t+1}$
- Positive Homogeneity: $\rho_t(\tau Z) = \tau\rho_t(Z)$, $\forall\ Z \in \mathcal{Z}_{t+1},\ \tau \ge 0$

Scandolo ('03), Riedel ('04), R.-Shapiro ('06), Cheridito-Delbaen-Kupper ('06), Föllmer-Penner ('06), Artzner-Delbaen-Eber-Heath-Ku ('07), Pflug-Römisch ('07)

## Example: Conditional Mean–Semideviation

$$\rho_t(Z_{t+1}) = \mathbb{E}[Z_{t+1}|\mathcal{F}_t] + \kappa\mathbb{E}\Big[\big(Z_{t+1} - \mathbb{E}[Z_{t+1}|\mathcal{F}_t]\big)_+^s\Big|\mathcal{F}_t\Big]^{\frac{1}{s}}$$

Here $s \in [1, \bar{s}]$ and $\kappa \in [0,1]$ may be $\mathcal{F}_t$-measurable

# Coherent One-Step Conditional Risk Measures

Stronger assumptions about one-step measures $\rho_t : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$:

- **Convexity:** $\rho_t(\lambda Z + (1-\lambda)W) \le \lambda\rho_t(Z) + (1-\lambda)\rho_t(W)$
  $\forall\ \lambda \in (0,1),\ Z, W \in \mathcal{Z}_{t+1}$
- **Monotonicity:** If $Z \le W$ then $\rho_t(Z) \le \rho_t(W),\ \forall\ Z, W \in \mathcal{Z}_{t+1}$
- **Predictable Translation Equivariance:**
  $\rho_t(Z+W) = Z + \rho_t(W),\ \forall\ Z \in \mathcal{Z}_t,\ W \in \mathcal{Z}_{t+1}$
- **Positive Homogeneity:** $\rho_t(\tau Z) = \tau\rho_t(Z), \forall\ Z \in \mathcal{Z}_{t+1},\ \tau \ge 0$

Scandolo ('03), Riedel ('04), R.-Shapiro ('06), Cheridito-Delbaen-Kupper ('06), Föllmer-Penner ('06), Artzner-Delbaen-Eber-Heath-Ku ('07), Pflug-Römisch ('07)

## Example: Conditional Mean–Semideviation

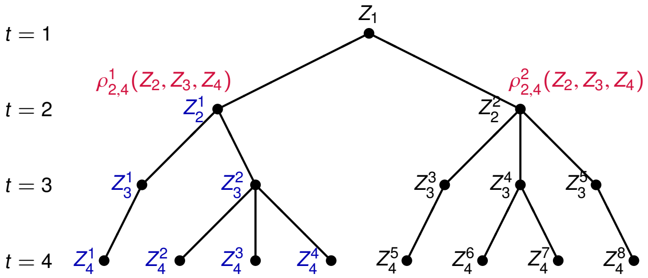
$$\rho_t(Z_{t+1}) = \mathbb{E}[Z_{t+1}|\mathcal{F}_t] + \kappa\mathbb{E}\Big[\big(Z_{t+1} - \mathbb{E}[Z_{t+1}|\mathcal{F}_t]\big)_+^s\Big|\mathcal{F}_t\Big]^{\frac{1}{s}}$$

Here $s \in [1, \bar{s}]$ and $\kappa \in [0,1]$ may be $\mathcal{F}_t$-measurable

# Local Property

A conditional risk measure $\rho_{t,T} : \mathcal{Z}_{t,T} \to \mathcal{Z}_t$ has the local property, if for every event $A \in \mathcal{F}_t$ we have the equation

$$\rho_{t,T}(\mathbb{1}_A Z_t, \mathbb{1}_A Z_{t+1}, \dots, \mathbb{1}_A Z_T) = \mathbb{1}_A \, \rho_{t,T}(Z_t, Z_{t+1}, \dots, Z_T)$$

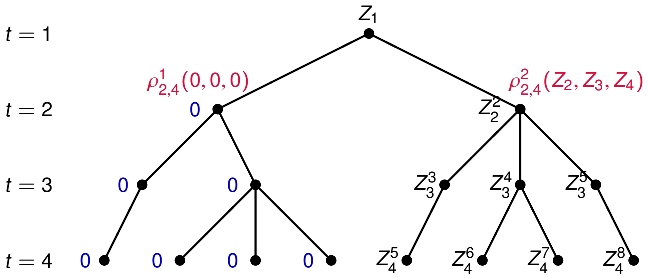



Automatic for coherent conditional risk measures

# Local Property

A conditional risk measure $\rho_{t,T} : \mathcal{Z}_{t,T} \to \mathcal{Z}_t$ has the local property, if for every event $A \in \mathcal{F}_t$ we have the equation

$$\rho_{t,T}(\mathbb{1}_A Z_t, \mathbb{1}_A Z_{t+1}, \dots, \mathbb{1}_A Z_T) = \mathbb{1}_A\, \rho_{t,T}(Z_t, Z_{t+1}, \dots, Z_T)$$




Automatic for coherent conditional risk measures

# Multistage Risk-Averse Optimization Problems

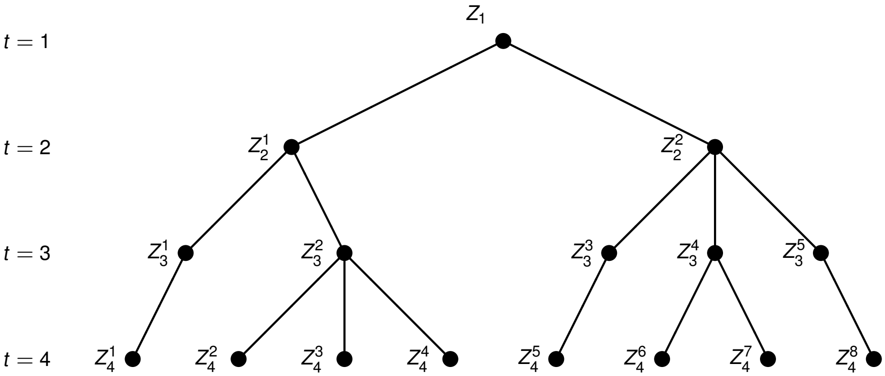
Probability Space: $(\Omega, \mathcal{F}, P)$ with filtration $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T \subset \mathcal{F}$
Decision Variables: $x_t(\omega)$, $\omega \in \Omega$, $t = 1, \ldots, T$
Nonanticipativity: Each $x_t$ is $\mathcal{F}_t$-measurable
Cost per Stage: $Z_t(x_t)$ with realizations $Z_t(x_t(\omega), \omega)$, $\omega \in \Omega$
Objective Function: Time-consistent dynamic measure of risk

## Interchangeability for Time-Consistent Measures

$$\min_{x_1, x_2(\cdot), \ldots, x_T(\cdot)} \Big\{ Z_1(x_1) + \rho_1\Big( Z_2(x_2) + \rho_2\Big( Z_3(x_3) + \ldots \cdots + \rho_{T-2}\Big( Z_{T-1}(x_{T_1} + \rho_{T-1}(Z_T(x_T))\Big) \cdots \Big)\Big)\Big\}$$

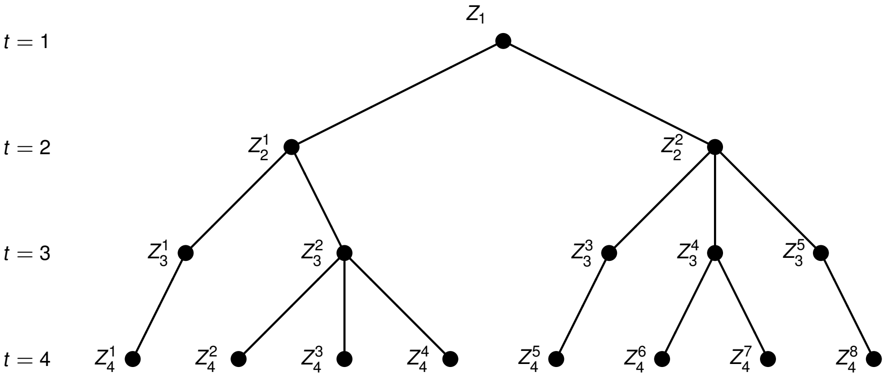
$$= \min_{x_1} \Big\{ Z_1(x_1) + \rho_1\Big[ \min_{x_2} \Big( Z_2(x_2) + \rho_2\Big[ \min_{x_3} \Big( Z_3(x_3) + \ldots \cdots + \rho_{T-2}\Big[ \min_{x_{T-1}} \Big( Z_{T-1}(x_{T_1}) + \rho_{T-1}\big(\min_{x_T} Z_T(x_T)\big)\Big)\Big] \cdots \Big)\Big]\Big)\Big]\Big\}$$

# Interchangeability on a Scenario Tree

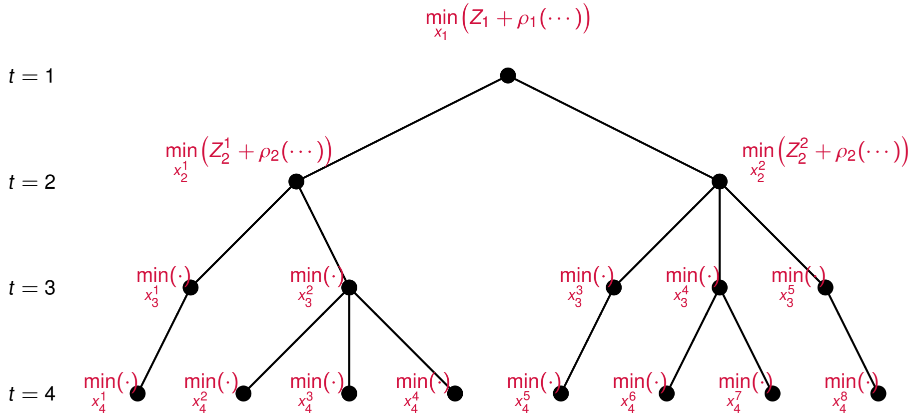


# Interchangeability on a Scenario Tree

$$\min_{x_1, x_2(\cdot), x_3(\cdot), x_4(\cdot)} \rho_{1,4}(Z_1, Z_2, Z_3, Z_4)$$



# Interchangeability on a Scenario Tree



# Linear Risk-Averse Multistage Optimization

$(\Omega, \mathcal{F}, P)$ - probability space with filtration $\{\emptyset, \Omega\} = \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_T = \mathcal{F}$.
A random $x = (x_1, \ldots, x_T)$ is a policy.
If each $x_t$ is $\mathcal{F}_t$-measurable, policy $x$ is implementable (belongs to $I$).
A policy $x$ is feasible (belongs to $F$), if it satisfies the conditions:

$$
\begin{array}{lllllll}
A_1 x_1 & & & & & = & b_1 \\
B_2 x_1 & + & A_2 x_2 & & & = & b_2 \\
\multicolumn{7}{c}{\dotfill} \\
 & & & B_T x_{T-1} & + \; A_T x_T & = & b_T \\
x_1 \in X_1, & & x_2 \in X_2, & \ldots & x_T \in X_T & &
\end{array}
$$

Each set $X_t$ is an $\mathcal{F}_t$-measurable convex and closed polyhedron.
Suppose $c_t$, $t = 1, \ldots, T$, is an adapted sequence of random cost vectors.
A policy $x$ results in a cost sequence $Z_t = \langle c_t, x_t \rangle$, $t = 1, \ldots, T$.

**Risk-averse multistage stochastic optimization problem**

$$
\min_{x \in I \cap F} \varrho(Z_1, Z_2, \ldots, Z_T) \qquad (\varrho \text{ - dynamic measure of risk})
$$

# Risk Evaluation on a Tree

## Scenario tree

Nodes $n \in \mathcal{N}$, organized in levels $\Omega_t$ corresponding to stages $1, \ldots, T$.
At level $t = 1$ - only one root node $n = 1$
Node $n$ at level $t$ is connected to an ancestor node $a(n)$ at level $t-1$
Node $n$ at level $t$ is connected to a set $C(n)$ of children nodes at $t+1$

## Value Function

$\mathcal{Q}_n\big(x_{a(n)}\big)$ - the best value of a subproblem rooted at node $n$, given $x_{a(n)}$
$Q_C$ vector of value functions at nodes in the set $C$

## Dynamic Programming Equations

$$\mathcal{Q}_n\big(x_{a(n)}\big) = \min_{x_n} \Big\{ \langle c_n, x_n \rangle : B_n x_{a(n)} + A_n x_n = b_n,\ x_n \in X_n \Big\}, \quad n \in \Omega_T,$$

$$\mathcal{Q}_n\big(x_{a(n)}\big) = \min_{x_n} \Big\{ \langle c_n, x_n \rangle + \rho_n\Big(\mathcal{Q}_{C(n)}\big(x_n\big)\Big) : \\ B_n x_{a(n)} + A_n x_n = b_n,\ x_n \in X_n \Big\}, \quad n \in \Omega_t, \quad t = T-1, \ldots, 1$$

The optimal value functions $\mathcal{Q}_n(\cdot)$ are convex.

# Controlled Markov Models

- State space $\mathcal{X}$ (Borel)
- Control space $\mathcal{U}$ (Borel)
- Feasible control set $U : \mathcal{X} \rightrightarrows \mathcal{U}$, $t = 1, 2, \ldots$
- Controlled transition kernel $Q : \text{graph}(U) \to \mathcal{P}(\mathcal{X})$, $t = 1, 2, \ldots$
  $\mathcal{P}(\mathcal{X})$ - set of probability measures on $\mathcal{X}$
- Cost functions $c : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$, $t = 1, 2, \ldots$
- State history $h_t = (x_1, \ldots, x_t) \in \mathcal{X}^t$ (up to time $t = 1, 2, \ldots$)
- Policy $\pi_t : \mathcal{X}^t \to \mathcal{U}$, $t = 1, 2, \ldots$ (always supported in $U(x_t)$)
- Markov policy $\pi_t : \mathcal{X} \to \mathcal{U}$, $t = 1, 2, \ldots$
  (stationary if $\pi_t = \pi_1$ for all $t$)

$$x_t \longrightarrow u_t = \pi_t(x_t)$$

$$(x_t, u_t) \longrightarrow x_{t+1} \sim Q(x_t, u_t)$$

# Risk-Neutral Total Cost Problem

Infinite horizon expected cost problem:

$$\min_{\pi_1,\pi_2,\dots} \mathbb{E}^{\Pi}\left[\sum_{t=1}^{\infty} \alpha^{t-1} c_t(x_t, u_t)\right], \quad \alpha \in (0,1]$$

with controls $u_t = \pi_t(x_1, \dots, x_t)$

**Two Cases:**

Discounted models (with $\alpha < 1$) and transient models (with $\alpha = 1$)

**Standard Results:**

- A deterministic Markov policy is optimal
- Optimal policy can be found by dynamic programming equations

**Our Intention**

Introduce risk aversion to the problem by replacing the expected value by dynamic risk measures

# Using Dynamic Risk Measures for Markov Decision Processes

- Controlled Markov process $x_t^{\Pi}$, $t = 1, \ldots, T$
- Policy $\Pi = \{\pi_1, \pi_2, \ldots, \pi_T\}$ with $u_t = \pi_t(x_t)$ implies measure $P^{\Pi}$
- Cost sequence $Z_t^{\Pi} = c(x_t^{\Pi}, \pi_t(x_t^{\Pi}))$, $Z_t \in \mathcal{Z}_t$, $t = 1, \ldots, T$,
- Dynamic time-consistent risk measure

$$J_T(\Pi) = Z_1^{\Pi} + \rho_1^{\Pi}\Big(Z_2^{\Pi} + \cdots + \rho_{T-1}^{\Pi}(Z_T^{\Pi}) \cdots\Big)\Big)$$

- Risk-averse optimal control problem

$$\min_{\Pi} \lim_{T \to \infty} J_T(\Pi)$$

**Difficulties**

- Probability measure $P^{\Pi}$, processes $x_t^{\Pi}$ and $Z_t^{\Pi}$ depend on policy $\Pi$
- The risk measures $\rho_t^{\Pi}(\cdot)$ depend on $\Pi$ and may depend on history; no Markov policies

- State space $\mathcal{X}$ (Borel)
- Control space $\mathcal{U}$ (Borel)
- State history $h_t = (x_1, \ldots, x_T) \in \mathcal{X}^t$ (up to time $t = 1, 2, \ldots$)
- Controlled transition kernels $Q_t : \mathcal{X}^t \times \mathcal{U} \to \mathcal{P}(\mathcal{X})$,
  $\mathcal{P}(\mathcal{X})$ - set of probability measures on $\mathcal{X}$
- Feasible control sets $U_t : \mathcal{X}^t \rightrightarrows \mathcal{U}$, $t = 1, 2, \ldots$
- Cost functions $c_t : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$, $t = 1, 2, \ldots$
- Policy $\pi_t : \mathcal{X}^t \to \mathcal{U}$, $t = 1, 2, \ldots$ (always supported in $U_t(h_t)$)

$$h_t \longrightarrow u_t = \pi_t(h_t)$$

$$(h_t, u_t) \longrightarrow x_{t+1} \sim Q_t(h_t, u_t) = Q_t^{\Pi}(h_t)$$

We only need to evaluate risk of processes $Z_t^{\Pi}(h_t) = c(x_t, \pi_t(h_t))$, $t = 1, \ldots, T$, which are measurable functions of the history $h_t$

History $h_t = (x_1, \dots, x_t)$. Process $Z_t^{\Pi}(h_t) = c(x_t, \pi_t(h_t))$, $t = 1, \dots, T$

A family of conditional risk measures $\{\rho_{t,T}^{\Pi}\}_{t=1,\dots,T}^{\pi \in \Pi}$ is stochastically conditionally time-consistent if for all feasible policies $\Pi, \Pi'$, all $1 \le t \le T-1$, and for all histories $h_t, h_t' \in \mathcal{X}^t$, the relations

$$Z_t^{\Pi}(h_t) = Z_t^{\Pi'}(h_t')$$

$$\Big(\rho_{t+1,T}^{\Pi}(Z_{t+1}^{\Pi}, \dots, Z_T^{\Pi}) \Big| H_t^{\Pi} = h_t\Big) \preceq_{\text{st}} \Big(\rho_{t+1,T}^{\Pi'}(Z_{t+1}^{\Pi'}, \dots, Z_T^{\Pi'}) \Big| H_t^{\Pi'} = h_t'\Big)$$

imply

$$\rho_{t,T}^{\Pi}(Z_t^{\Pi}, \dots, Z_T^{\Pi})(h_t) \le \rho_{t,T}^{\Pi'}(Z_t^{\Pi'}, \dots, Z_T^{\Pi'})(h_t').$$

The conditional stochastic order $\preceq_{\text{st}}$:

$$Q_t^{\Pi}(h_t)\Big(\{y : Z_t^{\Pi}(h_t) + \rho_{t+1,T}^{\Pi}(Z_{t+1}^{\Pi}, \dots, Z_T^{\Pi})(h_t, y) > \eta\}\Big)$$
$$\le Q_t^{\Pi'}(h_t')\Big(\{y : Z_t^{\Pi'}(h_t') + \rho_{t+1,T}^{\Pi'}(Z_{t+1}^{\Pi'}, \dots, Z_T^{\Pi'})(h_t', y) > \eta\}\Big)$$

The processes evaluated are $Z_t^{\Pi}(h_t) = c(x_t, \pi_t(h_t))$, $t = 1, \ldots, T$

A family of dynamic risk measures $\{(\rho_{t,T}^{\Pi})_{t=1,\ldots,T} : \Pi \in \Pi\}$ is translation-invariant and stochastically conditionally time-consistent if and only if there exist functionals

$$\sigma_t : \mathcal{V} \times (\cup_{\Pi \in \Pi} \text{Graph}(Q_t^{\Pi})) \to \mathbb{R}, \quad t = 1 \ldots T-1,$$

where $\mathcal{V}$ is the set of measurable functions on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, and

$$\rho_{t,T}^{\Pi}(Z_t^{\Pi}, \ldots, Z_T^{\Pi})(h_t) = Z_t^{\Pi}(h_t) + \sigma_t\Big(\rho_{t+1,T}^{\Pi}(Z_{t+1}^{\Pi}, \ldots, Z_T^{\Pi})(h_t, \cdot), h_t, Q_t^{\Pi}(h_t)\Big)$$

For all $\Pi \in \Pi$, $h_t \in \mathcal{X}^t$, the function $\sigma_t\Big(\cdot, h_t, Q_t^{\Pi}(h_t)\Big)$ is a law-invariant risk measure on $(\mathcal{X}, \mathcal{B}(\mathcal{X}), Q_t^{\Pi}(h_t))$.

The mapping $\sigma_t$ does not depend on $\Pi$: the policy only affects the equation through the next state's distribution $Q_t^{\Pi}(h_t)$.

A family of process-based dynamic risk measures $\left\{\rho^{\Pi}_{t,T}\right\}_{t=1,\dots,T}^{\Pi\in\Pi}$ for a Markov decision problem is Markov if for all Markov policies $\Pi \in \Pi$, for any measurable $c_1, \dots, c_T : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$, and for all $h_t = (x_1, \dots, x_t)$ and $h'_t = (x'_1, \dots, x'_t)$ such that $x_t = x'_t$, we have

$$\rho^{\Pi}_{t,T}\Big(c_t(X_t, \pi_t(X_t)), \dots, c_T(X_T, \pi_T(X_T))\Big)(h_t) = \rho^{\Pi}_{t,T}\Big(c_t(X_t, \pi_t(X_t)), \dots, c_T(X_T, \pi_T(X_T))\Big)(h'_t).$$

If the current state $x_t$ is the same, and the same Markov policy $\Pi$ is used, then the risk is the same.

# Structure of Markov Risk Measures (with Jingnan Fan)

For a fixed history-dependent policy $\Pi$ and every $h_t \in \mathcal{X}^t$, we write

$$v_t^{c,\Pi}(h_t) = \rho_{t,T}^{\Pi}\Big(c_t(X_t, \pi_t(H_t)), \ldots, c_T(X_T, \pi_T(H_T))\Big)(h_t)$$

If a family of process-based dynamic risk measures $\left\{\rho_{t,T}^{\Pi}\right\}_{t=1,\ldots,T}^{\Pi\in\Pi}$ is Markov, translation-invariant, and stochastically conditionally time-consistent, then there exist transition risk mappings

$$\sigma_t : \mathcal{V} \times \Big\{\Big(x, Q_t(x,u)\Big) \,:\, u \in U(x),\ x \in \mathcal{X}\Big\} \to \mathbb{R}, \quad t = 1, \ldots, T-1$$

such that for all $\Pi \in \Pi$, for all $t = 1, \ldots, T-1$, and all $h_t \in \mathcal{X}^t$, the functional $\sigma_t\Big(\cdot, x_t, Q_t(x_t, \pi_t(h_t))\Big)$ is a law-invariant risk measure on $\Big(\mathcal{X}, \mathcal{B}(\mathcal{X}), Q_t(x_t, \pi_t(h_t))\Big)$. Moreover, for any $c = \{c_t\}_{t=1\ldots T}$, we have

$$v_t^{c,\Pi}(h_t) = c_t(x_t, \pi_t(h_t)) + \sigma_t\Big(v_{t+1}^{c,\Pi}(h_t, \cdot), x_t, Q_t(x_t, \pi_t(h_t))\Big),\ t = 1, \ldots, T-1$$

# Dual Representation

From now on we assume that $\sigma_t(\cdot, x, m)$ is a coherent risk measure on $\mathcal{V} = \mathcal{L}_p(\mathcal{X}, \mathcal{B}, P_0)$.

## Dual representation of transition risk mappings

$$\sigma(v, x, m) = \max_{\mu \in \mathcal{A}(x,m)} \int_{\mathcal{X}} v(y)\, \mu(dy)$$

## Example: Mean–Semideviation

$$\sigma(v, x, m) = \int v\, dm + \kappa(x) \Big( \int \Big( v - \int v\, dm \Big)_+^p \, dm \Big)^{\frac{1}{p}}$$

For $p > 1$ we obtain

$$\mathcal{A}(x, m) = \Big\{ g = m\Big(1 + h - \int h\, dm\Big) : \|h\|_{\mathcal{L}_q(\mathcal{X},\mathcal{B},m)} \le \kappa(x),\ h \ge 0 \Big\}$$

# General Assumptions for Markov Models

**G0.** For all $x \in \mathcal{X}$, $u \in U_t(x)$ the measure $Q_t(x,u)$ is an element of $\mathcal{V}'$;

**G1.** The transition kernel $Q_t(\cdot,\cdot)$ is setwise continuous;

**G2.** The multifunctions $\mathcal{A}_t(\cdot,\cdot) \equiv \partial_\varphi \sigma_t(0,\cdot,\cdot)$ are lower semicontinuous;

**G3.** The functions $c_t(\cdot,\cdot)$ are measurable, $w$-bounded, and lower semicontinuous;

**G4.** The multifunctions $U_t(\cdot)$ are measurable and compact-valued.

# Finite Horizon Risk-Averse Control Problem

Consider a controlled Markov process $\{x_t\}$ with $u_t = \pi_t(x_1, \dots, x_t)$.

Risk-averse optimal control problem:

$$\min_{\Pi} \; J_T(\Pi, x_1) = c_1(x_1, u_1) + \rho_1^{\Pi}\Big( c_2(x_2, u_2) + \cdots + \rho_{T-1}^{\Pi}\Big( c_T(x_T, u_T) + \rho_T\big( c_{T+1}(x_{T+1}) \big) \cdots \Big)\Big).$$

**Theorem**

If the conditional measures $\rho_t^{\Pi}$ are Markov (+ general conditions), then the optimal solution is given by the dynamic programming equations:

$$v_{T+1}(x) = c_{T+1}(x), \quad x \in \mathcal{X}$$

$$v_t(x) = \min_{u \in U(x)} \Big\{ c_t(x, u) + \sigma_t\big( v_{t+1}, x, Q_t(x, u) \big) \Big\}, \quad x \in \mathcal{X}, \quad t = T, \dots, 1.$$

Optimal Markov policy $\hat{\Pi} = \{\hat{\pi}_1, \dots, \hat{\pi}_T\}$ - the minimizers above

# Finite Horizon Risk-Averse Control Problem

Consider a controlled Markov process $\{x_t\}$ with $u_t = \pi_t(x_1, \dots, x_t)$.

Risk-averse optimal control problem:

$$\min_{\Pi} \; J_T(\Pi, x_1) = c_1(x_1, u_1) + \rho_1^{\Pi}\Big(c_2(x_2, u_2) + \cdots + \rho_{T-1}^{\Pi}\Big(c_T(x_T, u_T) + \rho_T\big(c_{T+1}(x_{T+1})\big)\cdots\Big)\Big).$$

**Theorem**

If the conditional measures $\rho_t^{\Pi}$ are Markov (+ general conditions), then the optimal solution is given by the dynamic programming equations:

$$v_{T+1}(x) = c_{T+1}(x), \quad x \in \mathcal{X}$$

$$v_t(x) = \min_{u \in U(x)} \Big\{ c_t(x, u) + \max_{\mu \in \mathcal{A}_t(x, Q_t(x,u))} \mathbb{E}_\mu\big[v_{t+1}\big] \Big\}, \quad x \in \mathcal{X}, \quad t = T, \dots, 1.$$

Optimal Markov policy $\hat{\Pi} = \{\hat{\pi}_1, \dots, \hat{\pi}_T\}$ - the minimizers above

# Infinite Horizon Risk (for stationary models)

Discounted risk measure ($0 < \alpha < 1$)

$$J_T^{\alpha}(\Pi, x) = Z_1^{\Pi} + \rho_1^{\Pi}\Big(\alpha Z_2^{\Pi} + \cdots + \rho_{T-1}^{\Pi}\big(\alpha^{T-1} Z_T^{\Pi}\big)\cdots\Big)$$

Optimal cost: $J^*(x) = \inf_{\Pi} \lim_{T\to\infty} J_T^{\alpha}(\Pi, x)$

Assume that the model is stationary, the conditional risk measures $\rho_t$, $t = 1, \dots, T$, are Markov (+ technical conditions). Then a bounded function $v : \mathcal{X} \to \mathbb{R}$ satisfies the dynamic programming equations

$$v(x) = \min_{u \in U(x)} \Big\{ c(x,u) + \alpha\sigma\big(v, x, Q(x,u)\big)\Big\}, \quad x \in \mathcal{X},$$

if and only if $v(\cdot) \equiv J^*(\cdot)$. Moreover, the minimizer $\pi^*(x)$, $x \in \mathcal{X}$, on the right hand side exists and defines an optimal Markov policy $\Pi^* = \{\pi^*, \pi^*, \dots\}$.

If $\alpha = 1$ additional conditions of risk transient models

# Infinite Horizon Risk (for stationary models)

Discounted risk measure $(0 < \alpha < 1)$

$$J_T^\alpha(\Pi, x) = Z_1^\Pi + \rho_1^\Pi\Big(\alpha Z_2^\Pi + \cdots + \rho_{T-1}^\Pi\big(\alpha^{T-1} Z_T^\Pi\big)\cdots\Big)$$

Optimal cost: $J^*(x) = \inf_{\Pi} \lim_{T\to\infty} J_T^\alpha(\Pi, x)$

Assume that the model is stationary, the conditional risk measures $\rho_t$, $t = 1, \dots, T$, are Markov (+ technical conditions). Then a bounded function $v : \mathcal{X} \to \mathbb{R}$ satisfies the dynamic programming equations

$$v(x) = \min_{u\in U(x)} \Big\{ c(x,u) + \alpha \max_{\mu\in\mathcal{A}(x,Q(x,u))} \mathbb{E}_\mu[v] \Big\}, \quad x \in \mathcal{X},$$

if and only if $v(\cdot) \equiv J^*(\cdot)$. Moreover, the minimizer $\pi^*(x)$, $x \in \mathcal{X}$, on the right hand side exists and defines an optimal Markov policy $\Pi^* = \{\pi^*, \pi^*, \dots\}$.

If $\alpha = 1$ additional conditions of risk transient models

For every $x$ we define the set of probability measures:

$$\mathfrak{M}^{\pi}(x) = \mathcal{A}\Big(x, Q(x, \pi(x))\Big), \quad x \in \mathcal{X}$$

The multifunction $\mathfrak{M}^{\pi} : \mathcal{X} \rightrightarrows \mathcal{P}(\mathcal{X})$ is a risk multikernel, associated with the risk transition mapping $\sigma(\cdot,\cdot,\cdot)$, the kernel $Q$, and decision rule $\pi$.

**Key formula for Markov policy $\Pi = \{\pi, \pi, \ldots\}$**

$$\rho_t^{\Pi}\Big(v(x_{t+1})\Big) = \max_{M \in \mathfrak{M}^{\pi}(x_t)} \int_{\mathcal{X}} v(y)\; M(dy)$$

A Markov model is risk-transient if

$$\big\|M\big\|_w \leq K \quad \text{for all} \quad M \lessdot \sum_{j=1}^{T} \Big(\widetilde{\mathfrak{M}}^{\pi}\Big)^j \quad \text{and all} \quad T \geq 0$$

# Solution Methods

## Value iteration

$$v^{k+1}(x) = \min_{u \in U(x)} \Big\{ c(x,u) + \alpha\sigma\Big(v^k, x, Q(x,u)\Big) \Big\}, \quad x \in \mathcal{X}, \quad k = 1, 2, \ldots$$

## Policy iteration

- For $k = 0, 1, 2, \ldots$, given a stationary Markov policy $\{\pi^k, \pi^k, \ldots\}$, find the value function $v^k$ by solving (by a specialized Newton method) the nonsmooth equation

$$v(x) = c(x, \pi^k(x)) + \alpha\sigma\Big(v, x, Q(x, \pi^k(x))\Big), \quad x \in \mathcal{X}$$

- Find the next policy $\pi^{k+1}(\cdot)$ by one-step optimization

$$\pi^{k+1}(x) = \operatorname*{argmin}_{u \in U(x)} \Big\{ c(x,u) + \alpha\sigma\Big(v^k, x, Q(x,u)\Big) \Big\}, \quad x \in \mathcal{X}$$

For $\alpha = 1$ additional conditions of risk transient models
+ positive or negative $c(\cdot,\cdot)$ for the value iteration method

# Optimal Stopping - Asset Selling Example (with Özlem Çavuş)

Offers $Y_t$ arriving in time periods $t = 1, 2, \dots$ are i.i.d. integrable random variables. At each time we may accept the highest offer so far, or wait, at cost $c_0$.

The expected value solution: accept the first offer greater than or equal to the solution $\hat{x}$ of the equation

$$\mathbb{E}\Big[(Y - \hat{x})_+\Big] = c_0.$$

Risk-averse DP equation:

$$v(x) = \min\Big\{ -x, c_0 + \sigma\Big(v, x, Q(x)\Big)\Big\}, \quad x \in \mathbb{R}_+$$

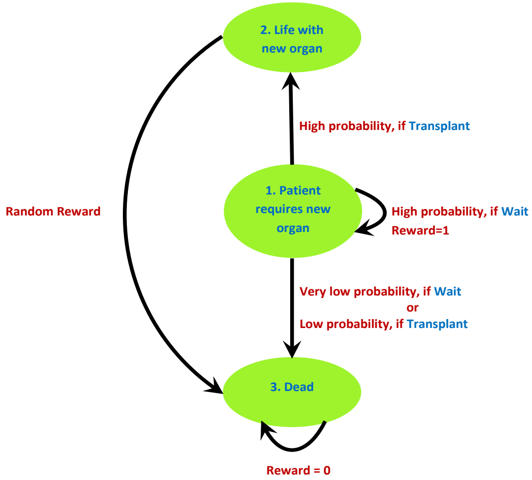
Suppose $\sigma$ is law invariant and does not depend on the second argument.

Risk-averse solution: accept any offer that is greater or equal to the solution $x^*$ of the equation

$$c_0 = \min_{\mu \in \mathcal{A}} \mathbb{E}_\mu\Big[(Y - x^*)_+\Big] \qquad (\mathcal{A} \text{ - subdifferential of } \sigma).$$

If $x < x^*$, then wait.

# Example: Organ Transplantation (with Özlem Çavuş)



# Example: Organ Transplantation

- **Expected Total Reward:**

  The optimal policy is to wait

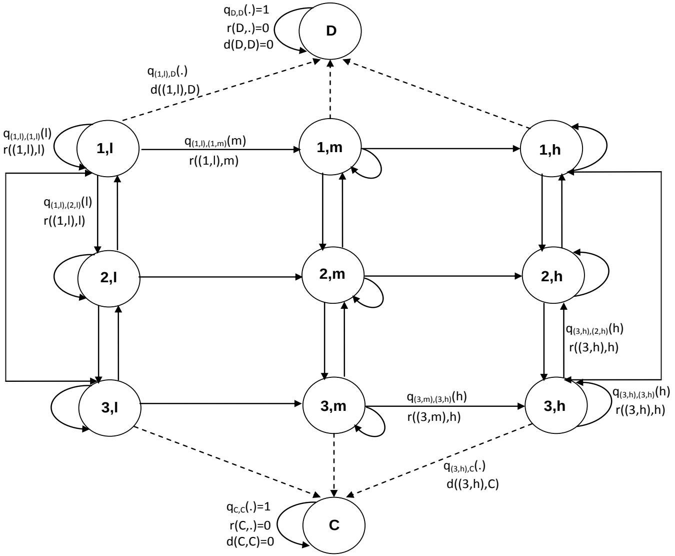
- **Mean Semi-Deviation with Deterministic Policies:**

  The optimal policy is to transplant

- **Mean Semi-Deviation with Randomized Policies:**

  Wait with probability 0.993983 and transplant with probability 0.006017

# Credit Card Example (with Özlem Çavuş)



# Optimal Policy

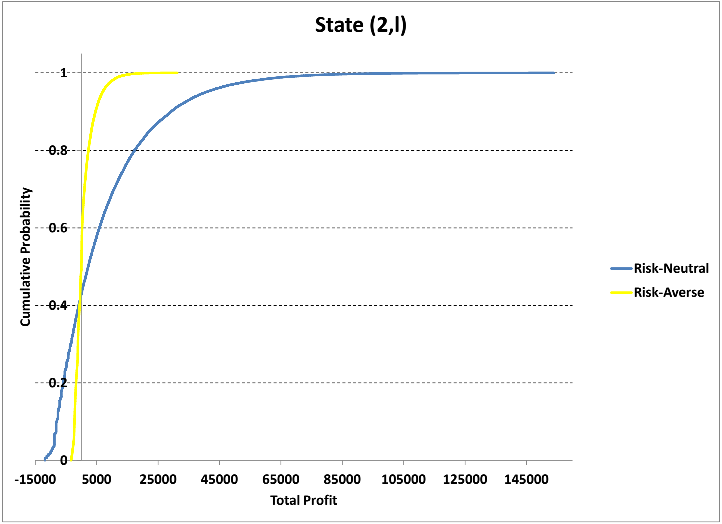
| $\kappa$ | (1,l) | (1,m) | (1,h) | (2,l) | (2,m) | (2,h) | (3,l) | (3,m) | (3,h) |
|---|---|---|---|---|---|---|---|---|---|
| 0.025 | m | h | h | m | h | h | m | m | h |
| 0.1 | l | h | h | m | h | h | m | m | h |
| 0.2 | l | h | h | m | h | h | m | m | h |
| 0.3 | l | h | h | m | h | h | m | m | h |
| 0.4 | l | h | h | m | h | h | m | m | h |
| 0.5 | l | h | h | m | h | h | m | m | h |
| 0.6 | l | h | h | m | h | h | m | m | h |
| 0.7 | l | h | h | m | h | h | m | m | h |
| 0.8 | l | m | h | l | h | h | m | m | h |
| 0.9 | l | m | h | l | m | h | m | m | h |
| 1 | l | m | h | l | m | h | m | m | h |

$\kappa$ - risk aversion coefficient

# Comparison of Methods

| $\kappa$ | # of Value Iterations | # of Policy Iterations | # of Newton Iterations |
|---|---|---|---|
| 0.025 | 869 | 3 | 4,3,3 |
| 0.1 | 797 | 4 | 3,3,2,3 |
| 0.2 | 746 | 4 | 3,3,2,2 |
| 0.3 | 689 | 4 | 4,2,2,2 |
| 0.4 | 658 | 4 | 4,2,2,2 |
| 0.5 | 661 | 4 | 4,2,2,2 |
| 0.6 | 761 | 3 | 4,3,3 |
| 0.7 | 893 | 3 | 4,2,3 |
| 0.8 | 525 | 3 | 4,3,2 |
| 0.9 | 1354 | 3 | 5,2,3 |
| 1 | 1231 | 3 | 6,2,3 |

# Profit Distribution



# References for Part II


- A. Ruszczyński and A. Shapiro, Conditional risk mappings, *Mathematics of Operations Research* 31 (2006) 544–561
- A. Ruszczyński, Risk-averse dynamic programming for Markov decision processes, *Mathematical Programming*, Series B 125 (2010) 235–261
- J. Fan and A. Ruszczyński, Process-based risk measures for observable and partially observable discrete-time controlled systems, *submitted for publication*.
- Ö. Çavuş and A. Ruszczyński, Computational methods for risk-averse undiscounted transient Markov models, *Operations Research*, 62 (2), 2014, 401–417.
- Ö. Çavuş and A. Ruszczyński, Risk-averse control of undiscounted transient Markov models, *SIAM J. on Control and Optimization*, in press.