Chapter 7

Unimodular random graphs

Roughly speaking, a unimodular random graph is a random graph where the origin plays no special role, where it has been chosen uniformly at random. Although easily defined for finite random graphs, such a notion seems vacuous in the infinite setting. Luckily, the so-called Mass Transport Principle is an equivalent property in the finite setting which still make sense for infinite graphs. We will see that unimodular random graphs can be seen as "stochastically transitive graphs" and obey very general structure theorems similar to those of transitive graphs.

7.1 Mass-transport principle

We recall the notion already implicitly encountered in the previous chapter:

Definition 15. *Let $G^\bullet$ be a finite (connected) random pointed graph. We say that $G^\bullet$ is* ***uniformly pointed*** *if the reference point is uniform over the vertices of G, namely if for any measurable $f : \mathcal{G}^\bullet \to \mathbb{R}_+$ we have*

$$\mathbb{E}[f(G^\bullet)] = \mathbb{E}\left[\frac{1}{\#\mathrm{V}(G)} \sum_{x \in \mathrm{V}(G)} f(G, x)\right].$$

In the notation of the previous chapter, a random finite pointed graph $G^\bullet = (G, \rho)$ is uniformly pointed if $\mathrm{U}^\bullet(G) = G^\bullet$ in distribution. The above notion is also called unbiased random graph in the pioneer work of Benjamini & Schramm [20].

7.1.1 Unimodular random graphs

The problem with the previous definition is that it a priori does not make sense for infinite graphs. We will thus give an equivalent definition which can be extended to the infinite case. To do so, we first introduce the set $\mathcal{G}^{\bullet\bullet}$ of (equivalence classes) of doubly-pointed graphs (g, x, y) i.e. given with two distinguished ordered vertices $x, y \in \mathrm{V}(\mathrm{g})$. More precisely, two doubly-pointed graphs (g, x, y) and (g', x', y') are identified if there exists a graph homomorphism from g to g' sending x to x' and y to y'. This set is endowed with a local topology for the notion of restriction $[(\mathrm{g}, x, y)]_r$ given by the graph made of all the vertices and edges which are at distance less than r from either x or y subject also to the condition[1] that $\mathrm{d_{gr}}(x, y) \le r$ and where the resulting graph is doubly pointed at x and y. Hence we can apply the general construction of Section 1.2.1 to get that $(\mathcal{G}^{\bullet\bullet}, \mathrm{d_{loc}})$ is a Polish space.

Exercise 19. Show that the projection $\pi : \mathcal{G}^{\bullet\bullet} \to \mathcal{G}^{\bullet}$ which forgets the second distinguished point is continuous with respect to the local distances.

A Borel function $f : \mathcal{G}^{\bullet\bullet} \to \mathbb{R}_+$ (thus invariant by homomorphism of doubly-pointed graph) is called a **transport function**: $f(\mathrm{g}, x, y)$ is interpreted as a quantity, a mass say, that the vertex x sends to the vertex y in the graph g.

Definition 16 (Unimodular random graph)**.** *A random pointed graph (G, ρ) is **unimodular** if it obeys the Mass-Transport Principle (MTP) i.e. if for any transport function f we have*

$$(MTP) \qquad \mathbb{E}\left[\sum_{x \in \mathrm{V}(G)} f(G, \rho, x)\right] = \mathbb{E}\left[\sum_{x \in \mathrm{V}(G)} f(G, x, \rho)\right]. \tag{7.1}$$

The preceding equation can be interpreted by saying that the average mass the reference point in G sends in total is equal to the average mass it receives from other vertices. The transport of mass using f is a fair game on average.

Remark 4. The terminology "unimodular" comes from group theory: If g is a graph we denote by Γ the group of all its automorphisms (homomorphisms from g to g). When Γ is locally compact, we know by general theory that there exists a left-invariant measure on it (called the Haar measure). The graph g is unimodular if this left-invariant measure is also right-invariant.

See Section 7.3 for applications of the MTP. They all consist in designing a clever mass transport function and applying (MTP). As a very first application, if $F(G, x, y) = 1$ if and only if $x \sim y$ and $\deg(x) = d$ we deduce that in a unimodular random graph (G, ρ) the expected number of neighbors of degree d of ρ is equal to $d\,\mathbb{P}(\deg(\rho) = d)$. As promised, the notion of unimodular random graph coincides with the notion of uniformly pointed random graph in the finite case:

[1] if the condition does not hold we put $[(\mathrm{g}, x, y)]_r = \dagger$ a cemetery point

Proposition 23. *A random finite pointed graph $G^\bullet$ is uniformly pointed if and only if it is unimodular.*

Proof. Suppose that $G^\bullet$ is uniformly pointed and let f be a transport function. Then noticing that $\sum_{x\in \mathrm{V(g)}} f(\mathrm{g},\rho,x) =: F(\mathrm{g},\rho)$ and $\sum_{x\in \mathrm{V(g)}} f(\mathrm{g},x,\rho) =: F'(\mathrm{g},\rho)$ are measurable functions for the single-pointed local topology we have

$$\mathbb{E}\left[\sum_{x\in \mathrm{V}(G)} f(G,\rho,x)\right] = \mathbb{E}[F(G,\rho)] = \mathbb{E}\left[\frac{1}{\#\mathrm{V}(G)}\sum_{x\in \mathrm{V}(G)} F(G,x)\right] = \mathbb{E}\left[\frac{1}{\#\mathrm{V}(G)}\sum_{x,y\in \mathrm{V}(G)} f(G,x,y)\right],$$
$$\mathbb{E}\left[\sum_{x\in \mathrm{V}(G)} f(G,x,\rho)\right] = \mathbb{E}[F'(G,\rho)] = \mathbb{E}\left[\frac{1}{\#\mathrm{V}(G)}\sum_{x\in \mathrm{V}(G)} F'(G,x)\right] = \mathbb{E}\left[\frac{1}{\#\mathrm{V}(G)}\sum_{x,y\in \mathrm{V}(G)} f(G,y,x)\right].$$

Since the right-most quantities are equal we conclude that $G^\bullet$ obeys the MTP. Conversely, if $G^\bullet$ is unimodular and almost surely finite, we choose a transport function of the form $f(\mathrm{g},x,y) = \frac{1}{\#\mathrm{V(g)}} h(\mathrm{g},x)$ where $h : \mathcal{G}^\bullet \to \mathbb{R}_+$ is a measurable function. We obtain by applying the MTP that

$$\mathbb{E}[h(G,\rho)] = \mathbb{E}\left[\sum_{x\in \mathrm{V}(G)} f(G,\rho,x)\right] = \mathbb{E}\left[\sum_{x\in \mathrm{V}(G)} f(G,x,\rho)\right] = \mathbb{E}\left[\frac{1}{\#\mathrm{V}(G)}\sum_{x\in \mathrm{V}(G)} h(G,x)\right],$$

which indeed shows that $G^\bullet$ is uniformly pointed. □

7.1.2 Mass-Transport principle and local limits

A key feature of unimodularity is its preservation under local limit:

Theorem 24 (*Stability under local convergence*)

Let $G_n^\bullet = (G_n, \rho_n)$ be a sequence of unimodular random graphs converging in distribution for $\mathrm{d_{loc}}$ towards $G_\infty^\bullet$. Then $G_\infty^\bullet = (G_\infty, \rho_\infty)$ is unimodular.

Proof. We start with a warmup. If f is a transport function with finite range, i.e. such that $f(\mathrm{g},x,y)$ is zero as soon as x and y are at distance larger than r_0 and that $f(\mathrm{g},x,y)$ only depends on $[(\mathrm{g},x,y)]_{r_0}$ then it follows that for every $k \geq 0$ the functions

$$F_k(\mathrm{g},\rho) = \sum_{x\in \mathrm{V(g)}} \big(k\wedge f(\mathrm{g},\rho,x)\big)\mathbf{1}_{\#\mathrm{V}([(\mathrm{g},\rho,x)]_{r_0})\leq k} \text{ and } F'_k(\mathrm{g},\rho) = \sum_{x\in \mathrm{V(g)}} \big(k\wedge f(\mathrm{g},x,\rho)\big)\mathbf{1}_{\#\mathrm{V}([(\mathrm{g},x,\rho)]_{r_0})\leq k},$$

are both bounded continuous functions for the local topology. Hence, applying the mass-transport principle on $G_n^\bullet$ we have that

$$\mathbb{E}[F_k(G_n^\bullet)] = \mathbb{E}[F'_k(G_n^\bullet)].$$

By the local convergence of $G_n^\bullet$ to $G_\infty^\bullet$ we thus get that $\mathbb{E}[F_k(G_\infty^\bullet)] = \mathbb{E}[F'_k(G_\infty^\bullet)]$. Letting $k \to \infty$ we get by monotone convergence that

$$\mathbb{E}\left[\sum_{x \in \mathrm{V}(G_\infty)} f(G_\infty, \rho_\infty, x)\right] = \mathbb{E}\left[\sum_{x \in \mathrm{V}(G_\infty)} f(G_\infty, x, \rho_\infty)\right].$$

The mass-transport principle is thus satisfied for all transport functions depending only on a finite range around the first point. However, there are transport functions which are not simple function of that sort, for example consider $f(\mathrm{g}, x, y) = \mathbf{1}_{x \sim y} \mathbf{1}_{\#\mathrm{V}(\mathrm{g}) = \infty}$ for which the condition $\#\mathrm{V}(\mathrm{g}) = \infty$ is not continuous for the local topology! Proving the general result necessitates a bit of abstract measure theory. We proceed as follows. Let $r_0, k \geq 0$ and denote by

$$D_{r_0,k} = \{(\mathrm{g}, x, y) : \mathrm{d_{gr}}(x, y) \leq r_0 \text{ and } \#\mathrm{V}([(\mathrm{g}, x, y)]_{r_0}) \leq k\} \subset \mathcal{G}^{\bullet\bullet}.$$

We then introduce the family of measurable sets

$$\mathcal{M}_{r_0,k} = \left\{ A \subset \mathcal{G}^{\bullet\bullet} \text{ measurable} : \mathbb{E}\left[\sum_{x \in \mathrm{V}(G_\infty)} \mathbf{1}_{(G_\infty, \rho_\infty, x) \in A \cap D_{r_0,k}}\right] = \mathbb{E}\left[\sum_{x \in \mathrm{V}(G_\infty)} \mathbf{1}_{(G_\infty, x, \rho_\infty) \in A \cap D_{r_0,k}}\right]\right\}.$$

By the above warmup, all the elementary sets $A = \{(\mathrm{g}, x, y) : [(\mathrm{g}, x, y)]_r = \mathrm{g}_0^{\bullet\bullet}\}$ when $\mathrm{g}_0^{\bullet\bullet} \in \mathcal{G}^{\bullet\bullet}$ is a finite bi-pointed graph are in $\mathcal{M}_{r_0,k}$ and those sets generate the Borel σ-field of $\mathcal{G}^{\bullet\bullet}$ and are stable under finite intersection. We claim that $\mathcal{M}_{r_0,k}$ is a monotone class: the stability under monotone union is clear using the monotone convergence theorem, and the stability under difference follows from the fact that

$$\mathbb{E}\left[\sum_{x \in \mathrm{V}(G_\infty)} \mathbf{1}_{(G_\infty, \rho_\infty, x) \in A^c \cap D_{r_0,k}}\right] = \mathbb{E}\left[\sum_{x \in \mathrm{V}(G_\infty)} \mathbf{1}_{(G_\infty, \rho_\infty, x) \in D_{r_0,k}}\right] - \mathbb{E}\left[\sum_{x \in \mathrm{V}(G_\infty)} \mathbf{1}_{(G_\infty, \rho_\infty, x) \in A \cap D_{r_0,k}}\right],$$

and similarly when the roles of ρ_∞ and x are exchanged. We notice that the first expectation in the right-hand side is finite (less than k from the definition of $D_{r_0,k}$). It follows that $\mathcal{M}_{r_0,k}$ is the Borel σ-field of $\mathcal{G}^{\bullet\bullet}$. Sending $r_0 \to \infty$ and $k \to \infty$, we deduce from the monotone convergence theorem that $G_\infty^\bullet$ obeys the mass-transport principle for any indicator function. Since positive functions are almost sure increasing limits of sum of indicator functions the full MTP follows from another application of the monotone convergence theorem. □

When G_n is a random finite graph, recall that if its uniformly pointed version $\mathrm{U}^\bullet(G_n)$ converges locally towards $G_\infty^\bullet$, we say that $G_\infty^\bullet$ is the Benjamini–Schramm limit of the random (unpointed) graphs G_n. In particular, by the above theorem, any Benjamini–Schramm limit is unimodular. The converse is a key open problem in the field. Its positive resolution would imply quite a few famous conjectures in group theory as this would imply that every group is "sofic":

Open Question 1 (Aldous–Lyons)**.** *Is every unimodular random graph a local limit in distribution of uniformly pointed random graphs ?*

This conjecture has been proved in special cases for example when the limiting random graph does not grow too fast [9] or the case of unimodular random trees [28, 37, 19, 26]. However, very recent works (still under review by the community at the time when those lines are written) announced a negative resolution of that conjecture, [29, 30].

Exercise 20. Recall the notation $G^\bullet(n, p)$ for the connected component of 1 where the vertex 1 is distinguished in an Erdős–Rényi random graph over n vertices and parameter p. Show that $G^\bullet(n, p)$ is a unimodular random graph and deduce that Poisson-Galton–Watson trees (pointed at the ancestor) are also unimodular. Show that a supercritical Poisson-Galton–Watson trees (pointed at the ancestor) conditioned to be infinite is still unimodular.

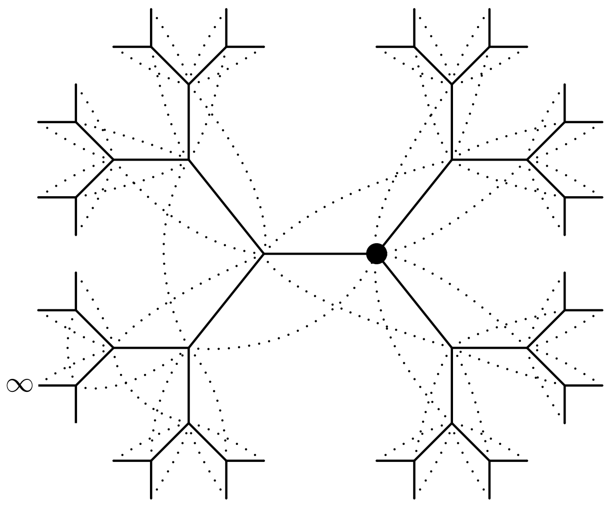
7.2 Examples

Before proving some very general properties of unimodular random graphs, let us give a few examples. The first one actually being a counter-example. We will see below that unimodular random graphs are stable under constructions which do not depend upon a distinguished origin.

7.2.1 The grand-father graph

The following graph is an example of a vertex-transitive graph (recall Definition 2) which is not unimodular. We start with a k-regular tree $\mathtt{t}$ with $k \geq 3$ (in the following $k = 3$) given together with a distinguished infinite ray (i.e. an infinite self-avoiding path converging to a point on the boundary). This ray enables us to speak about a limit point "∞" at its extremity. Hence, if x and y are neighbors in the graph, then one of the two vertices x or y is closer to "∞" than the other. The furthest of the two points is called the parent of the other and this induces a genealogical order on the vertices of the tree. In the original graph $\mathtt{t}$ we then add all the edges linking a vertex to its four grand-parents (hence the name of the graph). The graph $\mathtt{gf}$ obtained is clearly vertex-transitive.

However, this graph (pointed at any vertex) is not unimodular: consider the transport function $f(\mathrm{g}, x, y) = 1$ if y is a parent of x and 0 otherwise. We let the reader check that this is indeed a transport function (this is not trivial, we have to show that the graph



structure of gf enables to recover the distinguished ray hence the genealogical order). Then the MTP is violated since

$$2 = \mathbb{E}\left[\sum_{x\in \mathrm{V(gf)}} f(\mathsf{gf},\rho,x)\right] \neq \mathbb{E}\left[\sum_{x\in \mathrm{V(gf)}} f(\mathsf{gf},x,\rho)\right] = 1.$$

7.2.2 Cayley graphs

Recall the definition of Cayley graphs (Definition 1).

Proposition 25. *Any Cayley graph (pointed anywhere) is unimodular.*

Proof. Let g be the Cayley graph of (Gr, S) pointed at the identity e of the group. Then for any $x, y \in \mathsf{Gr}$ there is a homomorphism of bi-pointed graph $(\mathsf{g}, x, y) \to (\mathsf{g}, e, yx^{-1})$ where e is the identity of the group: this is the multiplication by x^{-1} on the right. Hence, since a transport function f is invariant under homomorphism of bi-pointed graphs, we have $f(\mathsf{g}, x, y) = \tilde{f}(yx^{-1})$ for some function $\tilde{f} : \mathsf{Gr} \to \mathbb{R}_+$. Hence we have

$$\mathbb{E}\left[\sum_{x\in\mathsf{Gr}} f(\mathsf{g},e,x)\right] = \sum_{x\in\mathsf{Gr}} f(\mathsf{g},e,x) = \sum_{x\in\mathsf{Gr}} \tilde{f}(x) = \sum_{x\in\mathsf{Gr}} \tilde{f}(x^{-1}) = \mathbb{E}\left[\sum_{x\in\mathsf{Gr}} f(\mathsf{g},x,e)\right],$$

where we used the fact that $x \mapsto x^{-1}$ is an involution of the group Gr. □

As a consequence of the last proposition we deduce that although being vertex-transitive, the grand-father graph is not a Cayley graph of any group (do you see another way to prove it?).

Exercise 21. Show that the Euclidean lattice $\mathbb{E}^d$ for $d \geq 1$ are unimodular. Show that the d-regular infinite tree $\mathbb{T}_d$ are unimodular (for d even these are Cayley graphs of the free group on $d/2$ generators, but for d odd that's not the case).

7.2.3 Construction from existing unimodular random graphs

We can obtain new unimodular random graphs from existing ones by modifications which do not depend on a base point. We present here the case of bond percolation but the result can be adapted to many other situations such as invariant percolation (see Section 7.3.2). Let $G^\bullet = (G, \rho)$ be a unimodular random graph and conditionally on $G^\bullet$ perform a bond percolation on G with parameter $p \in (0,1)$ (i.e. keep each edge independently with probability p). Denote by $\mathcal{C}^\bullet(\rho) = (\mathcal{C}(\rho), \rho)$ the cluster of the origin ρ pointed at ρ. Hence $\mathcal{C}^\bullet(\rho)$ is a random pointed graph.

Proposition 26. *The random graph $\mathcal{C}^\bullet(\rho)$ is unimodular.*

Proof. We directly verify the mass-transport principle (7.1): Fix a transport function f and compute

$$\mathbb{E}\left[\sum_{x \in \mathrm{V}(\mathcal{C}(\rho))} f(\mathcal{C}(\rho), \rho, x)\right] = \mathbb{E}\left[\sum_{x \in \mathrm{V}(G)} \mathbf{E}_G\left[f(\mathcal{C}(\rho), \rho, x)\mathbf{1}_{x \leftrightarrow \rho}\right]\right],$$

where $\mathbf{E}_{\mathrm{g}}$ is the probability measure underlying a bond percolation on the graph g and $x \leftrightarrow y$ means that x and y are in the same cluster after performing the percolation. We just have to realize that $F(\mathrm{g}, x, y) = \mathbf{E}_{\mathrm{g}}\left[f(\mathcal{C}(x), x, y)\mathbf{1}_{x \leftrightarrow y}\right]$ is a transport function and so applying the MTP with the initial graph (G, ρ) we get that

$$\mathbb{E}\left[\sum_{x \in \mathrm{V}(G)} \mathbf{E}_G\left[f(\mathcal{C}(\rho), \rho, x)\mathbf{1}_{x \leftrightarrow \rho}\right]\right] = \mathbb{E}\left[\sum_{x \in \mathrm{V}(G)} \mathbf{E}_G\left[f(\mathcal{C}(x), x, \rho)\mathbf{1}_{\rho \leftrightarrow x}\right]\right].$$

Noting that on the event $\{\rho \leftrightarrow x\}$ the clusters $\mathcal{C}(\rho)$ and $\mathcal{C}(x)$ containing respectively x and ρ are the same it remains just to apply Fubini's theorem to arrive at the desired equality. □

Exercise 22. Let (G, ρ) be a unimodular random graph. Let $A \subset \mathcal{G}^\bullet$ be an invariant event in the sense that if $(\mathrm{g}, x) \in A$ then $(\mathrm{g}, y) \in A$ for any $y \in \mathrm{g}$. Give examples of such A and prove that if $\mathbb{P}((G, \rho) \in A) > 0$ then (G, ρ) conditioned on being in A is again unimodular.

7.3 A few applications

We now present a couple of results which hold for any unimodular random graphs. We will see indeed that these random graphs cannot behave too widely and should be thought of as the stochastic analog of "regular" or "homogeneous" graphs. We focus mainly on the concept of ends. Then we present the initial use of the Mass-Transport [40] which does not directly lie in our framework but illustrates the power of the technique. We start with a useful proposition:

Proposition 27 (Everything shows at the origin)**.** *Let $G^\bullet$ be a unimodular random graph and $A \subset \mathcal{G}^\bullet$ be a Borel set such that $\mathbb{P}(G^\bullet \in A) = 0$. Then the probability that there exists a vertex $x \in \mathrm{V}(G)$ such that $(G, x) \in A$ is equal to zero.*

Proof. Simply consider the transport function $f(\mathrm{g}, x, y) = \mathbf{1}_{(\mathrm{g},x)\in A}$. The mass-transport principle entails that

$$0 = \mathbb{P}((G,\rho) \in A) = \mathbb{E}\left[\sum_{x\in \mathrm{V}(G)} f(G,\rho,x)\right] = \mathbb{E}\left[\sum_{x\in \mathrm{V}(G)} f(G,x,\rho)\right] = \mathbb{E}\left[\sum_{x\in \mathrm{V}(G)} \mathbf{1}_{(G,x)\in A}\right].$$

□

Proposition 28 (If it happens, it happens a lot)**.** *Let (G, ρ) be a unimodular random graph which is almost surely infinite. Then for any $A \subset \mathcal{G}^\bullet$ Borel we have*

$$\#\{x \in \mathrm{V}(G) : (G, x) \in A\} \in \{0, \infty\} \quad a.s.$$

Proof. Fix a measurable subset $A \subset \mathcal{G}^\bullet$ and for $\mathrm{g} \in \mathcal{G}^\bullet$ denote by $A_g = \{x \in \mathrm{V}(\mathrm{g}) : (\mathrm{g}, x) \in A\}$. We then consider the transport function

$$f(x, y, \mathrm{g}) = \frac{1}{\#A_{\mathrm{g}}} \mathbf{1}_{y\in A_{\mathrm{g}}} \mathbf{1}_{0<\#A_{\mathrm{g}}<\infty}.$$

In other words, on the event when A_{g} is non-empty and finite, each vertex x splits a unit mass between all the vertices of A_{g} and otherwise does nothing. Applying the mass-transport principle (7.1) we deduce that

$$\mathbb{P}(\#A_G \in \{1, 2, ...\}) = \mathbb{E}[\infty \cdot \mathbf{1}_{(G,\rho)\in A} \mathbf{1}_{0<\#A_{\mathrm{g}}<\infty}].$$

Since the left-hand side is bounded by one we deduce that the event $\{(G, \rho) \in A$ and $0 < \#A_{\mathrm{g}} < \infty\}$ has zero probability. By the same display it follows that $\mathbb{P}(0 < \#A_G < \infty) = 0$ also. This is the desired statement. □

Heuristically speaking, when the random graph G possesses some vertex x such that seen from x the graph G has a certain property, then there are infinitely such vertices and even with "positive" density, whatever it means. Beware, this is not a $0-1$ law for the event in question: if the graph (G, ρ) is equal to $\mathbb{E}^2$ with probability $1/2$ and to $\mathbb{E}$ with probability $1/2$ and if A is the event $\{\deg(\rho) = 2\}$ then $\mathbb{P}(A) = 1/2$.

7.3.1 Ends of unimodular random graphs

One first striking application of the MTP is about ends of unimodular random graphs. Let us recall the definition of this important notion.

Definition 17. *A ray in a graph* g *is an infinite self-avoiding path (i.e. a sequence of distinct neighboring edges). We say that two rays* r *and* r' *are equivalent if there exists a third ray* r'' *which shares an infinite number of edges with both* r *and* r'*. This is an equivalence relation. The number of ends of* g *is the cardinality of the quotient space of the space of rays by the above equivalence relation.*

Exercise 23. Show that the definition is well posed and show that:

- any finite graph has 0 end,
- any infinite graph has at least one end,
- $\mathbb{E}^1$ has two ends,
- $\mathbb{E}^d$ for $d \geq 2$ has one end
- the d-ary tree $\mathbb{T}_d$ with $d \geq 3$ has uncountably many ends.

Theorem 29 (*The degree tells us a lot*)

Let (G, ρ) *be a unimodular random graph.*

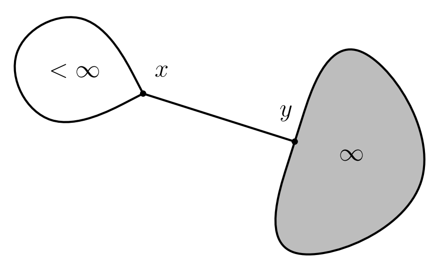
1. *If* G *is almost surely finite then we have* $\mathbb{E}[\deg(\rho)] \geq 2 \cdot \mathbb{E}[1 - \#\mathrm{V}(G)^{-1}]$.
2. *If* G *is almost surely infinite then* $\mathbb{E}[\deg(\rho)] \geq 2$.
3. *If* G *is almost surely infinite and* $\mathbb{E}[\deg(\rho)] = 2$ *then* G *is almost surely a tree with* 1 *or* 2 *ends.*

Proof. For the first point we consider the transport function $f(\mathrm{g}, x, y) = \deg(x)/\#\mathrm{V}(\mathrm{g})$. Applying (7.1) we get that

$$\mathbb{E}[\deg(\rho)] = \mathbb{E}\left[\sum_{x \in \mathrm{V}(G)} \frac{\deg(x)}{\#\mathrm{V}(G)}\right].$$

But in any graph the sum of all the degrees of the vertices is equal to twice the number of edges and since G is connected we have by Proposition 1 that $\#\mathrm{E}(G) \geq \#\mathrm{V}(G) - 1$. Combining these observations we get point 1.

For the second point we consider the transport function f defined by $f(\mathrm{g}, x, y) = 1$ whenever x and y are neighbors and linked by a single edge whose suppression isolates x in a finite connected component. We then apply the mass-transport principle (7.1) and split the cases depending on the degree of ρ. Remark first that ρ receives deterministically always strictly less than $\deg(\rho)$ unit of mass for otherwise g would be finite, furthermore:



- when $\deg(\rho) = 1$ (we say that ρ is a leaf) then ρ sends a unit of mass to its unique neighbor and receives nothing,
- when $\deg(\rho) \geq 2$ and ρ happens to send a unit of mass through some edge then it cannot receive mass through this same edge and thus receives in total less than $\deg(\rho) - 1$ unit of mass,
- when $\deg(\rho) \geq 2$ and ρ sends no mass at all then after a few drawings, one can convince oneself that ρ receives less than $\deg(\rho) - 2$ unit of mass.

In all cases we have deterministically

$$\deg(\rho) + \mathrm{Sent}(\rho) - \mathrm{Received}(\rho) \geq 2.$$

The mass-transport principle (7.1) precisely says that the averages of the last two terms of the left-hand side cancel so that $\mathbb{E}[\deg(\rho)] \geq 2$ as desired. The last point of theorem corresponds to the saturation of the above inequalities. More precisely, if

$$F(\mathrm{g}, x) = \deg(x) + \sum_{y \in \mathrm{V}(\mathrm{g})} f(\mathrm{g}, x, y) - \sum_{y \in \mathrm{V}(\mathrm{g})} f(\mathrm{g}, y, x)$$

then we deterministically have $F(\mathrm{g}, x) \geq 2$ by the above point and thus $\mathbb{E}[\deg(\rho)] = 2$ implies that $F(G, \rho) = 2$ almost surely. By Proposition 27 we deduce that almost surely we have $F(G, x) = 2$ simultaneously for all $x \in \mathrm{V}(G)$. However it is easy to see that if there is a non-trivial cycle in the graph G then $F(G, x) > 2$ for any vertex x on this cycle. The graph is thus a tree. Also, when x is a point where at least three infinite paths merge then we have $F(\mathrm{g}, x) > 2$, hence there are no such points and the tree has 1 or 2 ends. □

Exercise 24. Recall that a vertex x in a graph g is a leaf if it has degree 1 (that is adjacent to a unique edge which is not a loop). Show that in a unimodular random graph (G, ρ) we have

$$\mathbb{P}(\rho \text{ is a leaf}) = \mathbb{E}\left[\#\text{leaves neighboring } \rho\right].$$

The next result is a generalization of a well-known theorem for Cayley graphs:

Theorem 30 (***Number of ends***)

The number of ends of a unimodular random graph necessarily belongs to $\{0, 1, 2, \infty\}$ *(but can be a random variable).*

Proof. We can suppose that (G, ρ) is almost surely infinite, otherwise consider (G, ρ) conditioned on the event $\#\mathrm{V}(G) = \infty$ which is still a unimodular random graph by Exercise 22. For $r \geq 0$ we call a vertex $x \in \mathrm{V}(\mathrm{g})$ an r-trifurcation if $\mathrm{g} \backslash [(\mathrm{g}, x)]_r$ contains at least 3 infinite connected components. It is an easy exercise to see that if a locally finite infinite

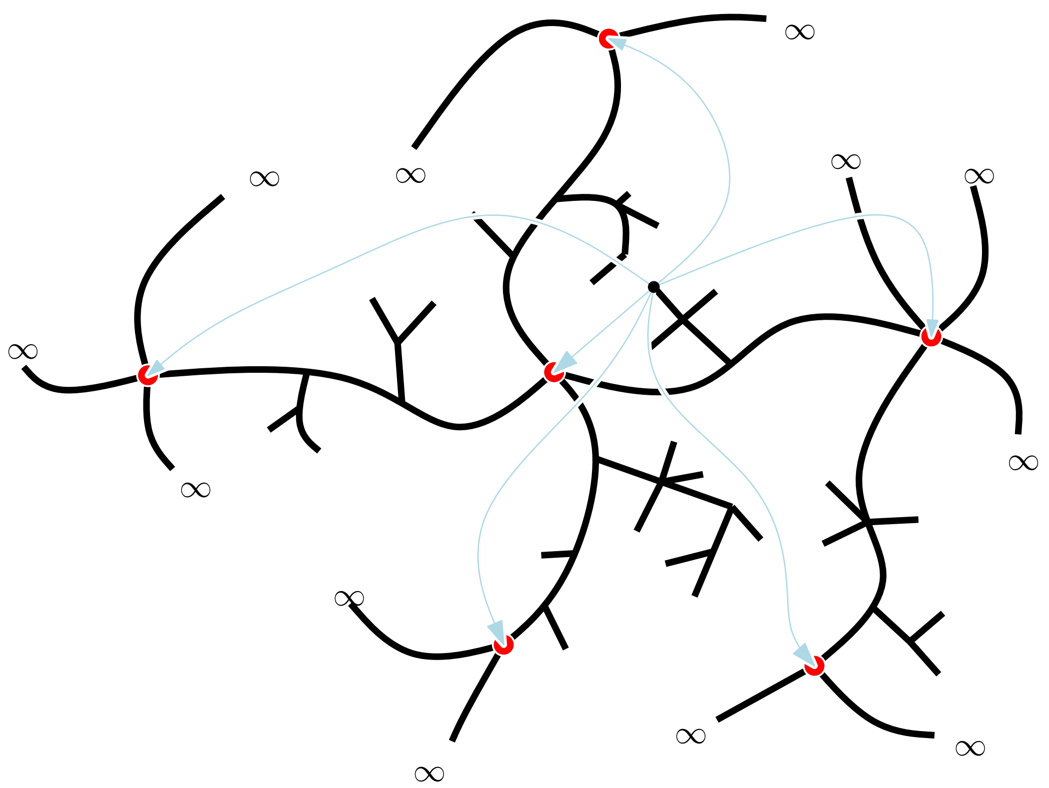


Figure 7.1: Illustration of the proof of Theorem 30. By (MTP), the number of r-trifurcations (in red on the figure) is either 0 or ∞. In the first case, the number of ends is $0, 1$ or 2, whereas it is ∞ is the second case.

graph has $k \in \{3, 4, ...\}$ ends then there must exists a r-trifurcation for some $r \geq 1$. Also if the number of r-trifurcations is infinite, then a simple geometric argument shows that g must have infinitely many ends, see Figure 30. By Proposition 28 for any $r \geq 1$, in a unimodular infinite random graph (G, ρ) the number of r-trifurcations is either 0 or ∞ almost surely. We deduce from the last two geometric remarks that the number of ends of (G, ρ) is necessarily in $\{1, 2, \infty\}$.

□

7.3.2 Invariant percolation on trees

We end this section by presenting the very first application of the MTP in the context of probability: Let $\mathbb{T}_4$ be the four-regular infinite tree (which is transitive and unimodular). An invariant (site) percolation on $\mathbb{T}_4$ is a probability measure on $\Omega : \{0, 1\}^{\mathrm{V}(\mathbb{T}_4)}$ which is invariant under any graph homomorphism of $\mathbb{T}_4$, in words, it is a random bicoloring of the vertices $v \in \mathrm{V}(\mathbb{T}_4)$ in black if $\omega(v) = 1$ and white if $\omega(v) = 0$ such that the law of coloring does not depend on the distinguished pointed vertex. For example a Bernoulli site percolation would do the job, but the interesting case is that this percolation may exhibit dependence between sites! By invariance, the percolation density

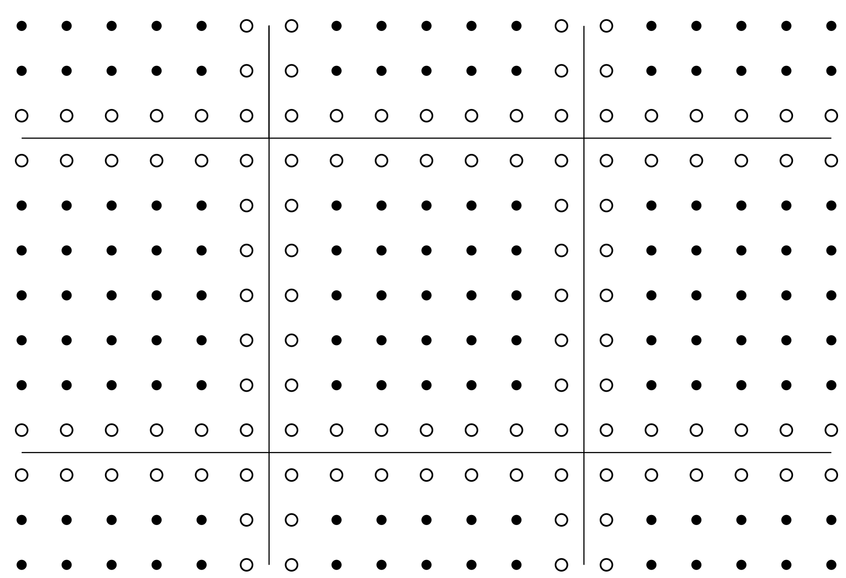
$$p = \mathbb{E}[\omega(v)] \in [0, 1],$$

does not depend on $v \in V(\mathbb{T}_4)$. The result we want to prove is the following:

Theorem 31 (*High density automatically implies percolation*)

If p is close enough to 1 then any invariant percolation on $\mathbb{T}_4$ with density p has at least one infinite cluster with positive probability.

This theorem may seem awkward at first glance. To understand its power we will construct a counterexample, i.e. a percolation on $\mathbb{E}^2$ that is an invariant dependent percolation with density as close to 1 as possible but with no infinite cluster. To do this, imagine that we tile $\mathbb{E}^2$ with squares of size $n \times n$ and color the vertices on the boundary of these squares in black (there are $4n-4$ such vertices per big square) and all the others in white (there are n^2-4n+4 such vertices per square). This gives a periodic coloring of $\mathbb{E}^2$. To transform it into an invariant percolation we just need to pick the distinguished vertex ρ of $\mathbb{E}^2$ uniformly at random inside a fixed big square.



Exercise 25. Check that the resulting random configuration of colors on $(\mathbb{E}^2, \rho)$ is invariant in law under all graph homomorphisms of $\mathbb{E}^2$.

Clearly the resulting percolation has no infinite cluster, but the density of the percolation is

$$\frac{n^2-4n+4}{n^2} \xrightarrow[n\to\infty]{} 1.$$

Hiding behind this construction is the fact that in $\mathbb{E}^2$ there are large sets whose boundary size is negligible with respect to their size. This fact is not true in regular trees (or more generally in non-amenable graphs) and we can check that for any connected set A of vertices in $\mathbb{T}_4$, if we denote by ∂A the set of vertices adjacent to A but not in A then we have for some constant $c > 0$ independent of A,

$$\frac{\#\partial A}{\#A} \geq c. \tag{7.2}$$

Proof of Theorem 31. Given the tree $\mathbb{T}_4$ and a percolation ω on it, we will define a transport function f_ω on the vertices of $\mathbb{T}_4$ as follows: Recall that $\mathcal{C}(x)$ is the cluster of x then for all $y \in \mathrm{V}(\mathbb{T}_4)$ we put

$$\begin{cases} \text{if } \omega(x) = 0 & \text{then} \quad f_\omega(x,y) = 0 \\ \text{if } \omega(x) = 1 \text{ and } \#\mathcal{C}(x) = \infty & \text{then} \quad f_\omega(x,y) = 0 \\ \text{if } \omega(x) = 1 \text{ and } \#\mathcal{C}(x) < \infty & \text{then} \quad f_\omega(x,y) = \frac{1}{|\partial\mathcal{C}(x)|}\mathbf{1}_{y\in\partial\mathcal{C}(x)}. \end{cases}$$

In words, if $\omega(x) = 0$ or if x is in an infinite cluster then x sends no mass at all. Otherwise, if x is in a finite connected component for ω he sends mass 1 which is spread over all the neighbor vertices of $\mathcal{C}(x)$. Recall that $\mathbb{T}_4$ can be seen as the Cayley graph of the free group over two elements (see Exercise 21) we deduce that $\mathbb{E}[f_\omega(x,y)]$ in fact only depends on xy^{-1} and by invariance of the percolation and involution invariance of the group we get that (exercise!)

$$\mathbb{E}\left[\sum_{x\in\mathrm{V}(\mathbb{T}_4)} f_\omega(\rho,x)\right] = \mathbb{E}\left[\sum_{x\in\mathrm{V}(\mathbb{T}_4)} f_\omega(x,\rho)\right]. \tag{7.3}$$

This is the version of (7.1) that we will use. On the left-hand side of the last display we have $\mathbb{P}(\omega(x) = 1 \text{ and } \#\mathcal{C}(x) < \infty)$. We will now bound the right-hand side. Remark first that to receive mass, the vertex ρ must be white $\omega(\rho) = 0$ and must lie on the boundary of a finite cluster. Deterministically ρ can be on the boundary of at most 4 finite clusters A_1, A_2, A_3 and A_4 and all the vertices in A_i sends a mass to ρ equal to

$$\frac{\#A_i}{\#\partial A_i}.$$

By (7.2) and (7.3) we thus deduce that

$$\mathbb{P}(\omega(\rho) = 1 \text{ and } \#\mathcal{C}(\rho) < \infty) \leq \frac{4}{c}\cdot\mathbb{P}(\omega(\rho) = 0).$$

If the density $p = \mathbb{P}(\omega(\rho) = 1)$ is close enough to 1 then the left-hand side becomes smaller than p and thus ρ is in an infinite cluster with positive probability as desired. □

Exercise 26 (From [42]). The goal of this exercise is to prove that a unimodular random graph $G_\infty^\bullet = (G_\infty, \rho_\infty)$ which is almost surely recurrent and satisfies $\mathbb{E}[\deg(\rho_\infty)] < \infty$ has the infinite collision property, that is, two independent random walkers will collide (be at the same vertex at the same time) infinitely many often. Let us start with some notation. Fix the underlying graph $\mathbf{g}$ and denote by $p_n(\cdot,\cdot)$ the n-step transition probabilities for the simple random walk. For $u \in \mathrm{V}(\mathbf{g})$, let $q_{\mathrm{fin}}(u)$ the probability that two independent random walks started at u collide only finitely often, $q_0(u)$ the probability that they not collide at all after time zero, and for $u \in \mathrm{V}(\mathbf{g})$ let $q_{\mathrm{last}}(u,v)$ be the probability that two independent random walks started at u collide for the last time at v.

1. Show that
$$q_{\text{fin}}(u) = \sum_{v \in \mathrm{V}(\mathrm{g})} q_{\text{last}}(u,v) = \sum_{v \in \mathrm{V}(\mathrm{g})} \sum_{n \geq 0} p_n(u,v)^2 q_0(v).$$

2. Apply (MTP) with the transport function $f(G,u,v) = \deg(u) q_{\text{last}}(u,v)$ and deduce that in (G_∞, ρ_∞) we have
$$\mathbb{E}[\deg(\rho_\infty) q_{\text{fin}}(\rho_\infty)] = \mathbb{E}\left[q_0(\rho_\infty) \deg(\rho_\infty) \sum_{n \geq 0} p_{2n}(\rho_\infty, \rho_\infty) \right].$$
Hint: use $\deg(u) p_n(u,v) = \deg(v) p_n(v,u)$.

3. Conclude.

7.4 Palm distribution

A unimodular random graph $G^\bullet$ can thus be thought of as a random graph where the pointed vertex is "uniform" over the vertices of G and the Mass–Transport–Principle is a way to make this rigorous. In this informal section, we draw the parallel with the Palm distribution in the theory of point processes on Euclidean or Hyperbolic spaces. Indeed the Palm version can be understood as a point process with a distinguished point chosen uniformly at random.

Suppose $\mathbb{P}$ is a law on point processes $\mathcal{P}$ of $\mathbb{R}^d$ say, that are collection of countable points. We suppose that $\mathbb{P}$ has a constant intensity, that is, for any $A \in \mathbb{R}^d$
$$\mathbb{E}[\#\{\mathcal{P} \cap A\}] = c \cdot \mathrm{Leb}(A). \tag{7.4}$$
For any function $F(x, \mathcal{P})$ that depends on a point x in first coordinate, and a point process on the second coordinate (we spare the measurability issues of those definitions) one can define a infinite measure by putting
$$\int \mathrm{d}\mu(x,\mathcal{P}) F(x,\mathcal{P}) = \mathbb{E}\left[\sum_{x \in \mathcal{P}} F(x, \mathcal{P} - x) \right].$$
By stationarity (7.4) the measure μ decomposes as $c \cdot \mathrm{Leb} \otimes \mathbb{P}^\bullet$ where $\mathbb{P}^\bullet$ is a law on point process containing the point 0. This law is called the Palm version of $\mathbb{P}$ and can be morally interpreted as the law $\mathbb{P}$ seen from a uniform random point (or conditioned to contain the point 0). The equivalent of the Mass Transport Principle in this geometric version is the Mecke–Campbell theorem, which is immediate from the above discussion and reads
$$c \cdot \mathbb{E}^\bullet \left[\int_{\mathbb{R}^d} F(x,\mathcal{P}) \mathrm{d}x \right] = \mathbb{E}\left[\sum_{x \in \mathcal{P}} F(x, \mathcal{P} - x) \right].$$

One can then formally recover the Mass Transport principle and get that

$$\mathbb{E}^{\bullet}\left[\sum_{x\in\mathcal{P}} F(\mathcal{P},0,x)\right] = \mathbb{E}^{\bullet}\left[\sum_{x\in\mathcal{P}} F(\mathcal{P},x,0)\right].$$

Biliographical references. Olle Haggstrom [40] first used a form of the Mass-transport principle to study (dependent) invariant percolation on regular trees and show Theorem 31. This triggered the systematic study of percolation on Cayley graph using the mass-transport principle as done by Benjamini, Lyons, Peres and Schramm [18]. The most general form of unimodular random graph and the MTP (as well as the local topology) have been introduced by Benjamini and Schramm [20]. Most of this chapter is adapted from the wonderful survey paper [9] which regroups and extends the results of [18] to random graphs. We refer to [9] for original references and pointers. The Open Question 1 is due to Aldous & Lyons [9]. The answer is known to be true in the case of trees [37, 19]. The reader interested in connection with measured equivalence relations should consult [9, 44, 59]

Chapter 8

The random walk point of view

When the distinguished vertex in a pointed graph is replaced by a distinguished oriented edge (we speak then of rooted graphs), the theory can be naturally extended using a Mass Transport Principle (MTP) adapted to directed edges. A key observation is that the uniform measure on oriented edges corresponds to the stationary distribution of the simple random walk on the graph. This insight allows the theory to be reformulated in terms of random walks.

This reformulation leads to the notions of stationarity and reversibility in the context of random rooted graphs: A random rooted graph is said to be stationary if the distribution of the graph as seen from a particle performing a simple random walk remains statistically unchanged as the particle moves. It is reversible if the process looks the same when the direction of time is reversed.

Though conceptually simple, this perspective allows powerful tools from ergodic theory to be imported into the probabilistic framework, enriching the analysis of random graphs and processes on them.

8.1 Stationary (and reversible) random graphs

A **rooted graph** is a pair $\vec{\mathsf{g}} = (\mathsf{g}, \vec{e})$ where g is a (locally finite, connected) graph and $\vec{e}$ is a distinguished oriented edge (that is given with a direction) that we call the **root edge**. If $\mathsf{g} = (\mathrm{V}(\mathsf{g}), \mathrm{E}(\mathsf{g}))$ is a graph, we denote by $\vec{\mathrm{E}}(\mathsf{g})$ the set of all oriented edges of the graph, which is obtained informally by duplicating each non-oriented edge (including self-loops) into two oriented edges. If $\vec{e} \in \vec{\mathrm{E}}(\mathsf{g})$ we denote by $\vec{e}_*$ (resp. $\vec{e}^*$) the origin (resp. the target) vertex of $\vec{e}$. If $\vec{e}$ is the root edge, we usually write $\vec{e}_* = \rho$. Two rooted graphs $\vec{\mathsf{g}}_1 = (\mathsf{g}_1, \vec{e}_1)$ et $\vec{\mathsf{g}}_2 = (\mathsf{g}_2, \vec{e}_2)$ are equivalent if there exists a graph homomorphism $\mathsf{g}_1 \to \mathsf{g}_2$ which sends $\vec{e}_1$ onto $\vec{e}_2$. As usual we will implicitly identify such graphs and work on $\vec{\mathcal{G}}$, the space of equivalence classes of (locally finite, connected) rooted graphs. For $r \geq 0$, the restriction of radius r in $\vec{\mathrm{g}}$, denoted by $[\vec{\mathrm{g}}]_r$, is obtained by keeping only those vertices and edges of g which are at distance less than or equal to r from the origin of the root edge and keeping the root edge as the distinguished oriented edge. As previously, we consider the local distance $\mathrm{d_{loc}}$ on $\vec{\mathcal{G}}$ defined by the procedure of Section 1.2.1. If $\vec{\mathrm{g}} = (\mathrm{g}, \vec{e})$ is a rooted graph, we denote by $\pi^\bullet(\vec{\mathrm{g}})$ the pointed graph obtained by distinguished in g the origin vertex of the root edge. Conversely, if $G^\bullet = (G, \rho)$ is a deterministic or random pointed graph, we denote by $\vec{\pi}(G^\bullet)$ the *random* rooted graph obtained by distinguishing an oriented edge emanating from ρ uniformly at random, conditionally on $G^\bullet$. Although $\vec{\pi}$ contains some additional randomness, the mapping $\pi^\bullet \circ \vec{\pi}$ is the identity on $\mathcal{G}^\bullet$. If G is a finite random graph, we denote by $\vec{\mathrm{U}}(G)$ the random rooted graph obtained from G by distinguishing an oriented edge uniformly at random.

Exercise 27. Show that $\pi^\bullet : \vec{\mathcal{G}} \to \mathcal{G}^\bullet$ is continuous.

8.1.1 Uniformly rooted random graphs

We start with the exact analog of Definition 4 in the context of rooted graphs:

Definition 18. *Let $\vec{G} = (G, \vec{e})$ an almost surely finite random rooted graph. We say that $\vec{G}$ is **uniformly rooted** if the root edge is chosen uniformly on $\vec{\mathrm{E}}(G)$, i.e. $\vec{G} = \vec{\mathrm{U}}(G)$ in law that is if for all Borel function $f : \vec{\mathcal{G}} \to \mathbb{R}_+$ we have*

$$\mathbb{E}[f(\vec{G})] = \mathbb{E}\left[\frac{1}{\#\vec{\mathrm{E}}(G)} \sum_{\vec{\sigma} \in \vec{\mathrm{E}}(G)} f(G, \vec{\sigma})\right].$$

We now explain how to pass from a uniformly rooted graph to a uniformly pointed graph and vice-versa. This uses the concept of law biased by a random variable. Recall that if X, Y are two random variables defined on a common probability space such that X takes its values in some abstract space (E, d) and such that Y is positive real-valued then we can construct a new random variable $\tilde{X}$ whose law is the law of *the random variable X biased by Y* characterized by

$$\mathbb{E}[f(\tilde{X})] = \frac{1}{\mathbb{E}[Y]} \mathbb{E}[f(X) \cdot Y],$$

for every Borel function $f : E \to \mathbb{R}_+$. Of course, this definition requires that Y has a finite non zero expectation. Equivalently, the law of $\tilde{X}$ is the distribution with Radon–Nikodym derivative equal to $Y(\omega)/\mathbb{E}[Y]$ with respect to the law of X. To simplify notation below, we use the shorthand notation

$$\Delta = \deg(\rho) = \deg(\vec{e}_*),$$

both for pointed graphs $G^\bullet$ or rooted graphs $\vec{G}$ (recall that we denoted by $\rho = \vec{e}_*$ the origin vertex of the root edge $\vec{e}$).

We will show that if $\vec{G}$ is a random uniformly rooted graph then the graph $\underline{\pi^\bullet(\vec{G})}$ obtained from $\pi^\bullet(\vec{G})$ by biasing by Δ^{-1} is a random uniformly pointed graph. Indeed, if f is a positive Borel function on $\mathcal{G}^\bullet$ we have

$$\begin{aligned}
\mathbb{E}[f(\underline{\pi^\bullet(\vec{G})})] &= \frac{1}{\mathbb{E}[\Delta^{-1}]}\mathbb{E}\left[\Delta^{-1} f(\pi^\bullet(\vec{G}))\right] \\
&\underset{\text{unif. rooted}}{=} \frac{1}{\mathbb{E}[\Delta^{-1}]}\mathbb{E}\left[\frac{1}{\#\vec{E}(G)}\sum_{\vec{\sigma}\in\vec{E}(G)}(\deg(\vec{\sigma}_*))^{-1} f(G,\vec{\sigma}_*)\right] \\
&= \frac{1}{\mathbb{E}[\Delta^{-1}]}\mathbb{E}\left[\frac{1}{\#\vec{E}(G)}\sum_{x\in \mathrm{V}(G)} f(G,x)\right] \\
&= \frac{1}{\mathbb{E}[\Delta^{-1}]}\mathbb{E}\left[\frac{1}{\#\vec{E}(G)}\sum_{\vec{\sigma}\in\vec{E}(G)}(\deg(\vec{\sigma}_*))^{-1}\frac{1}{\#\mathrm{V}(G)}\sum_{x\in \mathrm{V}(G)} f(G,x)\right].
\end{aligned}$$

The function $F(G,\vec{\sigma}) = (\deg(\vec{\sigma}_*))^{-1}\frac{1}{\#\mathrm{V}(G)}\sum_{x\in\mathrm{V}(G)} f(G,x)$ is measurable function from $\vec{\mathcal{G}}$ to $\mathbb{R}_+$ hence using once more the fact that $\vec{G}$ is uniformly rooted (Definition 18) we get that the last chain of equalities goes on with

$$\begin{aligned}
&= \frac{\mathbb{E}\left[F(\vec{\mathrm{U}}(\vec{G}))\right]}{\mathbb{E}[\Delta^{-1}]} \underset{\text{unif. rooted}}{=} \frac{\mathbb{E}\left[F(\vec{G})\right]}{\mathbb{E}[\Delta^{-1}]} \\
&= \frac{1}{\mathbb{E}[\Delta^{-1}]}\mathbb{E}\left[\Delta^{-1}\frac{1}{\#\mathrm{V}(G)}\sum_{x\in\mathrm{V}(G)} f(G,x)\right] \\
&= \mathbb{E}\left[f(\mathrm{U}^\bullet(\underline{\pi^\bullet(\vec{G})})\right].
\end{aligned}$$

The very same manipulations go backward: If $G^\bullet$ is a random uniformly pointed graph with $\mathbb{E}[\deg(\rho)] < \infty$, then the random rooted graph $\vec{\pi}(\overline{G^\bullet})$ obtained by biaising $G^\bullet$ with respect to $\Delta = \deg(\rho)$ and then picking an oriented edge coming from ρ at random is uniformly rooted. Indeed, we can write

$$\begin{aligned}
\mathbb{E}\left[\deg(\rho)\cdot\frac{1}{\deg(\rho)}\sum_{\vec{\sigma}:\vec{\sigma}_*=\rho} f(G,\vec{\sigma})\right] &\underset{\text{unif. pointed}}{=} \mathbb{E}\left[\frac{1}{\#\mathrm{V}(G)}\sum_{x\in\mathrm{V}(G)}\sum_{\vec{\sigma}:\vec{\sigma}_*=x} f(G,\vec{\sigma})\right] \\
&= \mathbb{E}\left[\frac{1}{\#\mathrm{V}(G)}\sum_{\vec{\sigma}\in\vec{\mathrm{E}}(G)} f(G,\vec{\sigma})\right],
\end{aligned}$$

whereas on the other hand we also have

$$\mathbb{E}\left[\deg(\rho)\frac{1}{\#\vec{\mathrm{E}}(G)}\sum_{\vec{\sigma}\in\vec{\mathrm{E}}(G)} f(G,\vec{\sigma})\right] \underset{\text{unif. pointed}}{=} \mathbb{E}\left[\frac{1}{\#\mathrm{V}(G)}\sum_{x\in\mathrm{V}(G)}\deg(x)\frac{1}{\#\vec{\mathrm{E}}(G)}\sum_{\vec{\sigma}\in\vec{\mathrm{E}}(G)} f(G,\vec{\sigma})\right]$$
$$= \mathbb{E}\Big[\frac{1}{\#\vec{\mathrm{E}}(G)}\sum_{\vec{\sigma}\in\vec{\mathrm{E}}(G)} f(G,\vec{\sigma})\underbrace{\frac{1}{\#\mathrm{V}(G)}\sum_{x\in\mathrm{V}(G)}\deg(x)}_{\#\vec{\mathrm{E}}(G)/\#\mathrm{V}(G)}\Big],$$

so that the previous two displays agree.

8.1.2 Invariance along the random walk

We face the same problem as in Section 7.1: the notion of uniformly rooted graph cannot trivially be extended to infinite graphs because of the presence of $\frac{1}{\#\vec{\mathrm{E}}(G)}$. We could do the same trick as in the last chapter and define "edge-unimodular" random graphs as those which satisfy the "edge-mass transport principle":

$$(\vec{E}-MTP) \qquad \mathbb{E}\left[\sum_{\vec{\sigma}\in\vec{\mathrm{E}}(G)} f(G,\vec{e},\vec{\sigma})\right] = \mathbb{E}\left[\sum_{\vec{\sigma}\in\vec{\mathrm{E}}(G)} f(G,\vec{\sigma},\vec{e})\right],$$

for any transport function f which associates a mass to any bi-rooted graph $(\mathrm{g},\vec{e},\vec{\sigma})$... We will however following a different concept yielding to a more general concept.

If $\vec{g}=(g,\vec{e})$ is a fixed rooted graph, we denote by $\mathbf{P}_{\vec{g}}$ the law of the simple random walk on g starting from the target of the root edge in g. More precisely this yields a sequence $\vec{E}_0,\vec{E}_1,...$ of oriented edges where $\vec{E}_0=\vec{e}$ and recursively for $i\geq 0$ we choose independently of the past the next oriented edge $\vec{E}_{i+1}$ uniformly among all the oriented edges emanating from the target vertex of $\vec{E}_i$. If $\vec{G}=(G,\vec{e})$ is a random rooted graph the random walk on $\vec{G}$ is the law of $(G,(\vec{E}_i)_{i\geq 0})$ under the probability

$$\int \mathrm{d}\mathbb{P}(\vec{G})\int \mathrm{d}\mathbf{P}_{\vec{G}}((\vec{E}_i)_{i\geq 0}),$$

see Section 9.1 below for the formalization of the state space.

Exercise 28. Let $\vec{G}_n$ be a sequence of random rooted graphs converging in distribution for the local distance towards $\vec{G}_\infty$. If conditionally on $\vec{G}_n$ we denote by $(\vec{E}_i^{(n)})_{i\geq 0}$ the oriented edges traversed by a simple random walk on $\vec{G}_n$ show that for any $k\geq 0$ we have

$$(G_n,\vec{E}_k^{(n)}) \xrightarrow[n\to\infty]{(d)} (G_\infty,\vec{E}_k^{(\infty)}),$$

where $(\vec{E}_i^{(\infty)})_{i\geq 0}$ is a simple random walk on $\vec{G}_\infty$ (Hint: use Skorokhod's representation theorem to assume that $\vec{G}_n$ converge almost surely towards $\vec{G}_\infty$).

Definition 19. *Let $\vec{G} = (G, \vec{e})$ be a random rooted graph and denote by $(\vec{E}_i)_{i\geq 0}$ the sequence of edges visited by a simple random walk on it. We say that $\vec{G}$ is **stationary** (or* stationary *along simple random walk) if for every $k \geq 0$ the law of $(G, \vec{E}_k)$ is the same as that of $\vec{G}$. It is furthermore **reversible** if on top of it we have $(G, \vec{E}_0) = (G, \overleftarrow{E}_0)$ in law, where $\overleftarrow{e}$ is the edge $\vec{e}$ with reversed orientation.*

In the context of random walk in random environment (here the random environment is the underlying random graph) we often speak of a stationary environment seen from the particle. In other words, when the random walk displaces, it sees at each step the same surrounding random graph in distribution.

We begin with a few elementary remarks:

- By an easy induction we deduce that $\vec{G}$ is stationary if and only if $(G, \vec{E}_1) = (G, \vec{E}_0)$ in distribution.
- In the definition of reversibility, we first ask for stationarity: there are examples of random rooted graphs such that $(G, \vec{E}_0) = (G, \overleftarrow{E}_0)$ in distribution but which are not stationary (for example consider two copies of $\mathbb{Z}$ joined by an oriented edge).
- For a stationary and reversible random graph we get that $(G, \vec{E}_0) = (G, \vec{E}_k) = (G, \overleftarrow{E}_k)$ for every k. Also conditionally on the graph, the probability that $\vec{E}_0 = \vec{e}_0, ..., \vec{E}_k = \vec{e}_k$ for a fixed path $\gamma = (\vec{e}_0, ..., \vec{e}_k)$ is given by

  $$\left(\prod_{i=1}^{k} \deg((\vec{e}_i)_*)\right)^{-1}.$$

  Combining this with the above remark we deduce that for a stationary and reversible random graph we have

  $$(G, \vec{E}_0, ..., \vec{E}_k) \overset{(d)}{=} (G, \overleftarrow{E}_k, ..., \overleftarrow{E}_0), \tag{8.1}$$

  or in words that the first k steps of a random walk have the same law seen from either tip on a stationary and reversible random graph.
- We will often use that for a stationary random rooted graph $\vec{G} = (G, \vec{E}_0)$ we have

  $$\vec{\pi} \circ \pi^{\bullet}(\vec{G}) = \vec{G}, \quad \text{in distribution.} \tag{8.2}$$

  In words, this means that we can start the simple random walk from the origin of the root edge instead of the target of the root edge and get the same distribution. To see this, denote $(\vec{E}_0, \vec{E}_1)$ and $(\vec{E}_0, \vec{E}'_1)$ two independent one-step simple random walks on $\vec{G}$. Since these two walks start with the same oriented edges we clearly have

$$\pi_{\to} \circ \pi_{\bullet}((G, \vec{E}_1)) \overset{(d)}{=} (G, \vec{E}_1').$$

But by stationarity we also have $(G, \vec{E}_1') = (G, \vec{E}_1) = \vec{G}$ in distribution. Combining the two statements we indeed get (8.2).

As promised, in the case of almost surely finite random graphs, stationarity along the simple random walk is equivalent to uniform rooting.

Proposition 32. *Let $\vec{G}$ be an almost surely finite random rooted graph. Then $\vec{G}$ is uniformly rooted if and only if $\vec{G}$ is stationary along the simple random walk.*

Proof. We first recall a well-known fact. If g is a fixed finite connected graph then the invariant probability measure for the oriented edges visited by simple random walk is nothing but the uniform measure on $\vec{\mathrm{E}}(\mathrm{g})$. In other words, if we pick a uniform oriented edge $\vec{E}_0$ of g and then perform $k \geq 0$ steps of random walk starting from the extremity of $\vec{E}_0$ then the distribution of the last oriented edge visited is again uniform over $\vec{\mathrm{E}}(\mathrm{g})$. It easily follows from this observation that if $(\vec{E}_k)_{k\geq 0}$ is a simple random walk on a random uniformly rooted graph $\vec{G}$ then $(G, \vec{E}_k) = (G, \vec{E}_0)$ in distribution as desired.

To show the converse we again recall that if $\vec{\mathrm{g}}$ is a finite connected rooted graph and $(\vec{E}_i)_{i\geq 0}$ has law $\mathbf{P}_{\vec{\mathrm{g}}}$ then by the classical ergodic theorem for irreducible Markov chains on a finite state space we have the almost sure weak convergence

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{\vec{E}_k} \xrightarrow[n\to\infty]{a.s.} \frac{1}{\#\vec{\mathrm{E}}(\mathrm{g})} \sum_{\vec{\sigma}\in\vec{\mathrm{E}}(\mathrm{g})} \delta_{\vec{\sigma}}.$$

Then, if $\phi : \vec{\mathcal{G}} \to \mathbb{R}_+$ is a bounded Borel function then we have

$$\frac{1}{r}\sum_{k=0}^{r-1} \phi(\mathrm{g}, \vec{E}_k) \xrightarrow[r\to\infty]{\mathbf{P}_{\vec{\mathrm{g}}}-a.s.} \frac{1}{\#\vec{\mathrm{E}}(\mathrm{g})} \sum_{\vec{\sigma}\in\vec{\mathrm{E}}(\mathrm{g})} \phi(\mathrm{g}, \vec{\sigma}) = \mathbb{E}[\phi(\vec{\mathrm{U}}(\vec{\mathrm{g}}))].$$

If now $\vec{G}$ is a finite stationary random graph then for any $k \geq 0$ we have $\int \mathrm{d}\mathbb{P}(\vec{G}) \int \mathrm{d}\mathbf{P}_{\vec{G}}(\vec{E}_k)\phi(G, \vec{E}_k) = \mathbb{E}[f(\vec{G})]$ so that

$$\mathbb{E}[\phi(\vec{G})] \underset{\text{stationarity}}{=} \mathbb{E}\left[\int \mathrm{d}\mathbf{P}_{\vec{G}}((\vec{E}_i)_{i\geq 0}) \frac{1}{n}\sum_{k=0}^{n-1} \phi(G, \vec{E}_k)\right] \xrightarrow[n\to\infty]{\text{dom. conv.}} \mathbb{E}[\mathbb{E}[f(\vec{\mathrm{U}}(\vec{G})) \mid \vec{G}]] = \mathbb{E}[f(\vec{\mathrm{U}}(\vec{G}))],$$

which proves that $\vec{G}$ is indeed uniformly rooted. □

Notice that a uniformly rooted random graph is automatically reversible and by the above result any stationary random graph which is almost surely finite is automatically reversible. However, this is not always the case in the infinite setting: Recall the grand-father graph gf of Section 7.2.1. Since gf is transitive, it is easy to see that $\vec{\pi}(\mathsf{gf})$ is stationary. However, $\vec{\pi}(\mathsf{gf})$ is not reversible since the root edge is more likely to link a vertex to one of its two grand-fathers (probability 1/3) rather than to its unique grand-son (probability 1/6).

We have the analog of Theorem 24:

Theorem 33 (*Stability under local limits*)

Let $\vec{G}_n$ be a sequence of stationary (resp. stationary and reversible) random graphs which converges locally in distribution towards $\vec{G}_\infty$. Then $\vec{G}_\infty$ is also stationary (resp. stationary and reversible).

Proof. **[N : Alternative : A Skorokhod proof.]** Let $A_r \subset \vec{\mathcal{G}}$ be a Borel subset of rooted graphs such that $\vec{\mathfrak{g}} \in A_r$ only depends on the restriction of radius r around the root edge of $\vec{\mathfrak{g}}$. Then by stationarity of the graphs $\vec{G}_n$ we get that for any $k \geq 0$ we have

$$\mathbb{E}\left[\mathbf{P}_{\vec{G}_n}((G_n, \vec{E}_k^{(n)}) \in A_r)\right] = \mathbb{E}\left[\mathbf{P}_{\vec{G}_n}((G_n, \vec{E}_0^{(n)}) \in A_r)\right]$$

where $\vec{E}_i^{(n)}$ are the oriented edges traversed by the simple random walk on $\vec{G}_n$. By our assumption, the function $\mathbf{1}_{A_r}$ is a bounded continuous function for the local topology hence using Exercise 28 we can pass to the limit in the last display and deduce that

$$\mathbb{E}\left[\mathbf{P}_{\vec{G}_\infty}((G_\infty, \vec{E}_k^{(\infty)}) \in A_r)\right] = \mathbb{E}\left[\mathbf{P}_{\vec{G}_\infty}((G_\infty, \vec{E}_0^{(\infty)}) \in A_r)\right]$$

with obvious notation. Using Proposition 4 we conclude that $(G_\infty, \vec{E}_0^{(\infty)}) = (G_\infty, \vec{E}_k^{(\infty)})$ in distribution. The case of stationary and reversible random graphs is treated similarly. □

The analog of the Open question 1 also holds in this context: is any stationary and reversible random graphs a local limit of uniformly rooted random graphs (this is not true for stationary random graphs only as we have seen above that there are stationary infinite random graphs which are not reversible)?

8.1.3 Involution invariance

We now make the connection between stationary and reversible random rooted graphs and unimodular random pointed graphs directly in the (possibly) infinite case. We start with the easier direction: from unimodular random graphs to stationary and reversible random graphs.

Proposition 34 (From unimodular to stationary and reversible)**.** *Let $G^\bullet = (G,\rho)$ be a unimodular random pointed graph. Consider first the random pointed graph $\overline{G}^\bullet = (\overline{G},\overline{\rho})$ obtained from (G,ρ) after biasing by the degree of its origin (provided that $\mathbb{E}[\deg(\rho)] < \infty$). Then the random graph $\vec{\pi}(\overline{G}^\bullet)$ obtained by distinguishing an oriented edge emanating from $\overline{\rho}$ uniformly at random is stationary and reversible.*

Proof. We will first show that $\vec{\overline{G}} = (\overline{G},\vec{E}) = \vec{\pi}(\overline{G}^\bullet)$ has the same law as $(\overline{G},\overleftarrow{E})$. For this, let $h(\mathrm{g},\vec{e})$ be a function $\vec{\mathcal{G}} \to \mathbb{R}_+$ and denote by

$$f(\mathrm{g},x,y) = \mathbf{1}_{x\sim y} \sum_{x \xrightarrow[\vec{e}]{} y} h(\mathrm{g},\vec{e}), \tag{8.3}$$

the associated transport function obtained by summing over all choice of an oriented edge linking x to y in g (in particular x,y must be neighbors). Recalling the notation $\Delta = \deg(\rho)$. Then applying the mass-transport principle we get that

$$\begin{array}{ccccc} \mathbb{E}[\Delta]\cdot\mathbb{E}\left[h(\overline{G},\vec{E})\right] = \mathbb{E}\left[\Delta\cdot\dfrac{1}{\Delta}\displaystyle\sum_{\vec{e}\text{ s.t. }\vec{e}_*=\rho} h(G,\vec{e})\right] & = & \mathbb{E}\left[\displaystyle\sum_{x\in \mathrm{V}(G)} f(G,\rho,x)\right] \\ & & \| \\ \mathbb{E}[\Delta]\cdot\mathbb{E}\left[h(\overline{G},\overleftarrow{E})\right] = \mathbb{E}\left[\Delta\cdot\dfrac{1}{\Delta}\displaystyle\sum_{\vec{e}\text{ s.t. }\vec{e}_*=\rho} h(G,\overleftarrow{e})\right] & = & \mathbb{E}\left[\displaystyle\sum_{x\in \mathrm{V}(G)} f(G,x,\rho)\right]. \end{array}$$

But in our context this statement also implies stationary. Indeed if $(\vec{E}_0 = \overleftarrow{E},\vec{E}_1)$ are the first two steps of a random walk on $(\overline{G},\overleftarrow{E})$ (started from the extremity of the root edge) then it is clear that $(\overline{G},\vec{E}_1)$ has the same distribution as $(\overline{G},\vec{E})$ since $\vec{E}_1$ is nothing but a uniform edge emanating from $\overline{\rho}$. Since $(\overline{G},\overleftarrow{E})$ has the same law as $(\overline{G},\vec{E})$ we indeed deduce that $(\overline{G},\vec{E}_0) = (\overline{G},\vec{E}_1)$ in distribution and hence the desired stationarity. □

Theorem 35 (*From stationary reversible to unimodular*)

Let $\vec{G} = (G,\vec{E})$ be a stationary and reversible random graph. We consider $\underline{\vec{G}} = (\underline{G},\underline{\vec{E}})$ the graph obtained from $\vec{G}$ after biasing by Δ^{-1}, the inverse of the degree of the origin of $\vec{E}$. Then $\pi_\bullet(\underline{\vec{G}})$ is a unimodular random graph.

Proof. We will verify the mass-transport principle. Let $f(\mathrm{g},x,y)$ be a transport function which is zero as soon as x and y are not neighbors. We then form the function $h(\mathrm{g},\vec{e})$ such that (8.3) holds. Then the same calculation as in the previous proposition shows that the

MTP is verified in $\pi^\bullet(\vec{G}) = (\underline{G}, \rho)$ for such functions f: if we write $\Delta = \deg(\vec{E}_*)$ then

$$\mathbb{E}\left[h(G,\vec{E})\right] \underset{(8.2)}{=} \mathbb{E}\left[\Delta^{-1}\sum_{\vec{\sigma}_*=\vec{E}_*} h(G,\vec{\sigma})\right] = \mathbb{E}\left[\Delta^{-1}\sum_{x\in V(G)} f(G,\vec{E}_*,x)\right] = \mathbb{E}\left[\Delta^{-1}\right]\mathbb{E}\left[\sum_{x\in V(\underline{G})} f(\underline{G},\rho,x)\right]$$

(rev.) $\|$

$$\mathbb{E}\left[h(G,\overleftarrow{E})\right] \underset{(8.2)}{=} \mathbb{E}\left[\Delta^{-1}\sum_{\vec{\sigma}_*=\vec{E}_*} h(G,\overleftarrow{\sigma})\right] = \mathbb{E}\left[\Delta^{-1}\sum_{x\in V(G)} f(G,x,\vec{E}_*)\right] = \mathbb{E}\left[\Delta^{-1}\right]\mathbb{E}\left[\sum_{x\in V(\underline{G})} f(\underline{G},x,\rho)\right].$$

Actually, this suffices to imply the full MTP:

Lemma 36. *Let (G,ρ) be a random pointed graph satisfying the MTP for transport functions $f(\mathrm{g},x,y)$ which are null as soon as x and y are not neighbors in g. Then (G,ρ) is unimodular.*

Proof. Suppose that $f(\mathrm{g},x,y)$ is a transport function that is zero unless $\mathrm{d_{gr}}(x,y) = k$ for some $k \geq 1$ (any transport function is a sum of functions of the last sort). We denote by $\mathscr{P}(\mathrm{g},x,y)$ the number of geodesic paths going from x to y in g and $\mathscr{P}_j(\mathrm{g},x,y;u,v)$ the number of such paths such that the jth step links u to v where $1 \leq j \leq k$. We then form the transport functions for $1 \leq j \leq k$

$$f_j(\mathrm{g},u,v) = \sum_{x,y\in V(\mathrm{g})} f(\mathrm{g},x,y)\frac{\mathscr{P}_j(\mathrm{g},x,y;u,v)}{\mathscr{P}(\mathrm{g},x,y)}.$$

Clearly these are transport functions which are null except if u and v are neighbors in g. Since the MTP is valid for such functions we have

$$\mathbb{E}\left[\sum_{v\in V(G)} f_j(G,\rho,v)\right] = \mathbb{E}\left[\sum_{u\in V(G)} f_j(G,u,\rho)\right].$$

But on the other hand we have the following deterministic equalities:

$$\sum_{y\in V(G)} f(G,\rho,y) = \sum_{y\in V(G)} f(G,\rho,y)\sum_{v:\mathrm{d_{gr}}(\rho,v)=1}\frac{\mathscr{P}_1(G,\rho,y;\rho,v)}{\mathscr{P}(G,\rho,y)} = \sum_{v\in V(G)} f_1(G,\rho,v),$$

$$\sum_{x\in V(G)} f(G,x,\rho) = \sum_{x\in V(G)} f(G,x,\rho)\sum_{u:\mathrm{d_{gr}}(u,\rho)=1}\frac{\mathscr{P}_k(G,x,\rho;u,\rho)}{\mathscr{P}(G,x,\rho)} = \sum_{u\in V(G)} f_k(G,u,\rho),$$

and for $1 \leq j < k$ we have

$$\begin{aligned}\sum_{u\in V(G)} f_j(G,u,\rho) &= \sum_{x,y\in V(G)} f(G,x,y)\sum_{u\in V(G)}\frac{\mathscr{P}_j(G,x,y;u,\rho)}{\mathscr{P}(G,x,y)}\\ &= \sum_{x,y\in V(G)} f(G,x,y)\sum_{v\in V(G)}\frac{\mathscr{P}_{j+1}(G,x,y;\rho,v)}{\mathscr{P}(G,x,y)} = \sum_{v\in V(G)} f_{j+1}(G,\rho,v),\end{aligned}$$

since the second sum over paths is just the proportion of those paths going from x to y through the vertex ρ in $j+1$th position. Combining the last displays yields the MTP for the function f. □

8.2 Structure theorems for stationary random graphs

We now develop the analogs of the propositions we derived for unimodular random graphs in the "weaker" context of stationary random graphs only. When the graph is stationary and reversible one can consider its unimodular version and apply the results of Section 7.3.

8.2.1 Tightness

Stationary already gives a nice criterion for tightness:

Proposition 37. *A family $(G_i, \vec{E}^{(i)})_{i\in I}$ of stationary random graphs is tight in $\vec{\mathcal{G}}$ if and only if the family $(\deg(\vec{E}_*^{(i)}))_{i\in I}$ is tight.*

Proof. The characterization of pre-compact sets in $\vec{\mathcal{G}}$ is analogous to the one in $\mathcal{G}^\bullet$, i.e. they are those sets $A \subset \vec{\mathcal{G}}$ such that there exist $n_0, n_1, n_2, ..., n_r, ...$ satisfying

$$\sup\{\deg(x) : x \in [\vec{g}]_r,\ \vec{g} \in A\} \leq n_r.$$

In the following we denote by $M_r(\vec{g})$ the maximum vertex degree inside $[\vec{g}]_r$. In particular $(\vec{G}_i)_{i\in I}$ is tight if for every $\varepsilon > 0$ there exist $n_0, n_1, ..., n_r$ such that for all $i \in I$ we have $\mathbb{P}(\forall r \geq 0 : M_r(\vec{G}_i) \leq n_r) \geq 1-\varepsilon$. This is equivalent to the seemingly weaker condition:

$$\forall \varepsilon' > 0, \forall r \geq 0, \exists n_r' \geq 0, \quad \forall i \in I, \quad \mathbb{P}(M_r(\vec{G}_i) \leq n_r') \geq 1-\varepsilon'.$$

But taking $\varepsilon' = \varepsilon \cdot 2^{-r}$ and performing a union bound on r, we see that the second condition is indeed equivalent to the first one. We have thus shown that $(\vec{G}_i)_{i\in I}$ is tight if and only if for every $r \geq 0$ the family $(M_r(\vec{G}_i))_{i\in I}$ is tight. The argument so far is valid for any family of random rooted graphs, and it shows the necessity of the condition in the proposition. We will show now that if $(\vec{G}_i)_{i\in I}$ are stationary random graphs and $(M_0(\vec{G}_i))_{i\in I}$ is tight then $(M_r(\vec{G}_i))_{i\in I}$ is also tight for every $r \geq 0$. We treat the case $r = 1$ and leave the general case as an exercise to the reader. By assumption, for any $\varepsilon > 0$ there exists n_0 such that

$$\forall i \in I, \quad \mathbb{P}(M_0(\vec{G}_i) \geq n_0) \leq \varepsilon.$$

Now inside each $\vec{G}_i$ we perform a two-steps simple random walk started from the origin of the root edge and denote by $\vec{E}^{(i)'}$ the second directed edge visited. By (8.2) the graph

$(G_i, \vec{E}^{(i)\prime})$ has the same distribution as $\vec{G}_i$ and in particular the origin vertex $\vec{E}_*^{(i)\prime}$ of $\vec{E}^{(i)\prime}$ is a random uniform neighbor of $\vec{E}_*^{(i)}$ and its degree has the same law as that of $\deg(\vec{E}_*^{(i)})$. We choose $\ell \geq n_0$ large enough so that if we sample independently ℓ variables uniformly in $\{1, 2, ..., n_0\}$ then all the outcome have been seen with probability at least $1 - \varepsilon$. We deduce that in $\vec{G}_i$, if we pick ℓ independent neighbors $\rho_1, ..., \rho_\ell$ of $\vec{E}_*^{(i)}$ then on the event where $\deg(\vec{E}_*^{(i)}) \leq n_0$ the set $\{\rho_1, ..., \rho_\ell\}$ covers all the neighbors of $\vec{E}_*^{(i)}$ with probability at least $1 - \varepsilon$. Combining these observations we get that

$$\begin{aligned}\mathbb{P}(M_1(\vec{G}_i) \geq n_1) &\leq \mathbb{P}(M_0(\vec{G}_i) \geq n_0) + \varepsilon + \ell \mathbb{P}(M_0(\vec{G}_i) \geq n_1) \\ &\leq 2\varepsilon + \ell \mathbb{P}(M_0(\vec{G}_i) \geq n_1).\end{aligned}$$

Choosing n_1 large enough we can make the above display less than 3ε and that completes the proof for the case $r = 1$. □

Exercise 29. Using Proposition 37 and the above translation between unimodular random graphs and stationary and reversible random graphs, show that if $((G_i, \rho_i))_{i \in I}$ is a family of unimodular random pointed graphs such that $(\deg(\rho_i))_{i \in I}$ is uniformly integrable then $((G_i, \rho_i))_{i \in I}$ is tight in $\mathcal{G}^\bullet$. Show that the latter condition is not necessary and that the condition $\sup_i \mathbb{E}[\deg(\rho_i)] < \infty$ is not sufficient.

8.2.2 $0 - 1$ laws

We now have the analogs of Proposition 27 and 28:

Proposition 38 (Everything shows at the origin)**.** *Let $\vec{G}$ be a stationary random graph and $A \subset \vec{\mathcal{G}}$ be a Borel set such that $\mathbb{P}(\vec{G} \in A) = 0$. Then the probability that there exists an edge $\vec{e} \in \vec{\mathrm{E}}(G)$ such that $(\vec{G}, \vec{e}) \in A$ is equal to zero.*

Proof. For every $n \geq 0$ if $\vec{E}_n$ is the n-th edge visited by a simple random walk on $\vec{G}$ we have

$$\mathbb{P}((G, \vec{E}_n) \in A) \underset{\text{stat.}}{=} \mathbb{P}((G, \vec{E}_0) \in A) = 0.$$

Summing-up over all $n \geq 0$, we deduce by connectedness of the graph that

$$0 = \mathbb{E}\left[\sum_{n \geq 0} \mathbf{1}_{(G, \vec{E}_n) \in A}\right] = \int \mathrm{d}\mathbb{P}(\vec{G}) \sum_{\vec{e} \in \vec{\mathrm{E}}(G)} \mathbf{1}_{(G, \vec{e}) \in A} \underbrace{\int \mathrm{d}\mathbf{P}_{\vec{G}}((\vec{E}_i)_{i \geq 0}) \sum_{i \geq 0} \mathbf{1}_{\vec{E}_i = \vec{e}}}_{>0}.$$

This proves the desired statement. □

Proposition 39 (If it happens, it happens a lot)**.** *Let $\vec{G}$ be a stationary random graph which is almost surely infinite. Then for any $A \subset \vec{\mathcal{G}}$ Borel we have*

$$\#\{\vec{e} \in \vec{\mathrm{E}}(G) : (G, \vec{e}) \in A\} \in \{0, \infty\} \quad a.s.$$

Proof. Let $A \subset \vec{\mathcal{G}}$ and for a graph $\mathbf{g}$ denote by $\vec{\mathcal{E}}_A(\mathbf{g})$ the set of all oriented edges $\vec{e} \in \vec{\mathrm{E}}(\mathbf{g})$ such that $(g, \vec{e}) \in A$. We argue by contradiction and suppose that with positive probability we have

$$0 < \#\vec{\mathcal{E}}_A(G) < \infty.$$

After conditioning on the above event (which preserves stationarity by an extension Exercise 22) we can suppose that $0 < \#\vec{\mathcal{E}}_A(G) < \infty$ almost surely. Hence we will suppose that the last display happens almost surely. By the last proposition we must have $\mathbb{P}(\vec{G} \in A) > 0$ and so by stationarity

$$0 < \mathbb{P}(\vec{G} \in A) = \mathbb{E}\left[\frac{1}{n}\sum_{i=0}^{n-1} \mathbf{1}_{(G,\vec{E}_i)\in A}\right] = \mathbb{E}\left[\frac{1}{n}\sum_{i=0}^{n-1} \mathbf{1}_{\vec{E}_i \in \vec{\mathcal{E}}_A(G)}\right].$$

However for any infinite graph $\mathbf{g}$ and any finite subset $F \subset \vec{\mathrm{E}}(\mathbf{g})$ the proportion of time spend by a simple random walk in F almost surely goes to 0 (indeed, otherwise the walk would be positive recurrent and it is not possible that simple random walk on an infinite graph is positive recurrent because the only invariant measure is proportional to the degree of vertices which is an infinite measure). Hence by dominated convergence we get that

$$0 < \mathbb{P}(\vec{G} \in A) = \mathbb{E}\left[\frac{1}{n}\sum_{i=0}^{n-1} \mathbf{1}_{(G,\vec{E}_i)\in A}\right] = \mathbb{E}\left[\frac{1}{n}\sum_{i=0}^{n-1} \mathbf{1}_{\vec{E}_i \in \vec{\mathcal{E}}_A(G)}\right] \xrightarrow[n\to\infty]{} 0,$$

which yields a contradiction. □

Using the above propositions, one can adapt the proof of Theorem 30 and deduce:

Corollary 40. *The number of ends of a stationary random graph is almost surely in* $\{0, 1, 2\} \cup \{\infty\}$.

The analog of Theorem 29 in the weaker context of stationary random graphs will be proved below in Theorem 46.

Biliographical references. The connection between stationary reversible random graphs and unimodular random graphs is explicit in [9] in particular thanks to the involution invariant property (Lemma 36 above). The exposition here is adapted from [17]. The tightness criterion for stationary random graphs is taken from [19], see also [12]. Theorem 46 seems to be new.

Chapter 9

Connection with ergodic theory

In this chapter, we reformulate the notion of stationarity for random graphs in terms of shift-invariance for distributions of random graphs decorated with a random path. This viewpoint allows us to leverage the machinery of ergodic theory within our probabilistic framework. In particular, we show that sequences of finite random graphs that converge in the directed-edge variant of the Benjamini–Schramm sense to an ergodic limit also converge in the stronger quenched sense.

9.1 Framework

We now translate Definition 19 in the framework of ergodic theory in order to to apply powerful tools such as the subadditive ergodic theorem, Poincaré recurrence theorem... Recall that the basic ingredients of ergodic theory are 1) a probability space $(E, \mathcal{A}, \mu)$ and 2) an action $\theta : E \to E$ measurable which preserves the measure μ in the sense that $\mu(\theta^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{A}$.

We will work on the set of locally finite connected graphs endowed with a possibly infinite path made of oriented edges $(\mathrm{g}, (\vec{e}_i)_{a<i<b})$ where $a \in \{-\infty\} \cup \{..., -2, -1\}$ and $b \in \{1, 2, ...\} \cup \{\infty\}$. The oriented edge $\vec{e}_0$ could be seen as the root edge of the graph. As usual, we will identify $(\mathrm{g}, (\vec{e}_i)_{i\in(a,b)})$ with $(\mathrm{g}', (\vec{e}_i{}')_{i\in(a,b)})$ if there exists a graph homomorphism $\mathrm{g} \to \mathrm{g}'$ mapping $\vec{e}_i$ to $\vec{e}_i{}'$ for every $i \in (a, b)$. The quotient space is then denoted by $\mathcal{G}^{\leftrightarrow}$ and endowed with the local distance corresponding to the notion of restriction

$$\Big[(\mathrm{g}, (\vec{e}_i)_{a<i<b})\Big]_r = \Big([(\mathrm{g}, \vec{e}_0)]_r, (\vec{e}_i)_{i\in(a,b)\cap(-r,r)}\Big).$$

The resulting space $(\mathcal{G}^{\leftrightarrow}, \mathrm{d_{loc}})$ is again Polish by Section 1.2.1. We shall use the notation $x_i = (\vec{e}_i)_*$ for the vertices visited by the path. We focus in what follows on the subspace $\mathcal{G}^{\rightarrow}$ made of those graphs with a semi-infinite path indexed by $i \in \mathbb{Z}_{\geq 0}$ i.e. when $a = -1$

and $b = \infty$. This space comes with a natural shift

$$\theta(\mathrm{g}, (\vec{e}_i)_{i\geq 0}) \mapsto (\mathrm{g}, (\vec{e}_{i+1})_{i\geq 0}).$$

Recall that $\mathbf{P}_{(\mathrm{g},\vec{e})}$ is the law of the random walk, seen as a sequence of consecutive oriented edges, started from $\vec{e}$ on the graph g. Given a possibly random rooted graph $\vec{G} = (G, \vec{E})$ we define the law $\pi^{\rightarrow}(\vec{G})$ by launching a simple random walk on $\vec{G}$, more precisely

$$\mathbb{E}[F(\pi^{\rightarrow}(\vec{G}))] = \int \mathrm{d}\mathbb{P}(G, \vec{E}) \int \mathrm{d}\mathbf{P}_{(G,\vec{E})}((\vec{E}_i)_{i\geq 0})\, F(G, (\vec{E}_i)_{i\geq 0}).$$

We shall also use the shorthand notation $X_i = (\vec{E}_i)_*$ for the vertices visited by the simple random walk. It is an easy matter to translate the notion of stationary into θ-invariance:

Proposition 41. *The random graph $\vec{G}$ is stationary if and only if (the law of) $\pi^{\rightarrow}(\vec{G})$ is θ-invariant.*

Proof. If the law of $\pi^{\rightarrow}(\vec{G})$ is θ-invariant we in particular get that $(G, \vec{E}_0) = (G, \vec{E}_1)$ in law where $(\vec{E}_i)_{i\geq 0}$ is a simple random walk on $(G, \vec{E})$ and so the latter is stationary. Conversely, by the Markovian property of the simple random walk one can construct $\theta(G, (\vec{E})_{i\geq 0}) = (G, (\vec{E}_{i+1})_{i\geq 0})$ by first making one step of random walk to discover $\vec{E}_1$ then re-rooting at $\vec{E}_1$ to finally launch the rest of the walk independently of this step. Since $(G, \vec{E}_0) = (G, \vec{E}_1)$ this algorithm produces a random graph with a semi-infinite path which has the same law as $\pi^{\rightarrow}(\vec{G})$. □

Exercise 30 (Invertible shift). In ergodic theory, it is sometimes convenient that the shift operator θ be invertible (on a set of probability one). In our context, this requires to define a bi-infinite path $(\vec{E}_i : i \in \mathbb{Z})$ on the random graph $\vec{G}$ to that the origin can be shifted both ways. To do so, prove that if $(G, (\vec{E}_i)_{i\geq 0})$ is a stationary random graph given with a simple random walk on it, show that as $k \to \infty$ the variable $(G, (\vec{E}_{i+k})_{-k\leq i<\infty})$ converges in distribution in $\mathcal{G}^{\leftrightarrow}$ and that the resulting distribution is invariant by shifting along the path by ± 1. Show that the law obtained is invariant by time reversal of the path if and only if the original graph $(G, \vec{E}_0)$ is reversible.

Let us give a recreative application using the famous Poincaré recurrence theorem:

Proposition 42 (Recurrence of landscapes, by Poincaré)**.** *Let $(G, \vec{E})$ be a stationary random graph and denote by $(\vec{E}_i)_{i\geq 0}$ a simple random walk on it. Then we almost surely have*

$$\liminf_{n\to\infty} \mathrm{d}_{\mathrm{loc}}\big((G, \vec{E}_0), (G, \vec{E}_n)\big) = 0.$$

In words, the result says that almost surely when performing a simple random walk on a stationary random graph we will discover places where the landscape around the current oriented edge is arbitrarily close to the starting landscape. Notice that the previous proposition is trivial when the simple random walk on G is almost surely recurrent.

Proof. This is an application of Poincaré recurrence theorem: If $(X, \mathcal{A}, \mu)$ is a measurable space with a finite measure μ and $\theta : X \to X$ preserves μ then for any measurable $A \subset \mathcal{A}$ and for μ-almost all $x \in A$ there exists an infinite number of $n \geq 0$ such that $\theta^n(x) \in A$. We encourage the reader to try to prove this result as an interesting exercise. Hint: Consider $A' = \{x \in A : \theta^n(x) \notin A, \forall n \geq 1\}$ and show that $\theta^{-k}(A')$ are pairwise disjoint subspaces for $k \geq 0$. □

9.2 Ergodicity

One of the key notion of ergodic theory is that of ergodicity. Recall that $\theta : (E, \mu) \circlearrowleft$ is ergodic if for any measurable set A such that $\mu(A \Delta \theta^{-1}(A)) = 0$ then $\mu(A) \in \{0, 1\}$. In words, ergodicity means that the shift operation does not stabilize any non trivial event.

9.2.1 Equivalent definitions

Definition 20. *We say that a stationary random graph $(G, \vec{E})$ is ergodic (more precisely, its law is ergodic) if the law of $\pi^{\to}(\vec{G})$ on $\mathcal{G}^{\to}$ is ergodic for the shift θ.*

By extension, we shall say that a unimodular random graph satisfying $\mathbb{E}[\deg(\rho)] < \infty$ is ergodic is the stationary (and reversible) version obtained from Proposition 34 is ergodic. Many natural models of random stationary graphs are actually ergodic[1]. If the stationary random graph $\vec{G}$ is not ergodic that means that there is a large scale property of the graph (independent from the location of the root edge) which is genuinely random:

Proposition 43 (Geometric obstruction to ergodicity)**.** *Suppose that $\vec{G}$ is stationary. Then the random graph $\pi^{\to}(\vec{G})$ is not ergodic if and only if there exists a measurable event $A \subset \vec{\mathcal{G}}$ which that does not depend upon the location of the root edge so that*

$$\mathbb{P}(\vec{G} \in A) \in (0, 1).$$

Proof. If there exists such an event, then clearly it is shift-invariant (since it does not depend upon the location of the root edge) and so $\vec{G}$ cannot be ergodic. Reciprocally, if

[1] when $\mathbb{E}[\deg(\rho)] = \infty$ the following proposition can be taken as an alternative definition of ergodicity for unimodular random graphs

$\pi^{\to}(\vec{G})$ is not ergodic, there must exists $A^{\to} \subset \mathcal{G}^{\to}$ which is shift invariant. Since for any (connected locally finite) rooted graph $\vec{\mathrm{g}} = (\mathrm{g}, \vec{e})$ we have $\mathbf{P}_{\vec{\mathrm{g}}}(\vec{E}_4 = \vec{E}_0 = \vec{e})$, we deduce that the event $A^{\to}$ is invariant by pre-pending any edge pointed towards $\vec{E}_0$. By a standard measure-theory argument (see [46, Lemma 3.16]) one can approximate $A^{\to}$ by "finitary event": for any $\varepsilon > 0$ there exists $r \geq 1$ and an event $A_r^{\to} \subset \mathcal{G}^{\to}$ which only depends on the steps in $[[-r+1, r-1]]$ of the two-sided walk and on the ball of radius r around the origin of the graph such that we have

$$\mathbb{P}(\pi^{\to}(\vec{G}) \in A^{\to} \Delta A_r^{\to}) \leq \varepsilon.$$

Combining those two observations, we deduce that the event $\{\pi^{\to}(\vec{G}) \in A^{\to}\}$ is, up to zero measure, actually an event about the graph G itself, more precisely that there exists $A \in \vec{\mathcal{G}}$ such that

$$\{\pi^{\to}(\vec{G}) \in A^{\to}\} = \{\vec{G} \in A\},$$

and furthermore A does not depend upon the location of the root edge, i.e $\{(G, \vec{e}) \in A\} = \{(G, \vec{e}) \in A\}$ up to null events for any $\vec{e}, \vec{e}' \in \vec{\mathrm{E}}(G)$. □

Yet another equivalent characterization (exercise!) of ergodic stationary random graphs is via Birkhoff convergence theorem: If $G^{\to} = \pi^{\to}(\vec{G})$ is a stationary random graph decorated with a simple random walk, then $\vec{G}$ is ergodic if and only for any measurable $f : \mathcal{G}^{\to} \to \mathbb{R}_+$ we have

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ \theta^k(G^{\to}) \quad \xrightarrow[n\to\infty]{a.s.} \quad \mathbb{E}[f(G^{\to})]. \tag{9.1}$$

Even if a stationary random graph $\vec{G}$ is not ergodic, by the ergodic decomposition theorem, it can always be represented as a mixture of ergodic ones. Formally, that means that there is a probability distribution Θ on stationary distributions of random rooted graphs, such that

$$\mathbb{E}[f(\vec{G})] = \int_{\mathcal{M}_1(\vec{\mathcal{G}})} \mathrm{d}\Theta(\mu) \int_{\vec{\mathcal{G}}} \mathrm{d}\mu(\vec{\mathrm{g}})\ f(\vec{\mathrm{g}}),$$

for any $f : \vec{\mathcal{G}} \to \mathbb{R}_+$. This follows from a general fact that the extremal points of the convex set of invariant measures are precisely ergodic ones.

Exercise 31. Give an ergodic proof of Proposition 39.

9.2.2 Subadditive ergodic theorem

Let us now draw a few applications. Many of them will be based on Kingman's subadditive ergodic theorem which generalizes the pointwise ergodic theorem of Birkhoff (9.1).

Theorem 44 (*Kingman's subadditive ergodic theorem*)

If θ is a measure preserving transformation on a probability space $(E, \mathcal{A}, \mu)$ and $(h_n)_{n\geq 1}$ is a sequence of integrable functions satisfying for $n, m \geq 1$

$$h_{n+m}(x) \leq h_n(x) + h_m(\theta^n x),$$

then we have the following convergence almost sure and in L^1

$$\frac{h_n(x)}{n} \xrightarrow[n\to\infty]{a.s.\text{ and } L^1} h(x),$$

where $h(x)$ is a θ-invariant function (in particular constant if θ is ergodic).

When $h_n(x) = x + f(x) + f(\theta x) + ... + f(\theta^{n-1}x)$ for some integrable function $f : E \to \mathbb{R}$ we recover Birkhoff's ergodic theorem (9.1). There are many generalizations of the above subadditive ergodic theorem but this version is already sufficient to provide useful applications in the context of random graphs. In the following of this section $(G, \vec{E})$ is a stationary and ergodic random graph and we denote by μ the law of $\pi^{\rightarrow}(\vec{G}) = (G, (\vec{E}_i)_{i\geq 0})$. Let us recall our notation $x_i = (\vec{e}_i)_*$ and $X_i = (\vec{E}_i)_*$ in the random setting.

Speed of the random walk. We consider the function

$$h_n(\mathrm{g}, (\vec{e}_i)_{i\geq 0}) = \mathrm{d_{gr}}((\vec{e}_0)_*, (\vec{e}_n)_*) = \mathrm{d_{gr}}(x_0, x_n)$$

where $\mathrm{d_{gr}}$ is the graph distance. The triangular inequality shows that

$$\begin{aligned} h_{n+m}(\mathrm{g}, (\vec{e}_i)_{i\geq 0}) = \mathrm{d_{gr}}((\vec{e}_0)_*, (\vec{e}_{n+m})_*) &\leq \mathrm{d_{gr}}((\vec{e}_0)_*, (\vec{e}_n)_*) + \mathrm{d_{gr}}((\vec{e}_n)_*, (\vec{e}_{n+m})_*) \\ &= h_n(\mathrm{g}, (\vec{e}_i)_{i\geq 0}) + h_m(\mathrm{g}, (\vec{e}_i)_{i\geq n}) = h_n(\mathrm{g}, (\vec{e}_i)_{i\geq 0}) + h_m(\theta^n(\mathrm{g}, (\vec{e}_i)_{i\geq 0})). \end{aligned}$$

Clearly the functions h_n are bounded by n and so are integrable. We can thus apply Theorem 44, and get the existence of a constant $s \geq 0$ (called the **speed** of the random walk) such that

$$\frac{\mathrm{d_{gr}}(X_0, X_n)}{n} \xrightarrow[n\to\infty]{a.s.} s. \tag{9.2}$$

Linear growth of the range. We consider the function

$$r_n(\mathrm{g}, (\vec{e}_i)_{i\geq 0}) = \#\{x_0, ..., x_n\}$$

be the number of different vertices visited by the walk in the first n steps. It is plain that $\#\{(\vec{e}_0)_*, ..., (\vec{e}_{n+m})_*\} \leq \#\{(\vec{e}_0)_*, ..., (\vec{e}_n)_*\} + \#\{(\vec{e}_n)_*, ..., (\vec{e}_{n+m})_*\}$ and the second term is

$r_m(\theta^n(\mathrm{g}, (\vec{e}_i)_{i\geq 0}))$. Since the r_n are also integrable, we are in position to apply Theorem 44 again and get

$$\frac{\#\{X_0, ..., X_n\}}{n} \xrightarrow[n\to\infty]{a.s.} r, \tag{9.3}$$

for some $r \geq 0$. Actually we can in this case express exactly the constant r. Indeed, even if the graph is not ergodic, as long as it is stationary we can evaluate the expectation of the range of the random walk as follows

$$\begin{aligned}
\mathbb{E}\left[\#\{X_0, X_n\}\right] &= \mathbb{E}\left[\sum_{k=0}^{n} \mathbf{1}\{\text{it is the last visit to } X_k \text{ before time } n\}\right] \\
&= \sum_{k=0}^{n} \mathbb{P}\left(X_i \neq X_k, \forall k+1 \leq i \leq n\right) \\
&\underset{\text{stat.}}{=} \sum_{k=0}^{n} \mathbb{P}\left(X_0 \neq X_i, \forall 1 \leq i \leq k\right).
\end{aligned}$$

But by dominated convergence we have $\mathbb{P}(X_0 \neq X_i, \forall 1 \leq i \leq k) \to \mathbb{P}(X_0 \neq X_i, \forall i \geq 1)$ as $k \to \infty$. Performing Cesaro's summation we deduce that

$$n^{-1}\,\mathbb{E}\left[\#\{(X_0, ..., X_n\}\right] \xrightarrow[n\to\infty]{} \mathbb{P}\left(X_0 \neq X_i, \forall i \geq 1\right). \tag{9.4}$$

We deduce that if the graph is almost surely recurrent (not necessarily ergodic) then the range of the random walk grows sublinearly in probability, a fact that we will use later on. If the graph is ergodic, we can compare the last display together with (9.3) and get by dominated convergence that r is the averaged probability that a simple random walk on $\vec{G}$ does not come back to its starting point.

Shannon entropy. Another application of Kingman theorem for the function

$$\mathrm{Ent}_n(\mathrm{g}, (\vec{e}_i)_{i\geq 0}) = -\log \mathbf{P}_{(\mathrm{g},\vec{e}_0)}(\vec{E}_n = \vec{e}_n),$$

will be presented in the next chapter in connection with the notion of entropy.

9.3 Applications

Let us make use of the above machinery to prove interesting structure theorems for stationary random graphs.

9.3.1 Non-reversibility implies transience

Theorem 45 (*Non reversibility implies transience*)

Let $(G, \vec{E})$ be an ergodic stationary random graph. If $(G, \vec{E})$ is not reversible then it must also be almost surely transient.

Proof. We denote $(G, (\vec{E})_{i \geq 0})$ the random graph with the random walk path in $\mathcal{G}^{\rightarrow}$ obtained by launching the SRW on G. Since G is not reversible, there must exist a bounded function $F : \vec{\mathcal{G}} \to \mathbb{R}_+$ such that we have

$$\alpha = \mathbb{E}[F(G, \vec{E})] \neq \mathbb{E}[F(G, \overleftarrow{E})] = \beta.$$

The ergodic theorem hence ensures that

$$\frac{1}{n} \sum_{i=0}^{n-1} F(G, \vec{E}_i) \xrightarrow[n\to\infty]{a.s.} \alpha \quad \text{and} \quad \frac{1}{n} \sum_{i=0}^{n-1} F(G, \overleftarrow{E}_i) \xrightarrow[n\to\infty]{a.s.} \beta.$$

Now if we suppose by contradiction that G is recurrent with a positive probability, since this event is shift invariant then by ergodicity G must be almost surely recurrent. Writing τ_k for the successive return times of the simple random to the origin vertex we deduce from the last display and the dominated convergence theorem that

$$\mathbb{E}\left[\frac{1}{\tau_k} \sum_{i=0}^{\tau_k - 1} F(G, \vec{E}_i)\right] \xrightarrow[k\to\infty]{} \alpha \quad \text{and} \quad \mathbb{E}\left[\frac{1}{\tau_k} \sum_{i=0}^{\tau_k - 1} F(G, \overleftarrow{E}_i)\right] \xrightarrow[k\to\infty]{} \beta.$$

However, in any graph, by reversibility of the simple random walk path, the law of $(\vec{E}_0, ..., \vec{E}_{\tau_k - 1})$ is equal to that of $(\overleftarrow{E}_{\tau_k - 1}, ..., \overleftarrow{E}_0)$ since the density of these two paths is proportional to

$$\big(\deg(X_1) \times \cdots \times \deg(X_{\tau_k - 1})\big)^{-1}$$

where $X_0, X_1, ..., X_{\tau_k}$ are the vertices visited by the simple random walk. Combining this observation with the last display shows that $\alpha = \beta$ which is absurd! □

9.3.2 Degree of the root

We will now prove the analog of Theorem 29 in the weaker context of stationary random graphs. Recall that since we need to bias by the inverse degree of the origin vertex to go from stationary and reversible random graph to unimodular random graph, the analog of the equality $\mathbb{E}[\deg(\rho)] = 2$ in Section 7.3 becomes $\mathbb{E}[\deg(\rho)^{-1}] = \frac{1}{2}$.

Theorem 46 (*The (inverse) degree tells us a lot!*)

Let $(G, \vec{E})$ be an a.s. infinite stationary random graph. Then we have

$$\mathbb{E}[\deg(\vec{E}_*)^{-1}] \leq \frac{1}{2}.$$

Besides if the above inequality is an equality, then G is a.s. a random tree (with one or two ends, as proved below in Theorem 55).

Proof. By the ergodic decomposition, one may suppose that $\vec{G}$ is ergodic. We let $(\vec{E}_n)_{n\geq 0}$ be the sequence of oriented edges visited by the random walk on G and denote by $(X_n)_{n\geq 0}$ their origin vertices. By the above application of Kingman subadditive theorem, we know that the variables $H_n = \mathrm{d_{gr}}(X_0, X_n)$ satisfy

$$\frac{H_n}{n} \xrightarrow[n\to\infty]{a.s.\text{ and } L^1} s \geq 0.$$

One the other hand, given the graph G, we can condition on the σ-field $\mathcal{F}_n$ generated by the first $n-1$ steps (so that the walk sits on X_n) and compute

$$\mathbf{E}_{(G,\vec{E})}[H_{n+1} - H_n \mid \mathcal{F}_n] = \frac{1}{\deg(X_n)} \left(P_{G,X_0}(X_n) - N_{G,X_0}(X_n)\right),$$

where $P_{\mathrm{g},x}(y)$ is the number of edges in g starting from y pointing to a vertex at a distance strictly larger than $\mathrm{d_{gr}}(x,y)$ and similarly for $N_{\mathrm{g},x}(y)$ after replacing strictly larger than strictly less. Clearly we must have $N_{\mathrm{g},x}(y) \geq 1$ as long as $x \neq y$ since there must be a geodesic path going from y to x in g, and so we can always write:

$$\mathbf{E}_{(G,\vec{E})}[H_{n+1} - H_n \mid \mathcal{F}_n] \quad \leq \quad \frac{\deg(X_n) - 2}{\deg(X_n)} + \mathbf{1}_{X_n = X_0} \frac{2}{\deg(X_n)}. \tag{9.5}$$

By ergodicity, we then have

$$\sum_{k=1}^{n} \frac{\deg(X_k) - 2}{\deg(X_k)} \xrightarrow[n\to\infty]{a.s.\text{ and } L^1} 1 - 2\mathbb{E}[\deg(\vec{E}_*)^{-1}],$$

whereas in any infinite graph we have

$$\sum_{k=1}^{n} \mathbf{1}_{X_n = X_0} \frac{2}{\deg(X_n)} \xrightarrow[n\to\infty]{a.s.\text{ and } L^1} 0.$$

since the simple random walk is either transient or null recurrent (and never positive recurrent). Combining the above convergences and taking expectations we deduce that

$$0 \leq s \leq 1 - 2\mathbb{E}[\deg(\vec{E}_*)^{-1}],$$

as desired. If the inequality is strict, that means that the density of times for which we have $N_{G,X_n} \geq 2$ is equal to 0. A moment of thought shows that this implies that the density of times for which we see a cycle in the graph within the ball of radius r around the location of the random walk is also equal to 0. The result then follows by the ergodic theorem. We leave the details to the reader. □

9.3.3 Benjamini–Schramm quenched

Recall from Section 2.2.2 the definition of quenched Benjamini–Schramm convergence for a sequence of random pointed graphs. This notion can obviously be adapted to the case of uniformly rooted graphs (instead of uniformly pointed graphs). We show below that if a sequence converges in the Benjamini–Schramm sense towards an ergodic limit, then it automatically implies the quenched convergence:

Theorem 47 (*Ergodicity implies Benjamini–Schramm quenched*)

Suppose that $(\vec{G}_n : n \geq 0)$ is a sequence of uniformly rooted finite graphs such that

$$\vec{G}_n \xrightarrow[n\to\infty]{(d)} \vec{G}_\infty,$$

where the (automatically) stationary, reversible random rooted graph $\vec{G}_\infty$ is supposed to be ergodic. Then $(\vec{G}_n)_{n\geq 0}$ also converge in the quenched Benjamini–Schramm sense, that is for any bounded continuous $\phi : \vec{\mathcal{G}} \to \mathbb{R}_+$ are measurable, then

$$\mathbb{E}[\phi(\vec{\mathrm{U}}(\vec{G}_n)) \mid \vec{G}_n] \xrightarrow[n\to\infty]{(\mathbb{P})} \mathbb{E}[\phi(\vec{G}_\infty)].$$

Proof. Fix a continuous positive bounded functions $\phi : \vec{\mathcal{G}} \to \mathbb{R}_+$. For a fixed (possibly infinite) rooted graph $\vec{\mathrm{g}}$ and $r \geq 1$ let us denote by

$$\overline{\phi}_r(\vec{\mathrm{g}}) = \int \mathrm{d}\mathbf{P}_{\vec{\mathrm{g}}}(\vec{E}_i : i \geq 0)\ \frac{1}{r}\sum_{k=0}^{r-1}\phi(\mathrm{g}, \vec{E}_k).$$

In words, this consists in averaging over the first r steps of the random walk. Clearly, $\vec{g} \mapsto \overline{\phi}_r(\vec{g})$ is continuous for the local topology. On the one hand, notice that for any fixed finite graph $\vec{g}$, since the invariant measure of the random walk is the uniform measure on the oriented edges we have

$$\frac{1}{\vec{E}(\mathrm{g})}\sum_{\vec{\sigma}\in\vec{E}(\mathrm{g})}\phi(\mathrm{g},\vec{\sigma}) = \frac{1}{\vec{E}(\mathrm{g})}\sum_{\vec{\sigma}\in\vec{E}(\mathrm{g})}\overline{\phi}_r(\mathrm{g},\vec{\sigma}).$$

In particular, we can write for any finite graph $\vec{g}$

$$\begin{aligned}\left|\frac{1}{\vec{E}(\mathrm{g})}\sum_{\vec{\sigma}\in\vec{E}(\mathrm{g})}\phi(\mathrm{g},\vec{\sigma}) - \mathbb{E}[\phi(\vec{G}_\infty)]\right| &= \left|\frac{1}{\vec{E}(\mathrm{g})}\sum_{\vec{\sigma}\in\vec{E}(\mathrm{g})}\overline{\phi}_r(\mathrm{g},\vec{\sigma}) - \mathbb{E}[\phi(\vec{G}_\infty)]\right| \\ &\leq \frac{1}{\vec{E}(\mathrm{g})}\sum_{\vec{\sigma}\in\vec{E}(\mathrm{g})}\left|\overline{\phi}_r(\mathrm{g},\vec{\sigma}) - \mathbb{E}[\phi(\vec{G}_\infty)]\right|.\end{aligned}$$

Taking $\vec{g} = G_n$ and averaging over the choice of the graph, we deduce that

$$\mathbb{E}\left[\left|\frac{1}{\vec{E}(G_n)}\sum_{\vec{\sigma}\in\vec{E}(G_n)}\phi(G_n,\vec{\sigma}) - \mathbb{E}[\phi(\vec{G}_\infty)]\right|\right] \leq \mathbb{E}\left[\frac{1}{\vec{E}(G_n)}\sum_{\vec{\sigma}\in\vec{E}(G_n)}\left|\overline{\phi}_r(G_n,\vec{\sigma}) - \mathbb{E}[\phi(\vec{G}_\infty)]\right|\right].$$

Since $\overline{\phi}_r$ is a bounded continuous function, by the (annealed) Benjamini–Schramm convergence, the last expectation converges as $n \to \infty$ towards

$$\mathbb{E}\left[|\overline{\phi}_r(\vec{G}_\infty) - \mathbb{E}[\phi(\vec{G}_\infty)]|\right].$$

On the other hand, by Birkhoff ergodic theorem (9.1), for almost every realization of $\vec{G}_\infty$ we have

$$\overline{\phi}_r(\vec{G}_\infty) \xrightarrow[r\to\infty]{} \mathbb{E}[\phi(\vec{G}_\infty)], \tag{9.6}$$

and in particular, the penultimate display tends to 0 as $r \to \infty$. We deduce that

$$\lim_{r\to\infty}\limsup_{n\to\infty}\mathbb{E}\left[\left|\frac{1}{\vec{E}(G_n)}\sum_{\vec{\sigma}\in\vec{E}(G_n)}\phi(G_n,\vec{\sigma}) - \mathbb{E}[\phi(\vec{G}_\infty)]\right|\right] = 0,$$

which implies the convergence in probability of $\mathbb{E}[\phi(\vec{\mathrm{U}}(\vec{G}_n)) \mid \vec{G}_n]$ as desired. □

Using Exercise 8.1.1 and Proposition 34 we deduce the pointed version of the above theorem:

Corollary 48 (Pointed case)**.** *Suppose $G_n^\bullet = (G_n, \rho_n)$ is a sequence of uniformly pointed graphs which converge locally towards and infinite unimodular graph $G_\infty^\bullet = (G_\infty, \rho_\infty)$. If $\mathbb{E}[\deg(\rho_n])] \to \mathbb{E}[\deg(\rho_\infty)] < \infty$ and if $G_\infty^\bullet$ is ergodic, then $(G_n)_{n\geq 0}$ converges in the quenched Benjamini–Schramm sense.*

Biliographical references. The link with ergodic theory is classical and can be found e.g. in [17] where the application to the entropy of the walk was first derived following the approach of [45] in the case of fixed regular graphs. The proof of Proposition 43 is adapted from [34, Proposition 10] itself building upon [55, Theorem 5.1]. The proof of Theorem 45 is new but a stronger result has been proved in [17]: an ergodic stationary non reversible random graph (with bounded degrees) must actually have positive speed. Theorem 47 is, to the best of our knowledge, new.