SPECTRAL INVARIANTS AND LEGENDRIAN INTERSECTIONS

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1. **Persistence modules.** Consider the category Γ_2 with two objects:

$$A \xrightarrow{\quad c \quad} B$$

There are three morphisms $1_A, 1_B$ and c.

Let Vect be the category of finite dimensional vector spaces.

Question: Can one classify functors $\Gamma \to$ Vect up to isomorphism?

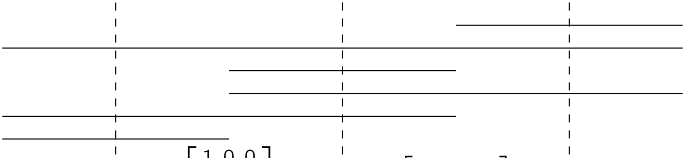
Note: A *functor* is a choice of two vector spaces $V(A)$ and $V(B)$ and a linear map $V(c)$ between them. An *isomorphism* is a commutative square:

$$\begin{array}{ccc} V_1(A) & \xrightarrow{V_1(c)} & V_1(B) \\ \downarrow\wr & & \downarrow\wr \\ V_2(A) & \xrightarrow{V_2(c)} & V_2(B) \end{array}$$

2. **Persistence modules.** Similarly with Γ_3:

$$A \xrightarrow{c_1} B \xrightarrow{c_2} C$$

Yes: using *barcodes*:



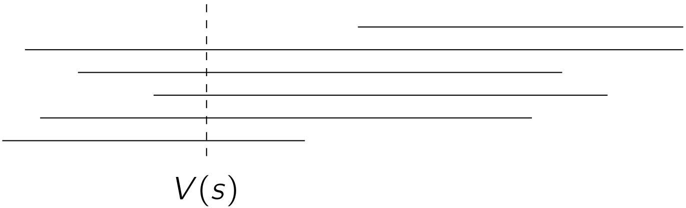
$$V(A) \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad V(B) \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad V(C)$$

Figure 1. Barcode classifying isomorphism class of functors $\Gamma_3 \to \mathrm{Vect}$

3. **Persistence modules.** Consider the category $\Gamma_{\mathbb{R}}$, with one object for each real number s, and an arrow $s_1 \to s_2$ if $s_1 \leq s_2$.

Definition: A functor $\Gamma_{\mathbb{R}} \to \text{Vect}$ is called a *persistence module*.

Functors $\Gamma_{\mathbb{R}} \to \text{Vect}$ can be classified by barcodes as well.



Typically one requires *upper continuity*: $V(s) = \lim_{s' \to s+} V(s')$.

Two interesting spaces: $V_\infty = \lim_{s \to \infty} V(s)$ and $V_{-\infty} = \lim_{s \to -\infty} V(s)$.

Modern language: the V_∞ is *colimit* of the functor and $V_{-\infty}$ is the *limit*.

4. **Persistence modules.** Persistence modules arise in the *topological analysis of data* (see, e.g., *DATASHAPE group at Orsay*).

Let $P \subset \mathbb{R}^n$ be a finite set (of data).

Let $X(s) = \{q \in \mathbb{R}^n : \inf_{p \in P} |q - p| < s\}$. *A thickened version of P*.

Let $V_d(s) = H_d(X(s))$ be the dth homology of $X(s)$.

Because $X(s_1) \subset X(s_2)$ if $s_1 \leq s_2$, there are maps $V_d(s_1) \to V_d(s_2)$.

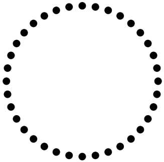


Figure 2. This data set will have a long bar in the persistence module using H_1.

5. **Spectral invariants.** Often it arises that one has many persistence modules V whose colimits are the same.

Let us therefore fix a vector space HW and consider some set V_Λ, $\Lambda \in \mathcal{L}$, of persistence modules, with isomorphisms $V_\infty \simeq \mathrm{HW}$.

Given a class $\zeta \in \mathrm{HW}$ and $\Lambda \in \mathcal{P}$, define the number:

$$c(\zeta; \Lambda) = \inf \left\{ s : V_\Lambda(s) \to \mathrm{HW} \text{ hits } \zeta \right\}.$$

When does the class ζ appear in the persistence module V_Λ.

6. **Spectral invariants for linear maps.** Let us consider $\mathcal{P}$ the class of symmetric linear endomorphisms $\Lambda : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$.

For each Λ, define the function $f : \mathbb{RP}^n \to \mathbb{RP}^n$ given by:

$$f(x) = \frac{\langle x, \Lambda x \rangle}{\langle x, x \rangle}.$$

Note: $\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{0\})/\mathbb{R}$, *so* f *is well-defined.*

Define a persistence module $V_\Lambda(x) := H_*(\{f \leq s\}\,; \mathbb{Z}/2)$.

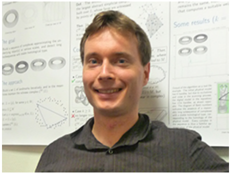
Then $V_{\Lambda,\infty} \simeq H_*(\mathbb{RP}^n; \mathbb{Z}/2) = \mathbb{Z}/2q^0 \oplus \mathbb{Z}/2q \oplus \cdots \oplus \mathbb{Z}/2q^n$.

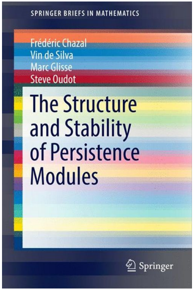
The class of q^k *represents the homology class of* $\mathbb{RP}^{n-k} \subset \mathbb{RP}^n$.

This gives $n+1$ spectral invariants: $\lambda_k = c(q^k; \Lambda)$.

Theorem. $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of Λ.

Chazal, Glisse, Oudot based at Paris-Saclay.



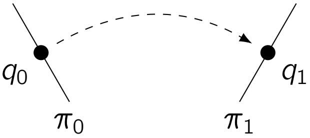


End of Part 1

7. **A particular contact manifold.** Let M^n be a smooth manifold, and let Y be the space of pairs (q, π) where π is a cooriented hyperplane.

Special hyperplane distribution $\xi \subset TY$; vectors $(\delta q, \delta\pi)$ based at the point (q, π) lie in the distribution if δq is tangent to π. *Skating.*

A Riemannian metric induces a flow: $R_s(q_0, \pi_0) = (q_1, \pi_1)$ holds if the unique geodesic of length s starting at q_0 positively orthogonal to π_0 ends at q_1.



Flow preserves ξ and is transverse ξ (Reeb flow R_s).

8. **Legendrians.** A *Legendrian* is an n-dimensional submanifold $\Lambda \subset Y$ so that $T\Lambda \subset \xi$ holds at all points.

Famous Conjecture. If Y, Λ are compact, and if R_s is a Reeb flow, and Λ is a Legendrian, there exists some $s \neq 0$ so that $R_s(\Lambda) \cap \Lambda \neq \emptyset$.

This conjecture is typically attributed to the famous Russian mathematician Arnol'd, and is called Arnol'd's chord conjecture.

9. **Normal chords.** Let $N \subset M$ be a submanifold of positive codimension. Define:

$$\Lambda = \{(q, \pi) : q \in N \text{ and } TN \subset \pi\}.$$

Then Λ solves the conjecture for the geodesic Reeb flow if and only if N admits a geodesic normal chord.

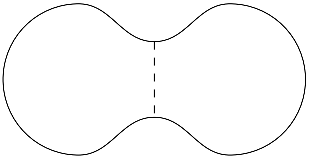
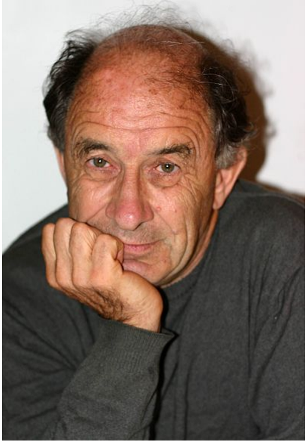
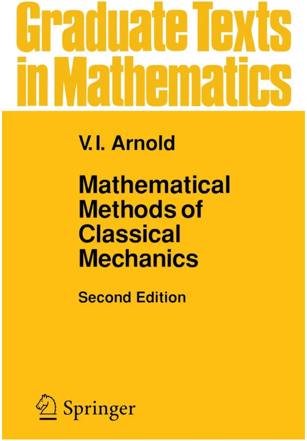


Figure 3. Normal chords are a special case of the conjecture

Arnol'd (not based at Paris-Saclay)





End of Part 2

10. **A category of Legendrians.** A path Λ_s of Legendrians is *non-negative* if any curve $x(s) \in \Lambda_s$ is never negatively transverse to ξ.

For instance, if R_s is a Reeb flow, then $\Lambda_s = R_s(\Lambda_0)$ is non-negative

Introduce $\mathcal{C}(\Lambda_0)$ the category whose objects are Legendrian isotopies Λ_t, and whose morphisms $\Lambda_t \to \Lambda'_t$ are homotopy classes of squares $\Lambda_{s,t}$:

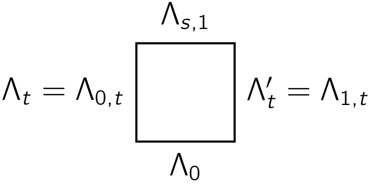


Figure 4. Morphisms in the category are homotopy classes of such squares where $\Lambda_{s,1}$ is a non-negative path.

11. **Discriminant functors.** A functor $\mathsf{HF} : \mathcal{C}(\Lambda_0) \to \mathsf{Vect}$ such that:

($*$) *The morphism* $\mathsf{HF}(\Lambda_{s,t}) : \mathsf{HF}(\Lambda_{0,t}) \to \mathsf{HF}(\Lambda_{1,t})$ *is an isomorphism provided* $\Lambda_{s,1} \cap \Lambda_0 \neq \emptyset$ *holds for all* s*.*

Idea: if one can construct a suitably rich discriminant functors, then one can prove $R_s(\Lambda_0) \cap \Lambda_0 \neq \emptyset$ for $s > 0$.

Modern techniques in *Floer theory* also suggest a way to construct these functors, and this is an active topic of research.

12. **Spectral invariants for Legendrian intersections.** Discriminant functor HF yields persistence modules.

Fix a Reeb flow R, and define:

$$V_{R,\Lambda_t}(s) = \mathrm{HF}(R_{st}(\Lambda_t)).$$

The colimit HW is independent of Λ_t and R.

Spectral invariants $c_R(\zeta;\Lambda_t)$ are numbers s such that $R_s(\Lambda_1) \cap \Lambda_0 \neq \emptyset$.

Thus, if $\Lambda_1 = \Lambda_0$, and $c_R(\zeta;\Lambda_t) \neq 0$, then the conjecture holds.

13. **Spectral invariants for Legendrian intersections.** It is convenient to consider classes of discriminant functors such that the colimit HW has a class ζ with:

$$(*) \qquad c_R(\zeta; \Lambda_0) = 0 \text{ for all } R.$$

(1) In search of a contradiction, suppose $R_s(\Lambda_0) \cap \Lambda_0 = \emptyset$ for all numbers $s \neq 0$. Then it is known that one can define a discriminant functor satisfying $(*)$.
(2) If $(*)$ holds, and Λ_t is a positive isotopy, then $c(\zeta, \Lambda_t) > 0$. In particular, if there exists a *positive loop* Λ_t, then the conjecture holds for Λ_0.

Similar ideas can be used to prove the conjecture if there exists a loop Λ_t such that paths $x(t) \in \Lambda_t$ are non-zero in $\pi_1(Y, \Lambda_0)$.

14. **Conclusion.** It is rather an interesting development that the idea of a persistence module (originating from topological data analysis) has achieved a prominent status in my field of research.

Symplectic topology as the geometry of generating functions

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Figure 5. Claude Viterbo, one of the pioneers of spectral invariants in symplectic geometry. Based at Paris-Saclay.