Stochastic optimal control for asset-liability management

PGMO seminar

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Setting

Consider:

- a brownian motion $(W_t)_{t\in[0,T]}$ and its filtration $(\mathcal{F}_t)_{t\in[0,T]}$
- the control space $\mathcal{U}$: all the square-integrable $\mathcal{F}_t$-measurable stochastic processes $(\nu_t)_{t\in[0,T]}$ with value in $U \subset \mathbb{R}^m$
- the state variable $(X_t)_{t\in[0,T]}$ driven by the SDE:

$$\begin{cases} dX_t = & f(X_t, \nu_t)\,dt + \sigma(X_t, \nu_t)\,dW_t, \\ X_{t_0} = & x, \end{cases}$$

with solution $X_t^{t_0,x,\nu}$.

Setting

The stochastic optimal control problem is

$$V(t,x) = \min_{\nu \in \mathcal{U}} \left\{ \mathbb{E}\left[\int_t^T \ell(X_s^{t,x,\nu}, \nu_s)\, \mathrm{d}s + \phi(X_T^{t,x,\nu}) \right] \right\}.$$

Assumptions: $\exists K \geq 0$ such that for all (t,x,y,u),

- $|f(x,u) - f(y,u)| + |\sigma(x,u) - \sigma(y,u)| \leq K|x-y|$
- $|f(x,u)| + |\sigma(x,u)| \leq K(1+|x|+|u|)$
- $|\ell(x,u)| + |\phi(x)| \leq K(1+|u|+|x|^2)$.
- f, σ, ℓ, and ϕ are continuous.

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- $|\ell(x,u)| + |\phi(x)| \leq K(1 + |u| + |x|^2)$.
- f, σ, ℓ, and ϕ are continuous.

Discretization

Consider:

- a time discretization $0 = t_0 < t_1 < \ldots < t_N = T$, with $\Delta_k = t_{k+1} - t_k$
- a sequence of i.i.d. random values $\xi_1, \ldots, \xi_N$ with value 1 or -1 with probability $1/2$
- a measurable control process $(\nu_k)_{k=0,\ldots,N-1}$ in $\tilde{\mathcal{U}}$
- the state variable $(\tilde{X}_t)_{t\in[0,T]}$ with the dynamic

$$\begin{cases} \tilde{X}_{k+1} = & \tilde{X}_k + f(y_k, \nu_k)\Delta_k + \sigma(y_k, \nu_k)\xi_{k+1}\sqrt{\Delta_k}, \\ \tilde{X}_{k_0} = & x, \end{cases}$$

with the solution $\tilde{X}_k^{k_0,x,\nu}$.

Dynamic programming

The discretized problem

$$\tilde{V}(k,x) = \min_{\nu \in \tilde{\mathcal{U}}} \left\{ \mathbb{E}\Big[\sum_{s=k}^{N} \ell(X_s^{k,x,\nu}, \nu_s) + \phi(X_N^{k,x,\nu}) \Big] \right\}$$

can be solved with the dynamic programming principle (DPP):

$$\begin{cases} \tilde{V}(N,x) = & \phi(x), \\ \tilde{V}(k,x) = & \min_{u \in U} \left\{ \ell(x,u) + \mathbb{E}\Big[\tilde{V}\left(k+1, X_{k+1}^{k,x,u}\right) \Big] \right\}. \end{cases}$$

By backward recursion, for $k = N, ..., 0$, we can compute $V(k, \cdot)$.

HJB equation (1)

In continuous time, the DPP writes, for any stopping time $\tau \geq t$,

$$\begin{cases} V(T,x) = & \phi(x) \\ V(t,x) = & \min_{\nu \in \mathcal{U}} \left\{ \mathbb{E}\left[\int_t^\tau \ell(X_s^{t,x,\nu}, \nu_s)\,\mathrm{d}s + V(\tau, X_\tau^{t,x,\nu}) \right] \right\}. \end{cases}$$

Formally, considering a constant control $\nu_t = u$ and $\tau = t + \mathrm{d}t$, we obtain with Itō's formula:

$$\begin{aligned} &\mathbb{E}\left[\int_t^\tau \ell(X_s^{t,x,\nu}, \nu_s)\,\mathrm{d}s + V(\tau, X_\tau^{t,x,\nu}) \right] \\ &= \mathbb{E}\Big[\ell(x,u)\,\mathrm{d}t + \partial_t V(t,x)\,\mathrm{d}t + V_x(t,x) f(x,u)\,\mathrm{d}t \\ &\qquad + V_x(t,x)\sigma(t,x)\,\mathrm{d}W_t + \tfrac{1}{2} V_{xx}(t,x)\sigma(t,x)^2\,\mathrm{d}t \Big]. \end{aligned}$$

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HJB equation (2)

This leads to the HJB equation:

$$\begin{cases} V(T,x) = & \phi(x), \\ -\partial_t V(t,x) = & H(x, V_x(t,x), V_{xx}(t,x)), \end{cases}$$

where the Hamiltonian is defined by

$$H(x,p,q) = \inf_{u \in U} \left\{ \ell(x,u) + p^T f(x,u) + \frac{1}{2}\sigma(x,u)^T q \sigma(x,u) \right\}.$$

We compute from backward the functions $V(t,\cdot)$.

Generalities

The probability constraint (PC)

$$\mathbb{P}\big[\Phi(X_T) \geq 0\big] \geq p$$

can be seen as an expectancy constraint:

$$\mathbb{E}\big[\mathbf{1}_{\{\Phi(X_T)\geq 0\}}\big] \geq p$$

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Characterization

Lemma

For all $\nu \in \mathcal{U}$, the PC holds if and only if there exist a square-integrable process α and a martingale Z satisfying

1. *the following dynamic:* $\begin{cases} dZ_t = & \alpha_t \, dW_t, \\ Z_0 = & p, \end{cases}$
2. *for all t,* $Z_t \in [0,1]$ *a.s.*
3. *the inequality:* $Z_T^{0,p,\alpha} \leq \mathbf{1}_{\{\Phi(X_T^{0,x,\nu}) \geq 0\}}$.

The martingale Z_t stands for the level of probability ensured at time t.

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The martingale Z_t stands for the level of probability ensured at time t.

Proof

($\Longleftarrow$) Let (α, Z, ν) such that (1-3) hold. Since Z is a martingale,

$$\mathbb{E}\left[\mathbf{1}_{\{\Phi(X_T)\geq 0\}}\right] \geq \mathbb{E}\left[Z_T\right] = p.$$

($\Longrightarrow$) Let ν be such that the PC holds. Set $p_0 = \mathbb{P}[\Phi(X_T) \geq 0]$. Note that $p_0 \geq p$. Define the martingale

$$Z_t = \mathbb{E}\left[\mathbf{1}_{\Phi(X_T)\geq 0} | \mathcal{F}_t\right].$$

Since $Z_T \leq 1$ a.s., then for all t, $Z_t \leq 1$. Set $Z^1 = Z - (p_0 - p)$. Let τ be the stopping time defined by

$$\tau = \inf \left\{t \in [0, T],\ Z_t^1 \leq 0\right\}.$$

Finally, $Z_t^2 = Z_t^1 \mathbf{1}_{\{t\leq\tau\}}$ satisfies (2-3). The control α is obtained by the martingale representation theorem.

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Application

Consequence of the lemma: the problem with probability constraint can be solved by dynamic programming!

- Add a state variable Z controlled by α,
- add to the final-cost function

$$\tilde{\phi}(x,z) = \begin{cases} 0 & \text{if } z \leq \mathbf{1}_{\{\Phi(x)\geq 0\}} \\ +\infty & \text{otherwise.} \end{cases}$$

Note that α is not bounded.

Setting

Consider the dynamics:

$$\begin{cases} dX_t = & f_1(X_t, \nu_t)\, dt + \sigma_1(X_t, \nu_t)\, dW_t, \\ dY_t = & f_2(X_t, Y_t, \nu_t)\, dt + \sigma_2(X_t, Y_t, \nu_t)\, dW_t \end{cases}$$

and the problem of minimum wealth

$$v(t,x) = \min \left\{ y,\ \exists \nu \text{ such that } \Phi(X_T^{t,x,\nu}, Y_T^{t,x,y,\nu}) \geq 0,\ \text{a.s.} \right\}.$$

We assume that for all $y \geq v(t,x)$, there exists ν such that

$$\Phi(X_T^{t,x,\nu}, Y_T^{t,x,y,\nu}) \geq 0,\ \text{a.s.}$$

Dynamic programming

In discrete time, with $0 = t_0 < t_1 < \dots t_N = T$, the DPP writes:

$$\begin{cases} \tilde{v}(N,x) = & \inf\{y,\ \Phi(x,y) \geq 0\} \\ \tilde{v}(k,x) = & \inf\left\{y,\ \exists u \text{ such that } \tilde{Y}_{k+1}^{k,x,y,u} \geq v(k+1, \tilde{X}_{k+1}^{k,x,u}), \text{a.s.}\right\}. \end{cases}$$

In continuous time, the inequality $\tilde{Y}_{k+1}^{k,x,y,u} \geq v(k+1, \tilde{X}_{k+1}^{k,x,u})$ writes

$$\begin{aligned} y &+ f_2(x,y,u)\,\mathrm{d}t + \sigma_2(x,y,u)\,\mathrm{d}W_t \\ &\geq y + \partial_t v(t,x)\,\mathrm{d}t + v_x(t,x) f_1(x,u)\,\mathrm{d}t + v_x(t,x)\sigma_1(x,u)\,\mathrm{d}W_t \\ &\quad + \tfrac{1}{2} v_{xx}(t,x)\sigma_1(x,u)^2\,\mathrm{d}t. \end{aligned}$$

and implies that the diffusions of the r.h.s. and the l.h.s. are equal:

$$\sigma_2(x,y,u) = v_x(t,x)\sigma_1(x,u).$$

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HJB equation

We obtain the following HJB equation:

$$\begin{cases} v(T,x) = & \inf\{y,\ \Phi(x,y) \geq 0\} \\ -\partial_t v(t,x) = & H(x, v(t,x), v_x(t,x), v_{xx}(t,x)) \end{cases}$$

with the Hamiltonian defined by

$$H(x,v,p,q) = \begin{cases} \min_{u\in U} \left\{ pf_1(x,u) - f_2(x,v,u) + \frac{1}{2} v_{xx}\sigma_1(x,u)^2 \right\} \\ \text{such that } \sigma_2(x,v,u) = v_x(t,x)\sigma_1(x,u). \end{cases}$$

An example

Consider the dynamic

$$\begin{cases} dX_t = & \alpha_t \, dW_t \\ X_0 = & x, \end{cases}$$

where the volatility α is not bounded, and the following problem:

$$V(t,x) = \min_{\alpha} \left\{ \mathbb{E}\left[\phi(X_T^{t,x,\alpha})\right] \right\}.$$

Then, the value function function is given by

$$V(t,x) = \phi^{\text{conv}}(x),$$

where ϕ^{conv} is the convex hull of ϕ.

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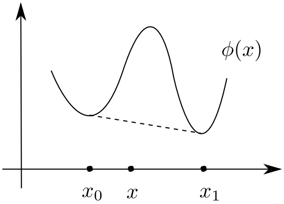
Proof (1)

- $V(t,x) \geq \phi^{\text{conv}}(x)$. Let α, then by Jensens' inequality,

$$\mathbb{E}(\phi(X_T^{t,x,\alpha}) \geq \phi^{\text{conv}}(\mathbb{E}X_T^{t,x,\alpha}) = \phi^{\text{conv}}(x).$$

- $V(t,x) \leq \phi^{\text{conv}}(x)$. Let $x \in \mathbb{R}$, let $x_0, x_1 \in \mathbb{R}$ and $\lambda \in [0,1]$ be such that

$$x = \lambda x_0 + (1-\lambda)x_1 \quad \text{and} \quad \phi^{\text{conv}}(x) = \lambda\phi(x_0) + (1-\lambda)\phi(x_1).$$



Proof (2)

Let $K > 0$, set $\nu_t = K$ and define the stopping time τ by

$$\tau = \inf \left\{ s \geq t, X_s^{t,x,s} \notin [x_0, x_1] \right\}$$

and the control $\nu'_t = K\mathbf{1}_{t \leq \tau}$. We can show that, when $K \to \infty$,

$$\mathbb{P}[X_T^{t,x,\nu'} = x_0] \to \lambda,$$
$$\mathbb{P}[X_T^{t,x,\nu'} = x_1] \to (1 - \lambda),$$
$$\mathbb{P}[X_T^{t,x,\nu'} \in (x_0, x_1)] \to 0.$$

Thus, $\mathbb{E}[\phi(X_T^{t,x,\nu'})] \to \phi^{\text{conv}}(x)$.

Remarks

Note that in this special case,

- the Hamiltonian is given by

$$H(q) = \min_{\alpha \in \mathbb{R}} \left\{ q\alpha^2 \right\} = \begin{cases} 0 & \text{if } q \geq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

- in the HJB equation, the final cost has been convexified
- the minimum in the HJB equation does not provide the optimal control.

Application

For our problem,

- the value function is convex w.r.t. to Z (and to any state variable with an unbounded volatility, dominating the drift)
- a convex hull must be computed at each step (in general)
- we may try and develop a cutting-plane method: a convex function is represented with the supremum of affine functions
- we may compute the HJB equation satisfied by the Legendre transform of V.

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Setting

Consider a risky asset of price S_t and a non-risky asset of price B_t with the dynamic:

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t, \quad \frac{\mathrm{d}B_t}{B_t} = r\mathrm{d}t.$$

The dynamics of the wealth A_t and the liability L_t is given by:

$$\begin{cases} \mathrm{d}L_t & = L_t(\mu' \mathrm{d}t + \sigma' \mathrm{d}W_t) \\ \mathrm{d}A_t & = A_t(\theta_t \frac{\mathrm{d}S_t}{S_t} + (1-\theta_t)\frac{\mathrm{d}B_t}{B_t}) + c_t \,\mathrm{d}t \\ & = \left[\theta_t A_t(\mu - r) + r + c_t\right]\mathrm{d}t + \theta_t A_t \sigma \mathrm{d}W_t, \end{cases}$$

where the controls θ_t and c_t satisfy:

$$\theta_t \in [0,1] \quad \text{and} \quad c_t \geq 0.$$

Setting

The probability constraint $\mathbb{P}\big[A_T/L_T \geq 1\big] \geq p$ is taken into account with the variable Z satisfying

$$\begin{cases} dZ_t = & \alpha_t dW_t \\ Z_0 = & p. \end{cases}$$

We set

$$\phi(a,l,z) = \begin{cases} 0 & \text{if } z \leq \mathbf{1}_{\{a/l \geq 1\}} \\ +\infty & \text{otherwise.} \end{cases}$$

The problem is the following:

$$V(t,a,l,z) = \min_{\theta,c,\alpha} \Big\{ \mathbb{E}\Big[\int_t^T e^{-\beta s} c_s \, ds + \phi(A_T, L_T, Z_T) \Big] \Big\}$$

Minimum wealth

The problem of minimum wealth necessary to ensure the PC without adding money is given by

$$v(t,l,z) = \min \Big\{ a, \exists(\theta,\alpha) \text{ such that } Z_T \le \mathbf{1}_{\{A_T/L_T \ge 1\}}, c = 0 \Big\}.$$

In this situation the algebraic constraint on the controls is given by

$$\theta a \sigma = l \sigma v_l(t,l,z) + \alpha v_z(t,l,z)$$

and the Hamiltonian by $H\big(t,(l,z),v,Dv,D^2 v\big) =$

$$\min_{\theta \in [0,1]} \Big\{ l\mu' v_l - [\theta v(\mu - r) + r] + \tfrac{1}{2} D^2 v(l\sigma', \alpha(\theta))^2 \Big\}.$$

Value function

For the initial problem, the value function $V(t, a, l, z)$ is

- convex w.r.t. z
- $\nearrow$ w.r.t. z, l
- $\searrow$ w.r.t. a.

Note that for all $a \geq v(t, l, z)$, $V(t, a, l, z) = 0$.

We minimize c and (θ, α) independently in the Hamiltonian. For c, we minimize $(e^{-\beta t} - V_a)c$, then

- $V_a > -e^{-\beta t} \implies c = 0$
- $V_a = -e^{-\beta t} \implies c = ?$
- $V_a < -e^{-\beta t} \implies c = +\infty$.

For (θ, α), we minimize

$$DV_a[\theta a(\mu - r) + r] + \frac{1}{2} D^2 V(\theta a\sigma, l\sigma', \alpha)^2.$$

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Future work

Theoretical issues:

- Well-posedness of HJB equation
- Proof of convergence of numerical schemes

Numerical issues:

- Convexification operations
- Implementation

General issue:

- More general risk constraints.

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Thank you for your attention!