Prophet Inequalities

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Material Perapared with
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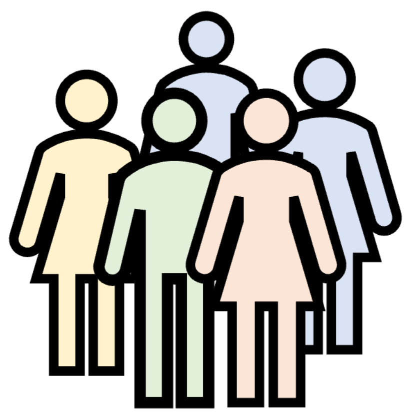
Motivation

Online platforms, e-commerce, etc

Flexible Model:

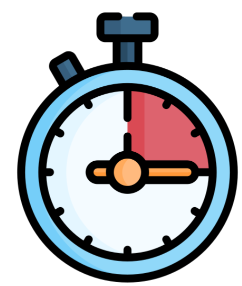
Multiple Goals

Incentives



Limited data

Sequential decisions



Course Overview

1. Classic single-choice problems:

   The classic prophet inequality, secretary problem, prophet secretary problem, etc

2. Data driven prophet inequalities:

   How can limited amount of data be nearly as useful as full distributional knowledge

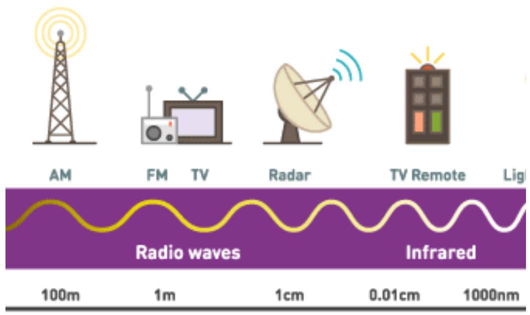
3. Combinatorial Prophet Inequalities

   Many ideas for single choice problems, extend to combinatorial contexts such as k-choice, Matching, hyper graph matching, and beyond

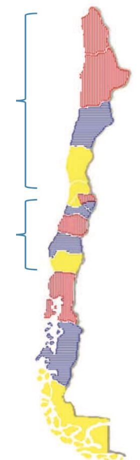
4. **Online Combinatorial Auctions**

   General Model that encompasses many online selection/allocation problems

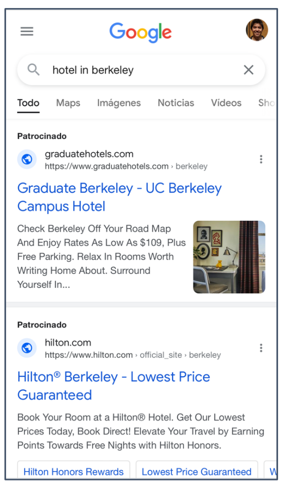
4. Online Combinatorial Auctions



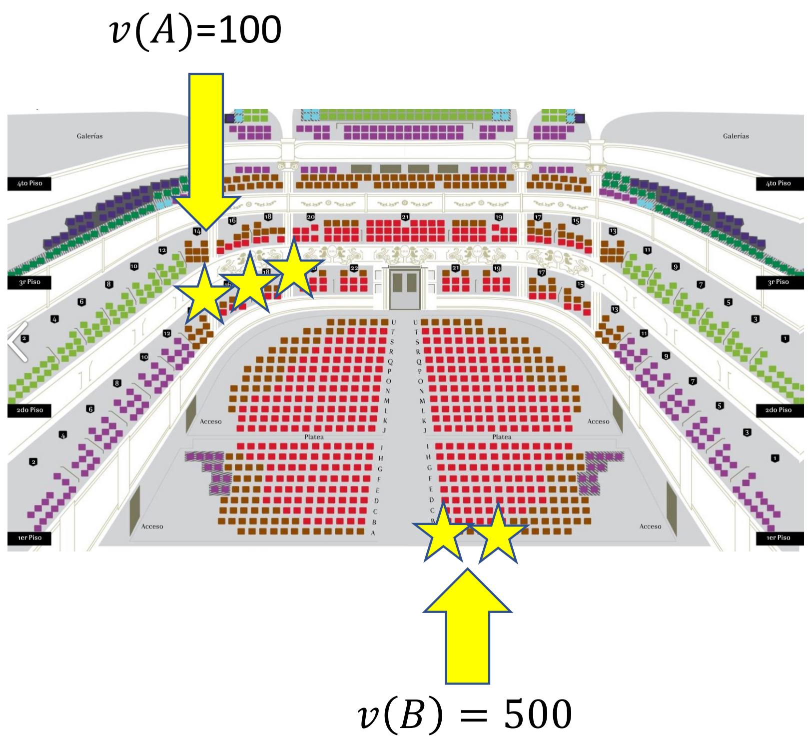
Auction for radio frequencies in the US (2017)
USD $19.8 billion

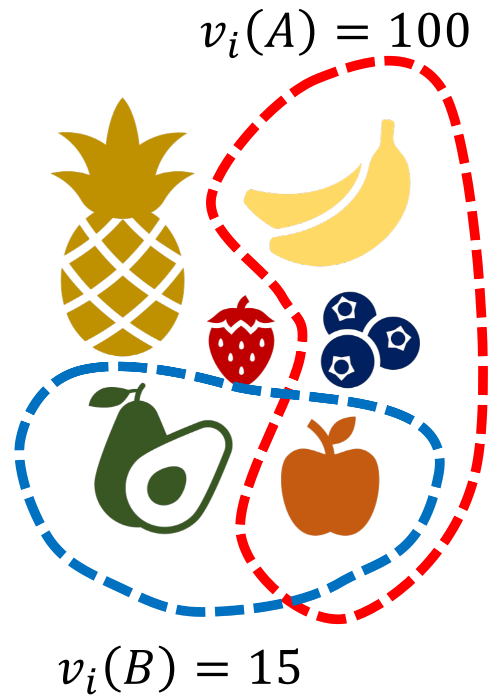


Procurement of meal providers in
Chilean public schools
USD $500+ million every year



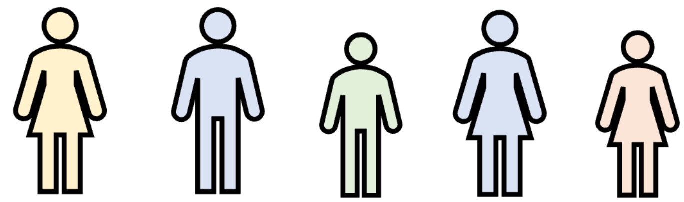
Online advertising
Google's and Meta's main
source of revenue


$v(A)$=100
$v(B) = 500$


$v_i(A) = 100$
$v_i(B) = 15$

Prophet Inequality

Set M with m items



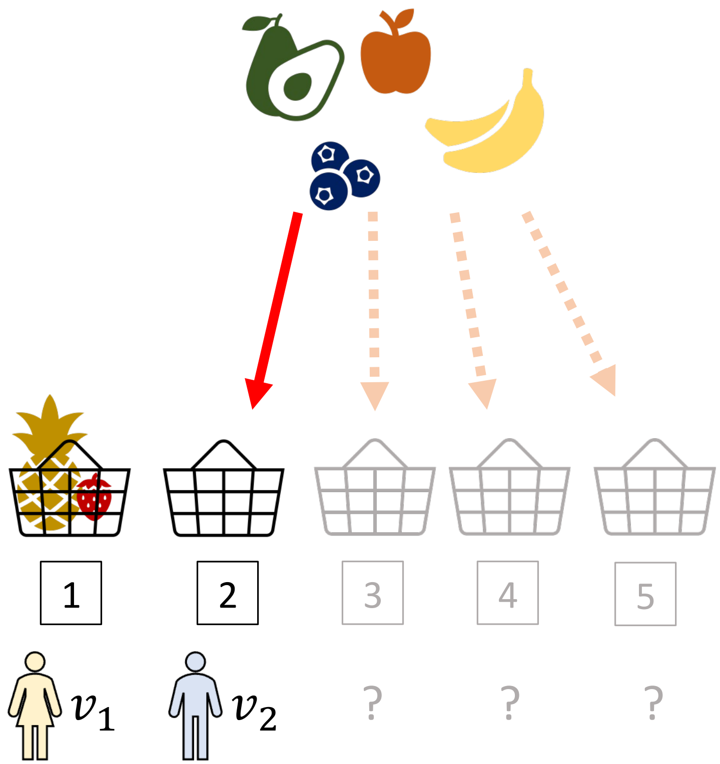
n agents arrive one by one

independent monotone valuations $v_i \sim F_i$

$v_i: 2^M \to \mathbb{R}_+$

(a random function for each agent)

Online welfare



agent i gets the set ALG_i

$$\mathbb{E}(ALG) = \mathbb{E}\left(\sum_i v_i(ALG_i)\right)$$

Incentive Compatible Dynamic Program

Optimal online solution:

$$V_{n+1}(R) = 0$$
$$V_i(R) = \mathbb{E}\left(\max_{X \subseteq R} \{v_i(X) + V_{i+1}(R \setminus X)\} \right)$$

When set R is available, offer agent i **per-bundle prices**

$$p_i(X, R) = V_{i+1}(R) - V_{i+1}(R \setminus X)$$

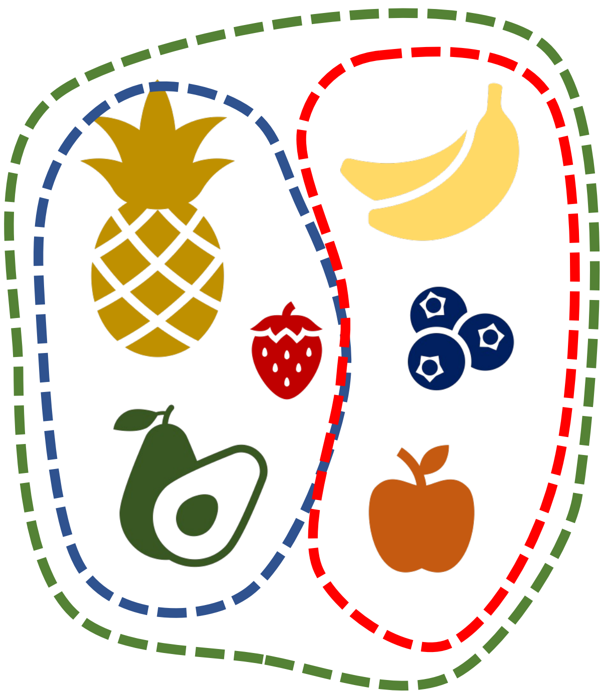
If the agent maximizes utility, then she **selects the same as the DP**:

$$\max_{X \subseteq R}\{v_i(X) - p_i(X, R)\} = \max_{X \subseteq R}\{v_i(X) + V_{i+1}(R \setminus X)\} - V_{i+1}(R)$$

Benchmark: Optimal offline welfare

$$\mathbb{E}(OPT) = \mathbb{E}\left(\max_{\substack{X_1,\ldots,X_n \\ \text{partition}}} \sum v_i(X_i) \right)$$

A simple case: additive valuations



If $A \cap B = \emptyset$, then
$\boldsymbol{v(A \cup B)} = \boldsymbol{v(A)} + \boldsymbol{v(B)}$

Valuations can be expresed by coefficients $v_{i,j}$:

$$v_i(A) = \sum_{j \in A} v_{i,j}$$

Thus, $\mathbb{E}(OPT) = \mathbb{E}\left(\sum_{j \in M} \max_i v_{i,j}\right)$

For each j, coefficients $v_{1,j}, v_{2,j}, \ldots, v_{n,j}$ are independent

Additive valuations

- ALG: set prices $p_j = \frac{1}{2} \cdot \mathbb{E}\left(\max_i v_{i,j}\right)$
- Each agent i takes all remaining items j such that $v_{i,j} \geq p_j$
- By linearity of expectation,

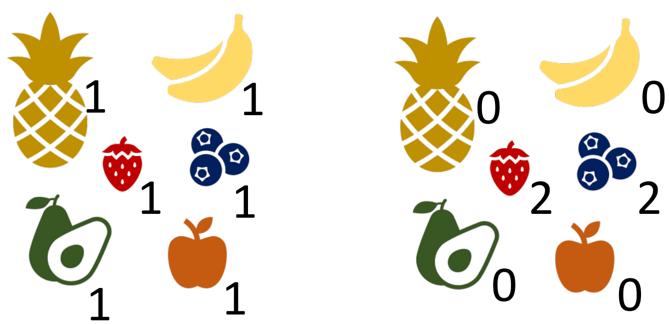
$$\mathbb{E}(ALG) \geq \sum_j \frac{1}{2} \cdot \mathbb{E}\left(\max_i v_{i,j}\right) = \frac{1}{2} \cdot \mathbb{E}(OPT)$$

XOS valuations

v is XOS if there are additive valuations $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$v(S) = \max_{1 \le \ell \le k} \alpha_\ell(S) = \max_{1 \le \ell \le k} \sum_{j \in S} \alpha_{\ell,j}$$

Example: I can make a fruit salad or a berry smoothie



$v\Big($  $\Big) = 3$

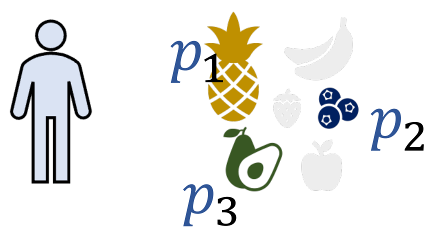
Theorem [Feldman, Gravin, Lucier, SODA'14]

There are item prices $(p_j)_{j \in M}$ that guarantee a 2-approximation.

Posted item-prices mechanism

We fix per-item prices $(p_j)_{j \in M}$

R_i = set of items available when buyer i arrives



buyers maximize utility

$$\underset{A \subseteq R_i}{\operatorname{argmax}} \left\{ v_i(A) - \sum_{j \in A} p_j \right\}$$

$\boldsymbol{ALG(p)}$ = welfare of resulting allocation

$$\mathbb{E}(OPT) = \mathbb{E}\left(\max_{\substack{X_1,\ldots,X_n \\ \text{partition}}} \sum v_i(X_i)\right) \longleftarrow OPT_i$$

Let β_i be the additive function such that $v_i(OPT_i) = \sum_{j \in OPT_i} \beta_{i,j}$

$$p_j = \frac{1}{2} \cdot \mathbb{E}\left(\sum_i \beta_{i,j} \cdot 1_{\{j \in OPT_i\}}\right)$$

$$\mathbb{E}\big(ALG(p)\big) = \mathbb{E}\left(\sum_{j\in \text{SOLD}} p_j + \sum_i \max_{A\subseteq R_i}\left\{v_i(A) - \sum_{j\in A} p_j\right\}\right)$$

$$= \mathbb{E}\left(\sum_{j\in \text{SOLD}} p_j\right) + \mathbb{E}\left(\sum_i \max_{A\subseteq R_i}\left\{v_i(A) - \sum_{j\in A} p_j\right\}\right)$$

revenue utility

$$u_i(X) = \mathbb{E}\left(\max_{A\subseteq X}\left\{v_i(A) - \sum_{j\in A} p_j\right\}\right), \qquad \mathrm{U}(\mathrm{X}) = \sum_i u_i(X)$$

$$\text{utility} = \sum_i \mathbb{E}\big(u_i(R_i)\big) \geq \sum_i \mathbb{E}\big(u_i(M \setminus \text{SOLD})\big) = \mathbb{E}\big(U(M \setminus \text{SOLD})\big)$$

$$U(X) = \sum_i \mathbb{E}\left(\max_{A\subseteq X}\left\{v_i(A) - \sum_{j\in A} p_j\right\}\right) = \mathbb{E}\left(\sum_i \max_{A\subseteq X}\left\{v_i(A) - \sum_{j\in A} p_j\right\}\right)$$

$$\geq \mathbb{E}\left(\sum_i\left(v_i(OPT_i\cap X) - \sum_{j\in OPT_i\cap X} p_j\right)\right) = \mathbb{E}\left(\sum_i\left(v_i(OPT_i\cap X)\right)\right) - \sum_{j\in X} p_j$$

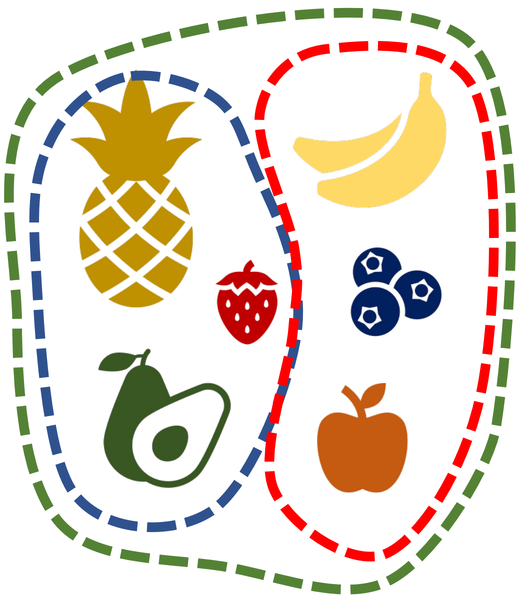
$$\geq \mathbb{E}\left(\sum_i \sum_{j\in OPT_i\cap X}\beta_{i,j}\right) - \sum_{j\in X} p_j = \sum_{j\in X}\left(\mathbb{E}\left(\sum_i \beta_{i,j}\cdot 1_{\{j\in OPT_i\}}\right) - p_j\right)$$

$$\mathbb{E}\big(ALG(p)\big) \geq \mathbb{E}\left(\sum_{j\in \text{SOLD}} p_j\right) + \mathbb{E}\left(\sum_{j\in M\setminus \text{SOLD}} \left(\mathbb{E}\left(\sum_i \beta_{i,j} \cdot 1_{\{j\in OPT_i\}}\right) - p_j\right)\right)$$

Taking $$p_j = \frac{1}{2} \cdot \mathbb{E}\left(\sum_i \beta_{i,j} \cdot 1_{\{j\in OPT_i\}}\right)$$

$$\mathbb{E}\big(ALG(p)\big) \geq \frac{1}{2} \cdot \sum_j \mathbb{E}\left(\sum_i \beta_{i,j} \cdot 1_{\{j\in OPT_i\}}\right) = \frac{1}{2} \cdot \mathbb{E}(OPT)$$

Subadditive Valuations
(a.k.a. complement-free valuations)



$$v(A \cup B) \leq v(A) + v(B)$$

Additive ⊆ XOS ⊆ Subadditive

Subadditive valuations

Offline:

Theorem. [Feige STOC'06]
If valuations are deterministic, we can find in polynomial time a 2-approximation.

Theorem. [Feldman, Fu, Gravin, Lucier STOC'13]
Simultaneous First-Price auctions result in a 2-approximation.

Online:

Theorem. [Dütting, Kesselheim, Lucier FOCS'20]
There is an $O(\log \log m)$ Prophet Inequality.

Theorem. [Correa and Cristi, STOC'23]
If valuations are subadditive, there is an online algorithm such that

$$\mathbb{E}(ALG) \geq \frac{1}{6} \cdot \mathbb{E}(OPT)$$

Same approach? cannot be approximated by XOS better than a factor $\log m$

Idea from sample-based Prophet Inequalities
+
Fixed-point argument

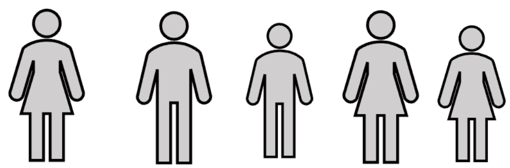
Who would win this battle?

$\mathbb{P}(\text{I win}) = 1/2$

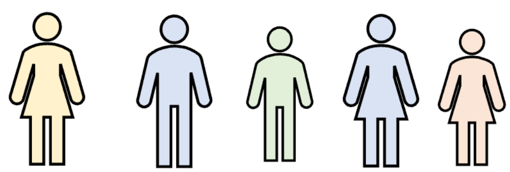
Single item

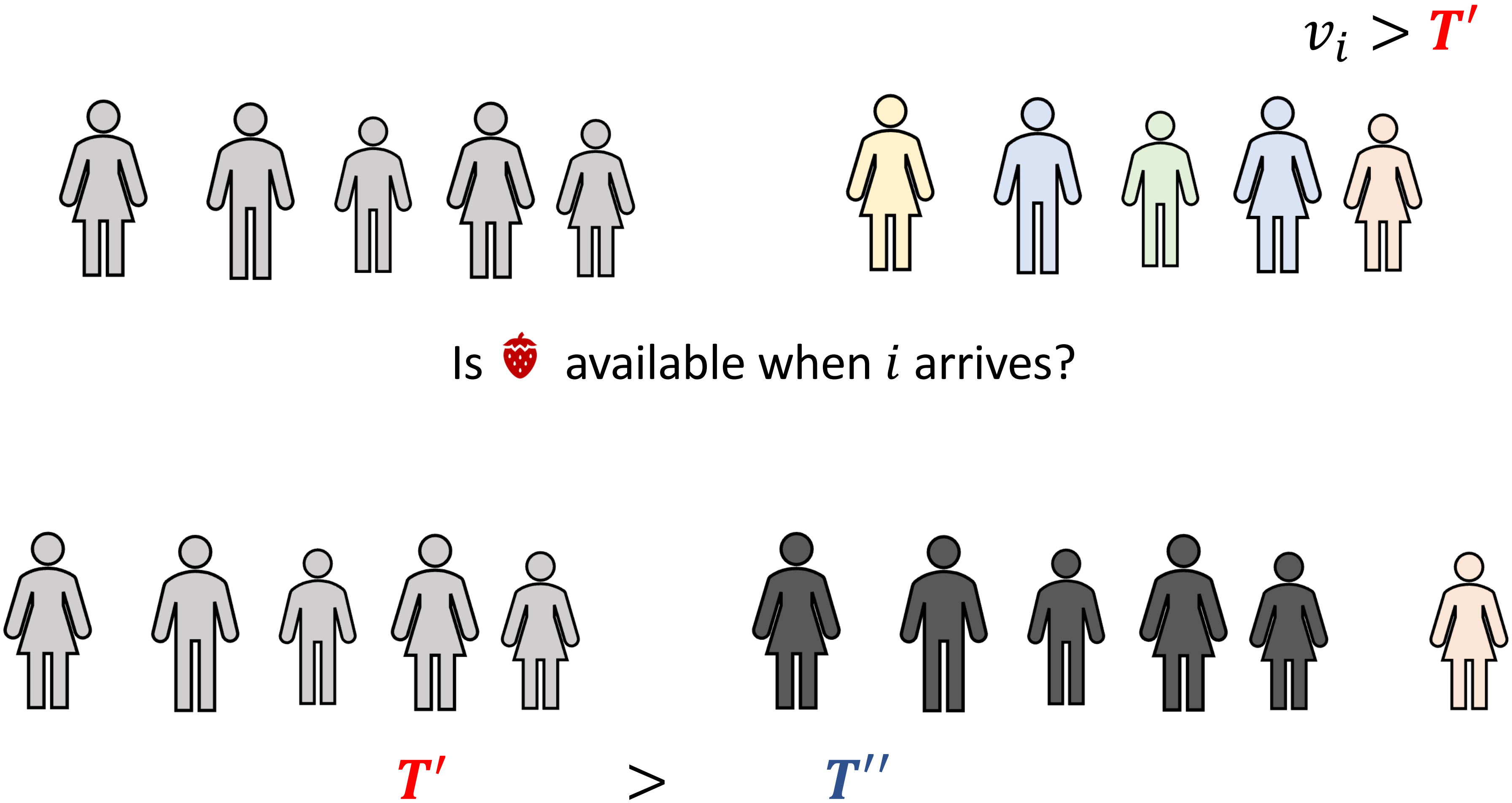
Algorithm:

- Sample $v_i' \sim F_i$ and set a threshold $T' = \max_i v_i'$



- Accept the first agent such that $v_i > T'$




$v_i > T'$
Is available when i arrives?
T'
$>$
T''

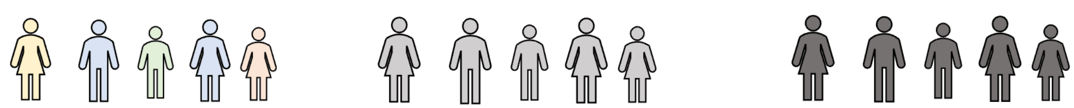
Short 6-approx. proof:

$\mathbb{E}(ALG) = \sum_i \mathbb{E}\big(v_i \cdot 1_{\{i \text{ gets}}$  $_{\}}\big)$

$= \sum_i \mathbb{E}\left(v_i \cdot 1_{\{v_i > T'\}} \cdot 1_{\{}\right.$  $\left._{\text{available for } i\}}\right)$

$$\geq \mathbb{E}\left(\sum_i v_i \cdot 1_{\{v_i > T' \geq T''\}}\right)$$

$$\geq \mathbb{E}\left(T \cdot 1_{\{T > T' \geq T''\}}\right) \geq \frac{1}{6} \cdot \mathbb{E}\left(\max_i v_i\right)$$



Define $T = \max_i v_i$ $\quad T' = \max_i v_i'$ $\quad T'' = \max_i v_i''$

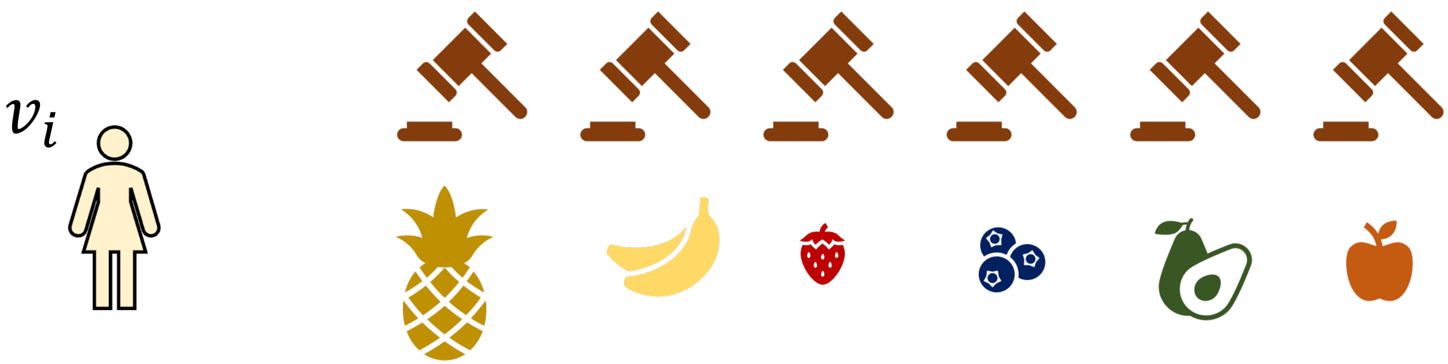
Idea:

Do the same for each item

Problem:

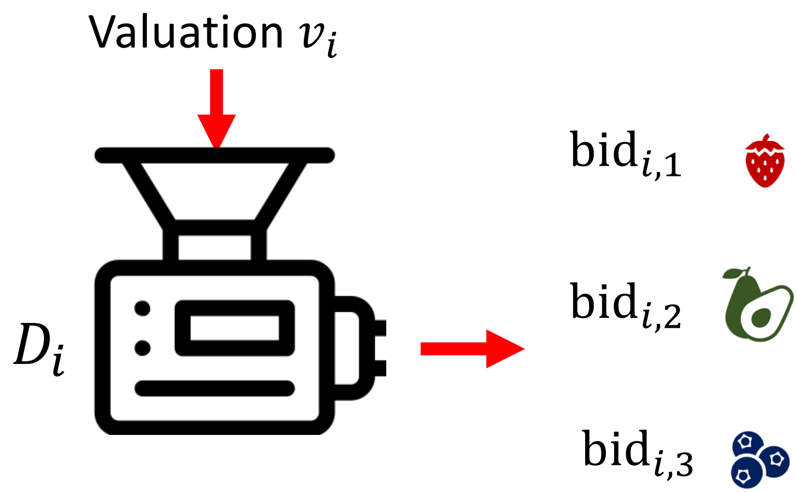
Valuations give a number *per subset*, not a number *per item*

Theorem. [Feldman, Fu, Gravin, Lucier STOC'13]
If valuations are subadditive and we run simultaneous First-Price auctions for **each item,** every equilibrium is in expectation a 2-approximation.



Random Score Generators (RSG)

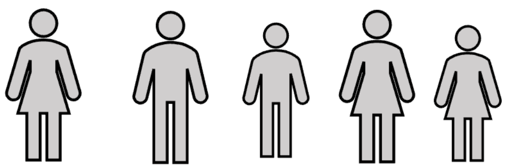
Take functions $D_i: V_i \to \Delta(R_+^M)$



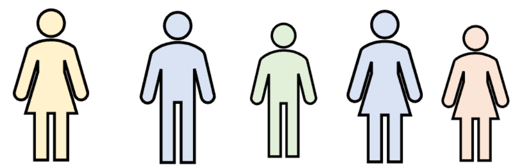
Random vector of scores
$\mathrm{bid}_i \in \mathbb{R}_+^M$
$\sim D_i(v_i)$

Algorithm

Simulate valuations v_i' and scores $\left(\mathrm{bid}_{i,j}'\right) \sim D_i(v_i')$

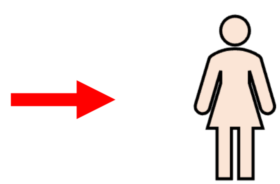


True valuations v_i and scores $\left(\mathrm{bid}_{i,j}\right) \sim D_i(v_i)$



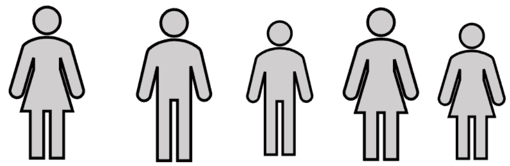
For each item in parallel:

Set threshold
$T_j' = \max_i \mathrm{bid}_{i,j}'$

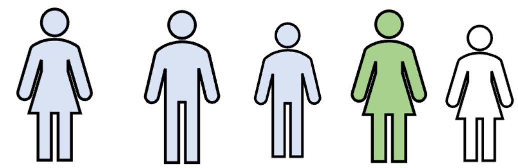


Give it to first agent such that $\mathrm{bid}_{i,j} > T_j'$

$T_j' = \max_i \text{bid}_{i,j}'$

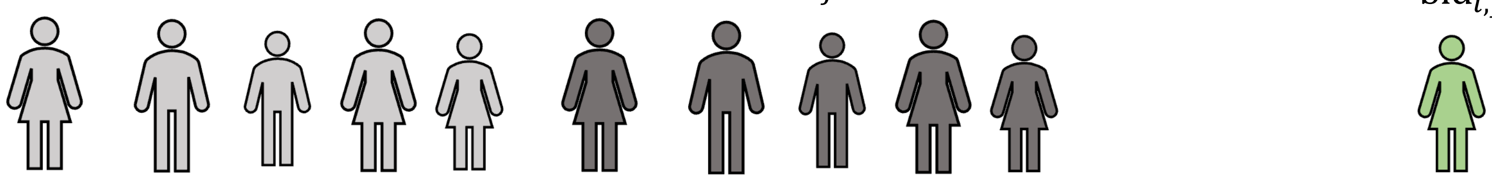


v_i and scores $(\text{bid}_{i,j}) \sim D_i(v_i)$



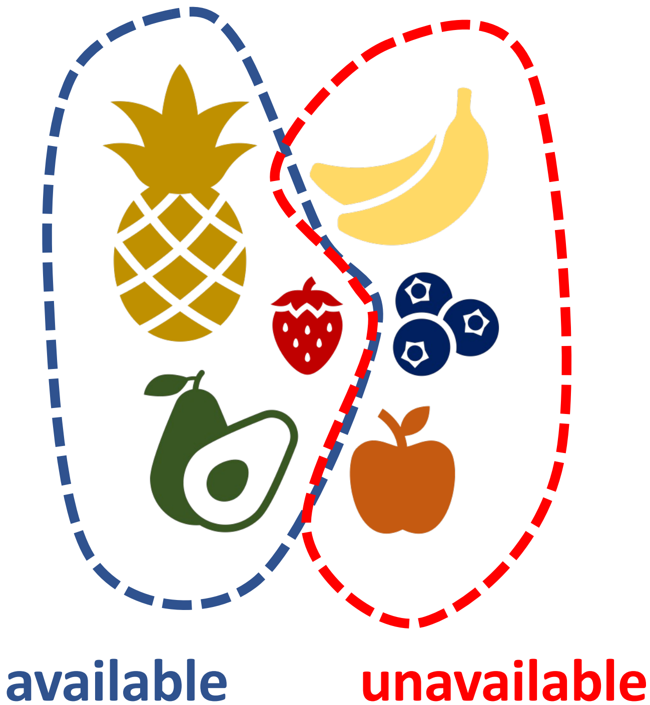
$$\mathbb{E}(v_i(ALG_i)) = \mathbb{E}\left(v_i\left(\left\{j\colon \text{bid}_{i,j} > T_j' \geq \max_{i' \prec i} \text{bid}_{i,j}\right\}\right)\right)$$

T_j' T_j'' $\text{bid}_{i,j}$



$$\geq \mathbb{E}\left(v_i(\{j\colon \text{bid}_{i,j} > T_j' \geq T_j''\})\right)$$

Key observation



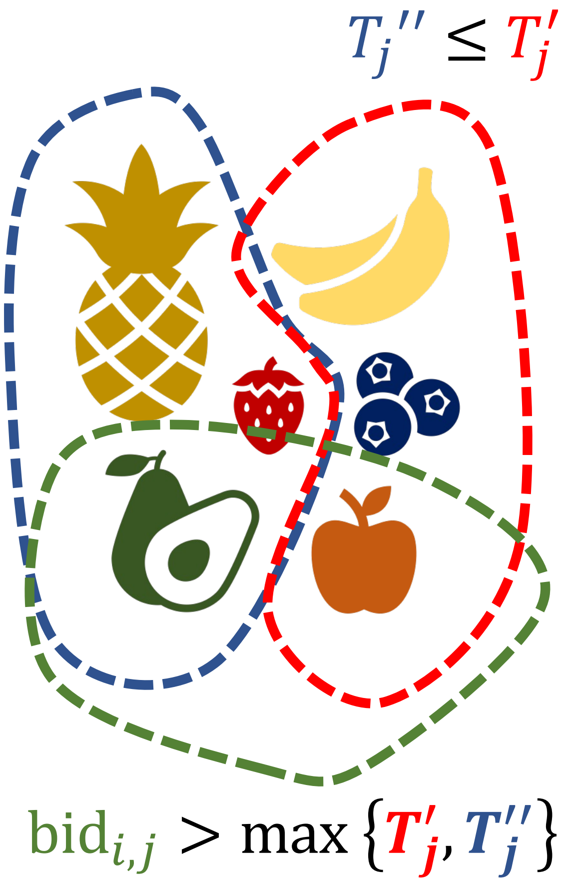
Set of **available** items

$T_j' \geq T_j''$

Set of **unavaliable** items

$T_j'' < T_j'$

The two sets have (essentially) the same distribution!



$$\mathbb{E}(v_i(ALG_i))$$
$$\geq \mathbb{E}\left(v_i(\{j: \mathrm{bid}_{i,j} > T_j' \geq T_j''\})\right)$$
$$= \mathbb{E}\left(v_i(\{j: \mathrm{bid}_{i,j} > T_j'' \geq T_j'\})\right)$$

$$= \frac{1}{2} \cdot \mathbb{E}\left(v_i(\{j: \mathrm{bid}_{i,j} > T_j' \geq T_j''\})\right)$$
$$+ \frac{1}{2} \cdot \mathbb{E}\left(v_i(\{j: \mathrm{bid}_{i,j} > T_j'' \geq T_j'\})\right)$$

$$\geq \frac{1}{2} \cdot \mathbb{E}\left(v_i\left(\left\{j: \mathrm{bid}_{i,j} > \max\{\boldsymbol{T_j'}, \boldsymbol{T_j''}\}\right\}\right)\right)$$

Mirror Lemma. For every agent i,

$$\mathbb{E}(v_i(ALG_i)) \geq \frac{1}{2} \cdot \mathbb{E}\left[v_i\left(\left\{\text{items } j\text{: } \mathbf{bid}_{i,j} > \max\left\{\boldsymbol{T'_j}, \boldsymbol{T''_j}\right\}\right\}\right)\right]$$

Where $\boldsymbol{T'_j} = \max_i \text{bid}'_{i,j}$ and $\boldsymbol{T''_j} = \max_i \text{bid}''_{i,j}$

Lemma 2. There are RSGs such that

$$\sum_i \mathbb{E}\left(v_i\left(\left\{j: \mathbf{bid}_{i,j} > \max\left\{\boldsymbol{T}'_j, \boldsymbol{T}''_j\right\}\right\}\right)\right) \geq \frac{1}{3} \cdot \mathbb{E}(OPT)$$

The proof uses a **fixed-point argument**.

Intuitively: we design a synthetic simultaneous auction with PoA $= 3$, and we take the equilibrium bids

Summary

- Computation of Online Combinatorial Auctions

→Can be implemented online in an incentive-compatible way (exponential DP)
→Unknown how to do this in Polynomial time
→Thus the problem reduces to an online allocation problem

- Approximation of Online Combinatorial Auctions

→For additive valuations, the problem is almost the same as single item
→For XOS valuations, known ½ approximation using balanced prices [Feldman, Gravin, Lucier, SODA 2014]
→For subadditive valuations, new 1/6 approximation [C., Cristi, STOC 2023]
→Improves upon O(log(log(m))) approximation [Dütting, Kesselheim, Lucier FOCS'20]
→[DKL20] approximation uses posted prices whereas [CC23] does not.
→Open: Get a constant factor for Online Combinatorial Auctions with prices.