Post-doctoral Research Project
Title: High frequency limits in the Kaluza-Klein setting

Higher dimensional gravity theories have attracted a great deal of attention in the physics community as possible models for quantum gravity and as candidates to give a unified description of the main forces in nature. The first of such theories was formulated by Kaluza in 1921 and revisited by Klein in 1926, with the aim of describing gravity and electromagnetism. It postulates that the universe is a 5-dimensional Lorentzian manifold $(\mathbb{R}^{1+3} \times \mathbb{S}^1, {}^{(5)}g)$, symmetric in the $\mathbb{S}^1$ direction and with ${}^{(5)}g$ decomposing as follows

$$^{(5)}g = g_{\alpha\beta}dx^\alpha dx^\beta + \Phi(dy + \mathcal{A}_\alpha dx^\alpha)^2.$$

In the above right hand side, x^α are rectangular coordinates in $\mathbb{R}^{1+3}$ and y is the periodic coordinate, g is a 4-dimensional metric, ϕ a scalar function and $\mathcal{A}$ a one-form. The 5-dimensional Einstein vacuum equations (EVE) for ${}^{(5)}g$ yield a Einstein-Maxwell-scalar field system on $\mathbb{R}^{1+3}$ for the new unknowns $\tilde{g} = \Phi^{\frac{1}{2}}g$, $F_{\alpha\beta} = \partial_\alpha \mathcal{A}_\beta - \partial_\beta \mathcal{A}_\alpha$ and $f = \ln \Phi$.

Additional compact directions are also considered in string theory and M-theory. If one believes that higher dimensional theories are truly fundamental, then it is important to study all their predictions, including higher dimensional solutions. In a recent work [6] in collaboration with Huneau and Wyatt, we proved that the Kaluza-Klein spacetime $(\mathbb{R}^{1+3} \times \mathbb{S}^1, g)$, with $g = -dt^2 + dx^2 + dy^2$, is stable as a solution to the 5-dimensional EVE under generic small perturbations, confirming a conjecture by Witten [8] and generalizing a result by Wyatt [9] in which only symmetric perturbations were considered. Our result also applies to metrics $g_\lambda = -dt^2 + dx^2 + \lambda^2 dy^2$ on $\mathbb{R}^{1+3} \times \mathbb{S}^1$, for any $\lambda > 0$. Non-symmetric perturbations of such metrics are highly oscillatory in the limit $\lambda \to 0$.

In recent works by Huneau-Luk [4, 5], the limiting behavior of high-frequency solutions to the 4-dimensional EVE is investigated. The so-called *back-reaction effect* was conjectured by Burnett in [1]: in the special regime where a sequence of metrics g_λ, solutions to the EVE, is such that

- g_λ converges uniformly to a smooth Lorentzian metric g_0, when $\lambda \to 0$;
- the derivatives ∂g_λ are locally bounded, uniformly with respect to λ,

the metric g_0 is not in general solution to the EVE. The nonlinearities in EVE, in particular terms of the form $(\partial g_\lambda)^2$, lead to a defect of convergence and an effective stress-energy tensor arises in the equations satisfied by g_0, corresponding to back-reaction. This tensor is conjectured by Burnett to be the stress-energy tensor of a massless Vlasov field. In [1], the reverse question is also formulated: given a solution g_0 to the Einstein equations coupled with a massless Vlasov field, is it possible to construct a sequence of metrics g_λ, solutions to the EVE, such that g_λ converges uniformly to g_0, with derivatives ∂g_λ uniformly locally bounded? Advances in that direction are to be found in a paper by Touati [7], where a regular solution to the Einstein-null dust systems - with restriction on the number of dusts - is shown to be the limit of a sequence of vacuum solutions in the sense described above.

In the Kaluza-Klein context, the presence of a compact direction shrinking to zero also leads to a back-reaction effect and first computations on a toy model suggest that the effective stress-energy tensor should be that of non-isotropic matter, which is a striking difference compared to the results in [4, 5]. In fact, in contrast to the 4-dimensional case in which the EVE reduce to quasilinear waves equations in the so-called *wave gauge*, the intrinsic nature of the EVE in this higher dimensional setting is that of a wave-Klein-Gordon type system, in which massless waves interact nonlinearly with massive (Klein-Gordon) ones. It is then very interesting to understand the differences that this feature brings in compared to the works mentioned above and to prove an analogue of Burnett's conjecture for the Kaluza-Klein theory.

In this post-doctoral research project, the first part of the work should be exploratory and aimed at showing a back-reaction effect, in the limit where the compact direction shrinks to zero, on some relevant examples (e.g. on quasilinear wave equations that are good toy models for the EVE) . Even on these examples, we foresee possible issues related to an apparent loss of derivatives in the limit equation and issues more related to the semi-classical limit. These should be tackled by adapting ideas developed in [3] and [2] respectively.

Once the nature of the limiting stress-energy tensor is understood on some toy models, the reverse question could be analyzed: if a certain matter model arises as such limit, is it true that all solutions (at least in certain regimes) to the Einstein equations coupled with that matter model can be achieved as a limit of vacuum higher-dimensional solutions in the sense described above?

References

[1] G. A. Burnett. The high-frequency limit in general relativity. *Journal of Mathematical Physics*, 30(1):90–96, 01 1989.

[2] R. Carles. *Semi-classical analysis for nonlinear Schrödinger equations—WKB analysis, focal points, coherent states*. World Scientific Publishing Co., Singapore, second edition, [2021] ©2021.

[3] M. Hadžić and J. Speck. The global future stability of the flrw solutions to the dust-einstein system with a positive cosmological constant. *Journal of Hyperbolic Differential Equations*, 12(01):87–188, 2015.

[4] C. Huneau and J. Luk. High-frequency backreaction for the Einstein equations under polarized $\mathbb{U}(1)$-symmetry. *Duke Math. J.*, 167(18):3315–3402, 2018.

[5] C. Huneau and J. Luk. Trilinear compensated compactness and Burnett's conjecture in general relativity. Preprint, arXiv:1907.10743, 2019.

[6] C. Huneau, A. Stingo, and Z. Wyatt. The global stability of the Kaluza-Klein spacetime. Preprint, arXiv:2307.15267, 2023.

[7] A. Touati. The reverse Burnett conjecture for null dusts. *Annals of PDE*, 11(2):22, 2025.

[8] E. Witten. Instability of the Kaluza-Klein vacuum. *Nuclear Physics B*, 195(3):481 – 492, 1982.

[9] Z. Wyatt. The weak null condition and Kaluza-Klein spacetimes. *J. Hyperbolic Differ. Equ.*, 15(2):219–258, 2018.