Mean-Field Games

Third lecture: Convergence Problem

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Based on joint works with E. Bayraktar, R. Carmona, P. Cardaliaguet, A. Cecchin, A. Cohen, D. Crisan, J.F. Chassagneux, R. Foguen, D. Lacker, J.M. Lasry, P.L. Lions, K. Ramaman

Part VII. Convergence Problem

Part VII. Convergence Problem

a. Connections between MFG and *N*-player game

Revisiting the N-player game

- Controlled dynamics

$$dX_t^i = \alpha_t^i dt + dW_t^i + \eta dB_t$$

- Cost functionals to player i

$$J^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^N) + \int_0^T \Big(f(X_s^i, \bar{\mu}_s^N) + \frac{1}{2}|\alpha_s^i|^2\Big)ds\Big]$$

- Rigorous connection between N-player game and MFG?

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- Rigorous connection between N-player game and MFG?

- Prove the convergence of the Nash equilibria as N tends to ∞

  ∘ difficulty $\rightsquigarrow$ no uniform smoothness on the optimal feedback function $\alpha^{\star,N}$ w.r.t to N

$$\underbrace{\alpha_t^{\star,i,N}}_{\text{optimal control to player } i} = \alpha^{\star,N}(X_t^i; \underbrace{X^1, \dots, X^{i-1}, X^{i+1}, \dots, X^N}_{\text{states of the others}})$$

$\rightsquigarrow$ no compactness on the feedback functions

  ∘ weak compactness arguments on the control (notion of relaxed controls) for equilibria over open loop controls [Lacker, Fischer] and, recently, closed loop [Lacker]

Revisiting the N-player game

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- Cost functionals to player i

$$J^i(\alpha^1, \dots, \alpha^N) = \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^N) + \int_0^T \Big(f(X_s^i, \bar{\mu}_s^N) + \frac{1}{2}|\alpha_s^i|^2\Big)ds\Big]$$

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$\rightsquigarrow$ no compactness on the feedback functions and, recently, closed loop [Lacker]

◦ use the master equation [C D L L]: expand $(\mathcal{U}(t, X_t^i, \bar{\mu}_t^N))_{0 \le t \le T}$ and prove $\approx$ equilibrium cost to player i (see the sequel)

Revisiting the N-player game

- Controlled dynamics

$$dX_t^i = \alpha_t^i dt + dW_t^i + \eta dB_t$$

- Cost functionals to player i

$$J^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^N) + \int_0^T \Big(f(X_s^i, \bar{\mu}_s^N) + \frac{1}{2}|\alpha_s^i|^2\Big)ds\Big]$$

- Rigorous connection between N-player game and MFG?
- Construct approximate Nash equilibria (easier)
  - limit setting $\rightsquigarrow$ optimal control has the form

$$\alpha_t^\star = -\partial_x \mathcal{U}(t, X_t, \underbrace{\mathcal{L}(X_t|B)}_{\text{population at equilibrium}})$$

  - in N-player game, use $\alpha_t^{N,i} = -\partial_x \mathcal{U}(t, X_t^i, \bar{\mu}_t^N)$
  - almost Nash $\rightsquigarrow$ cost decreases at most of ε_N under unilateral deviation where $\varepsilon_N \to 0$

Part VII. Convergence Problem

b. Construction of approximate equilibria

[Lasry-Lions, Caines-Huang-Malhamé]

Implementing the limit optimal feedback

- Choose $\eta = 0$
  - don't require uniqueness $\Rightarrow$ no Master equation!!
  - assume $\boldsymbol{\mu}$ equilibrium and $u^{\boldsymbol{\mu}}$ related value function
  - replace $\partial_x \mathcal{U}(t, x, \mu_t)$ by $\partial_x u^{\boldsymbol{\mu}}(t, x)$
  - optimal feedback in environment $\boldsymbol{\mu}$

$$-\partial_x u^{\boldsymbol{\mu}}(t, x)$$

  - under same assumptions as before $\rightsquigarrow \partial_x u^{\boldsymbol{\mu}}(t, \cdot)$ is Lipschitz continuous in x

- Go back to the dynamics of the finite player system
  - assume that σ is 1 (for simplicity)

$$dX_t^i = -\partial_x u^{\boldsymbol{\mu}}(t, X_t^i)dt + dW_t^i$$

  - compute first $\partial_x u^{\boldsymbol{\mu}}$ and $\boldsymbol{\mu}$ numerically and plug them!

Propagation of chaos

- N-player system

$$dX_t^i = -\partial_x u^{\mu}(t, X_t^i)dt + dW_t^i$$

  - fits the framework of MKV SDE

- As N tends to ∞
  - for k fixed

$$(X_t^1, \dots, X_t^k)_{0 \le t \le T} \underset{\mathcal{L}}{\longrightarrow} \mathcal{L}((X_t^\star)_{0 \le t \le T})^{\otimes k}$$

  - where $(X_t^\star)_{0 \le t \le T}$ optimal dynamics in the limit

$$dX_t^\star = -\partial_x u^{\mu}(t, X_t^\star)dt + dW_t$$

  - moreover,

$$\lim_{N \nearrow \infty} \sup_{0 \le t \le T} \mathbb{E}\Big[\Big(W_2(\bar{\mu}_t^N, \mu_t)\Big)^2\Big] = 0$$

Quasi-Nash property

- Notations

  ∘ $\alpha_t^i = -\partial_x u^{\mu}(t, X_t^i)$ controls taken from the limit feedback function

  ∘ call $J^\star$ the optimal cost in the MFG setting

  ∘ under mild assumptions (●)

$$J(\alpha^1, \dots, \alpha^N) \underset{N\to\infty}{\longrightarrow} J^\star$$

- Check that $(\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^N)$ forms a quasi-Nash equilibrium

  ∘ change $\boldsymbol{\alpha}^1$ into $\boldsymbol{\beta}^1$ and freeze the others (Nash over open loop)

  ∘ $\exists N_0$ s.t. for $N \geq N_0$, $A > 0$, $\exists C$ s.t.

$$\mathbb{E}\int_0^T |\beta_t^1|^2 dt \geq C \Rightarrow J^1(\beta^1, \alpha^2, \dots, \alpha^N) \geq J^\star + A$$

  ∘ for $A > 0$, $\exists (\varepsilon_N)_{N\geq 1} \downarrow 0$ s.t. 0, such that (●)

$$\mathbb{E}\int_0^T |\beta_t^1|^2 dt \leq A \Rightarrow \begin{array}{l} J^1(\beta^1, \alpha^2, \dots, \alpha^N) \geq J^\star - \varepsilon_N \\ J^i(\beta^1, \alpha^2, \dots, \alpha^N) \geq J^\star - \varepsilon_N, \quad 2 \leq i \leq N \end{array}$$

Part VII. Convergence Problem

c. Connection between MFG and Nash system

[Cardaliaguet, D. , Lasry, Lions]

Nash System

- N player game equilibrium
  - unique Markovian equilibrium with bounded feedback $\rightsquigarrow$

given by Nash system $\rightsquigarrow$ $v^{N,i}$ value function to player i (●)

$$\partial_t v^{N,i}(t,\boldsymbol{x}) + \frac{1}{2}\sum_j \Delta_{x_j} v^{N,i}(t,\boldsymbol{x}) + \frac{\eta^2}{2}\sum_{j,k} Tr\partial^2_{x_j,x_k} v^{N,i}(t,\boldsymbol{x})$$

$$- \sum_{j\neq i} \partial_{x_j} v^{N,j}(t,\boldsymbol{x}) \cdot \partial_{x_j} v^{N,i}(t,\boldsymbol{x})$$

$$- \frac{1}{2}|\partial_{x_i} v^{N,i}(t,\boldsymbol{x})|^2 + f(x_i, \bar{\mu}^N_{\boldsymbol{x}}) = 0$$

  - mean field interaction $\bar{\mu}^N_{\boldsymbol{x}} = \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$
  - boundary condition $v^{N,i}(T,\boldsymbol{x}) = g(x_i, \bar{\mu}^N_{\boldsymbol{x}})$

- The goal is to prove uniform convergence on $\mathbb{R}^d$

$$\lim_{N\to\infty} \sup_{0\leq t\leq T} \sup_{\boldsymbol{x}\in(\mathbb{R}^d)^N} |v^{N,i}(t,\boldsymbol{x}) - \mathcal{U}(t,x_i,\bar{\mu}^N_{\boldsymbol{x}})| = 0$$

Connection with the master equation

- Strategy is to compare $\boxed{u^{N,i}(t,\boldsymbol{x}) = \mathcal{U}(t,x_i,\bar{\mu}^N_{\boldsymbol{x}})}$ with $v^{N,i}$
- First-order terms

$$\partial_{x_j} u^{N,i}(t,\boldsymbol{x}) = \begin{array}{ll} \partial_x \mathcal{U}(t,x_i,\bar{\mu}^N_{\boldsymbol{x}}) + O(\frac{1}{N}) & \text{if} \quad j = i \\ \frac{1}{N}\partial_\mu \mathcal{U}(t,x_i,\bar{\mu}^N_{\boldsymbol{x}})(x_j) & \text{if} \quad j \neq i \end{array}$$

  - ○ Hamiltonian

$$-\frac{1}{2}|\partial_{x_i} u^{N,i}(t,\boldsymbol{x})|^2 + f(x_i,\bar{\mu}^N_{\boldsymbol{x}}) = -\frac{1}{2}|\partial_x \mathcal{U}(t,x_i,\bar{\mu}^N_{\boldsymbol{x}})|^2 + f(x_i,\bar{\mu}^N_{\boldsymbol{x}}) + O(\frac{1}{N})$$

  - ○ drift terms

$$-\sum_{j\neq i} \partial_{x_j} u^{N,j}(t,\boldsymbol{x}) \cdot \partial_{x_j} u^{N,i}(t,\boldsymbol{x})$$

$$= \frac{1}{N}\sum_{j\neq i} \partial_x \mathcal{U}(t,x_j,\bar{\mu}^N_{\boldsymbol{x}}) \cdot \partial_\mu \mathcal{U}(t,x_i,\bar{\mu}^N_{\boldsymbol{x}})(x_j) + O(\frac{1}{N})$$

$$= -\int_{\mathbb{R}^d} \partial_x \mathcal{U}(t,v,\bar{\mu}^N_{\boldsymbol{x}}) \cdot \partial_\mu \mathcal{U}(t,x_i,\bar{\mu}^N_{\boldsymbol{x}})(v)\, d\bar{\mu}^N_{\boldsymbol{x}}(v) + O(\frac{1}{N})$$

  - ○ up to $O(\frac{1}{N}) \rightsquigarrow$ fits 1st order of master eq. (●) at $x = x_i$, $\mu = \bar{\mu}^N_{\boldsymbol{x}}$

Connection with the master equation

- Strategy is to compare $\boxed{u^{N,i}(t,\boldsymbol{x}) = \mathcal{U}(t, x_i, \bar{\mu}^N_{\boldsymbol{x}})}$ with $v^{N,i}$
- Using smoothness of $\mathcal{U}$ at order 2 $\rightsquigarrow$ we show

$$
\begin{aligned}
\partial_t u^{N,i}(t,\boldsymbol{x}) &+ \frac{1}{2}\sum_j \Delta_{x_j} u^{N,i}(t,\boldsymbol{x}) + \frac{\eta}{2}\sum_{j,k} TrD^2_{x_j,x_k} u^{N,i}(t,\boldsymbol{x}) \\
&- \sum_{j\neq i} \partial_{x_j} u^{N,i}(t,\boldsymbol{x}) \cdot \partial_{x_j} u^{N,j}(t,\boldsymbol{x}) \\
&- \frac{1}{2}|\partial_{x_i} u^{N,i}(t,\boldsymbol{x})|^2 + \big(f(x_i, \bar{\mu}^N_{\boldsymbol{x}}) + \underbrace{r^{N,i}(t,\boldsymbol{x})}_{|r^{N,i}| \leq C/N} \big)
\end{aligned}
$$

  - with $\bar{x}^N = \frac{1}{N}\sum_{j=1}^N x_j$
- Propagation of reminder $O(1/N)$ among N players?

Part VII. Convergence Problem

d. A detour

Linear version

• Wish to use smoothness of the limiting problem to prove convergence of equilibria

• Illustration in the linear case without common noise $\rightsquigarrow$ focus on a standard McKean-Vlasov equation

$$dX_t = b(X_t, \mathcal{L}(X_t))dt + dW_t$$

• Analogue of the master equation?

  ◦ notice that $\mathcal{L}(X_t)$ only depends on $\mathcal{L}(X_0)$

  ◦ define the semi-group

$$(\mathcal{P}_t\phi)(\mathcal{L}(X_0)) = \phi(\mathcal{L}(X_t^\star)), \quad t \in [0,T], \quad \phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$$

  ◦ dynamics of $\mathbb{P}_2(\mathbb{R}^d) \ni \mu \mapsto \mathcal{P}_t\phi(\mu)$?

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  - dynamics of $\mathbb{P}_2(\mathbb{R}^d) \ni \mu \mapsto \mathcal{P}_t\phi(\mu)$?

- Form of the master equation ()

$$\partial_t(\mathcal{P}_t\phi)(\mu) - \int_{\mathbb{R}^d} b(v, \mu) \cdot \partial_\mu(\mathcal{P}_t\phi)(\mu, v)d\mu(v)$$
$$- \frac{1}{2}\int_{\mathbb{R}^d} Trace\Big[\partial_v\partial_\mu(\mathcal{P}_t\phi)(\mu, v)\Big]d\mu(v) = 0, \quad (\mathcal{P}_0\phi)(\mu) = \phi(\mu)$$

Linear version

- Illustration in the linear case without common noise $\rightsquigarrow$ focus on a standard McKean-Vlasov equation

$$dX_t = b(X_t, \mathcal{L}(X_t))dt + dW_t$$

- Revisit propagation of chaos for a particle system

$$dX_t^i = b(X_t^i, \bar{\mu}_t^N)dt + dW_t^i, \quad \bar{\mu}_t^N = \frac{1}{N}\sum_j \delta_{X_t^j}$$

  - expansion of $(\mathcal{P}_{T-t}\phi(\bar{\mu}_t^N))_{0 \le t \le T}$ ( ) ( )

$$d[\mathcal{P}_{T-t}\phi(\bar{\mu}_t^N)] = \frac{1}{N}\sum_{i=1}^N \partial_\mu \mathcal{P}_{T-t}\phi(\bar{\mu}_t^N)(X_t^i) \cdot dW_t^i$$

$$+ \frac{1}{2N^2}\sum_{i=1}^N Trace\Big[\partial_\mu^2 \mathcal{P}_{T-t}\phi(\bar{\mu}_t^N)(X_t^i, X_t^i)\Big]dt \rightarrow_{N \nearrow \infty} 0$$

  - $\sim$ behavior is close to that of original semi-group ( )

Part VI. Convergence Problem

e. Adaptation to the nonlinear framework

Comparison of value functions

- Equilibrium trajectories of the N player game

$$dX_t^{N,i} = -\partial_{x_i} v^{N,i}(t, X_t^{N,1}, \cdots, X_t^{N,N})dt + dW_t^i + \eta dB_t$$

- Value processes

$$Y_t^{N,i} = v^{N,i}(t, X_t^{N,1}, \cdots, X_t^{N,N}), \quad Z_t^{N,i,j} = \partial_{x_j} v^{N,i}(t, X_t^{N,1}, \cdots, X_t^{N,N})$$
$$\mathcal{Y}_t^{N,i} = u^{N,i}(t, X_t^{N,1}, \cdots, X_t^{N,N}), \quad \mathcal{Z}_t^{N,i,j} = \partial_{x_j} u^{N,i}(t, X_t^{N,1}, \cdots, X_t^{N,N})$$

- Itô's formula

$$dY_t^{N,i} = -(\frac{1}{2}|Z_t^{N,i,i}|^2 + f(X_t^{N,i}, \bar{\mu}_t^N))dt + \sum_j Z_t^{N,i,j} \cdot (dW_t^j + \eta dB_t)$$

$$d\mathcal{Y}_t^{N,i} = -(\frac{1}{2}|\mathcal{Z}_t^{N,i,i}|^2 + f(X_t^{N,i}, \bar{\mu}_t^N) + r^{N,i}(t, X_t^{N,i}))dt$$
$$+ \sum_j \mathcal{Z}_t^{N,i,j} \cdot (\mathcal{Z}_t^{N,j,j} - Z_t^{N,j,j})dt + \sum_j \mathcal{Z}_t^{N,i,j} \cdot (dW_t^j + \eta dB_t)$$

with $Y_T^i = g(X_T^i, \bar{\mu}_T^N)$ and $\mathcal{Y}_T^i = g(X_T^i, \bar{\mu}_T^N)$, and $\bar{\mu}_t^N = \frac{1}{N} \sum_j \delta_{X_t^{N,j}}$

Stability argument

- Difference between two dynamics

$$d(\mathcal{Y}_t^{N,i} - Y_t^{N,i})$$

$$= -\Big[\frac{1}{2}|\mathcal{Z}_t^{N,i,i}|^2 - \frac{1}{2}|Z_t^{N,i,i}|^2 + \underbrace{r^{N,i}(t, X_t^{N,i})}_{\sim C/N}\Big]dt$$

$$+ \sum_j \underbrace{\mathcal{Z}_t^{N,i,j}}_{\leq C/N \text{ si } i\neq j} (\mathcal{Z}_t^{N,j,j} - Z_t^{N,j,j})dt$$

$$+ \sum_j (\mathcal{Z}_t^{N,i,j} - Z_t^{N,i,j}) \cdot dW_t^j + (\sum_j \mathcal{Z}_t^{N,i,j} - \sum_j Z_t^{N,i,j}) \cdot \eta dB_t$$

- Observe that $\mathcal{Y}_T^{N,i} = Y_T^{N,i}$
  - if no dt terms except $O(1/N)$

$$\mathcal{Y}_t^{N,i} - Y_t^{N,i}$$

$$+ \int_t^T \sum_j (\mathcal{Z}_t^{N,i,j} - Z_t^{N,i,j}) \cdot dW_s^j + (\sum_j \mathcal{Z}_t^{N,i,j} - \sum_j Z_t^{N,i,j}) \cdot \eta dB_s = O(\tfrac{1}{N})$$

Stability argument

- Difference between two dynamics

$$d(\mathcal{Y}_t^{N,i} - Y_t^{N,i})$$
$$= -\Big[\frac{1}{2}|\mathcal{Z}_t^{N,i,i}|^2 - \frac{1}{2}|Z_t^{N,i,i}|^2 + \underbrace{r^{N,i}(t, X_t^{N,i})}_{\sim C/N}\Big]dt$$
$$+ \sum_j \underbrace{\mathcal{Z}_t^{N,i,j}}_{\leq C/N \text{ si } i\neq j} (\mathcal{Z}_t^{N,j,j} - Z_t^{N,j,j})dt$$
$$+ \sum_j (\mathcal{Z}_t^{N,i,j} - Z_t^{N,i,j}) \cdot dW_t^j + (\sum_j \mathcal{Z}_t^{N,i,j} - \sum_j Z_t^{N,i,j}) \cdot \eta dB_t$$

- Observe that $\mathcal{Y}_T^{N,i} = Y_T^{N,i}$
  - if no dt terms except $O(1/N)$

$$\mathbb{E}[|\mathcal{Y}_t^{N,i} - Y_t^{N,i}|^2]$$
$$+ \mathbb{E}\int_t^T \sum_j \big|\mathcal{Z}_t^{N,i,j} - Z_t^{N,i,j}\big|^2 + \eta\mathbb{E}\int_t^T \big|\sum_j \mathcal{Z}_t^{N,i,j} - \sum_j Z_t^{N,i,j}\big|^2 ds = O(\tfrac{1}{N^2})$$

Stability argument

- Difference between two dynamics

$$d(\mathcal{Y}_t^{N,i} - Y_t^{N,i})$$
$$= -\Big[\frac{1}{2}|\mathcal{Z}_t^{N,i,i}|^2 - \frac{1}{2}|Z_t^{N,i,i}|^2 + \underbrace{r^{N,i}(t, X_t^{N,i})}_{\sim C/N}\Big]dt$$
$$+ \sum_j \underbrace{\mathcal{Z}_t^{N,i,j}}_{\leq C/N \text{ si } i \neq j} (\mathcal{Z}_t^{N,j,j} - Z_t^{N,j,j})dt$$
$$+ \sum_j (\mathcal{Z}_t^{N,i,j} - Z_t^{N,i,j}) \cdot dW_t^j + (\sum_j \mathcal{Z}_t^{N,i,j} - \sum_j Z_t^{N,i,j}) \cdot \eta dB_t$$

- Do as if $|\cdot|^2$ is Lipschitz $\rightsquigarrow$ take the square and $\mathbb{E}$ (●)

$$\mathbb{E}\Big[\big|\mathcal{Y}_t^{N,i} - Y_t^{N,i}\big|^2 + \int_t^T \sum_{j=1}^N \big|\mathcal{Z}_s^{N,i,j} - Z_s^{N,i,j}\big|^2 ds\Big]$$
$$\leq \frac{C_\epsilon}{N^2} + \epsilon \mathbb{E} \int_t^T \big|\mathcal{Z}_s^{N,i,i} - Z_s^{N,i,i}\big|^2 ds + \frac{\epsilon}{N} \sum_j \mathbb{E} \int_t^T \big|\mathcal{Z}_s^{N,j,j} - Z_s^{N,j,j}\big|^2 ds$$

Stability argument

- Difference between two dynamics

$$d(\mathcal{Y}_t^{N,i} - Y_t^{N,i})$$

$$= -\Big[\frac{1}{2}|\mathcal{Z}_t^{N,i,i}|^2 - \frac{1}{2}|Z_t^{N,i,i}|^2 + \underbrace{r^{N,i}(t, X_t^{N,i})}_{\sim C/N}\Big]dt$$

$$+ \sum_j \underbrace{\mathcal{Z}_t^{N,i,j}}_{\leq C/N \text{ si } i\neq j} (\mathcal{Z}_t^{N,j,j} - Z_t^{N,j,j})dt$$

$$+ \sum_j (\mathcal{Z}_t^{N,i,j} - Z_t^{N,i,j}) \cdot dW_t^j + \Big(\sum_j \mathcal{Z}_t^{N,i,j} - \sum_j Z_t^{N,i,j}\Big) \cdot \eta dB_t$$

- To handle the square $\rightsquigarrow$ exponential transform $\Rightarrow$ final result

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |\mathcal{Y}_t^i - Y_t^i|^2\Big] + \mathbb{E}\int_0^T |\mathcal{Z}_t^i - Z_t^{i,i}|^2 dt \leq \frac{C}{N^2}$$

- Inserting in the forward equation

$$dX_t^{N,i} = -Z_t^{N,i,i}dt + dW_t^i + \eta dB_t \approx -\mathcal{Z}_t^{N,i,i}dt + dW_t^i + \eta dB_t$$

Part VI. Convergence Problem

f. Fluctuations and deviations

[D., Lacker, Ramanan]

Nash Equilibrium

- N fixed $\rightsquigarrow$ N player game equilibrium described by PDE system
- $v^{N,i}(t,\boldsymbol{x})$ = equilibrium cost to player i when

  the system starts from $\boldsymbol{x} = (x_1, \cdots, x_N) \in (\mathbb{R}^d)^N$ at time t
- Trajectories at equilibrium

$$dX_t^i = -\partial_{x_i} v^{N,i}(t, X_t^1, \cdots, X_t^N)dt + dW_t + \eta dB_t$$

  - $v^{N,i}(t,\boldsymbol{x}) \approx \mathcal{U}(t, x_i, \bar{\mu}_{\boldsymbol{x}}^N) \qquad \bar{\mu}_{\boldsymbol{x}}^N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$
  - and $\frac{1}{N}\sum_{i=1}^N \delta_{X_t^i} \approx \frac{1}{N}\sum_{i=1}^N \delta_{\overline{X}_t^i}$

with $\quad d\overline{X}_t^i = -\partial_x \mathcal{U}\Big(t, \overline{X}_t^i, \frac{1}{N}\sum_{i=1}^N \delta_{\overline{X}_t^i}\Big)dt + dW_t^i + \eta dB_t$

- Philosophy is LDP/CLT are the same for $\frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$ and $\frac{1}{N}\sum_{i=1}^N \delta_{\overline{X}_t^i}$

Principle CLT

- Call

$$\boxed{\nu_t^N = \sqrt{N}(\mu_t^N - \mu_t)}$$

  - with $\mu_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^{N,i}}$ and $\mu_t = \mathcal{L}(X_t^\star \,|\, B)$

$$dX_t^{N,i} = -\partial_{x_i} v^{N,i}(t, X_t^{N,1}, \cdots, X_t^{N,N})dt + dW_t + \eta dB_t$$

$$dX_t^\star = -\partial_x \mathcal{U}(t, X_t^\star, \mathcal{L}(X_t^\star|B))dt + dW_t + \eta dB_t$$

- Typical question $\rightsquigarrow$ take ϕ smooth

$$\langle \phi, \nu_t^N \rangle = \int_{\mathbb{R}^d} \phi(x) d\nu_t^N(x) \Rightarrow ???$$

  - more generally

$$\nu^N \Rightarrow ??? \quad \text{in} \quad C([0,T]; H^*)$$

  - H^* dual space of $H \rightsquigarrow H$ Hilbert space of test functions

H = Sobolev space of functions having λ_d derivatives with prescribed polynomial growth [Hitsuda-Mitoma, Kurtz-Xiong, Méléard]

Principle

- General principle
  - compute the cost for $(\mu_t^N)_{0\leq t\leq T}$ to be away from $(\mu_t)_{0\leq t\leq T}$

$$\mathbb{P}\Big((\mu_t^N)_{0\leq t\leq T} \approx (\nu_t)_{0\leq t\leq T}\Big) \sim \exp\Big[-N I\Big((\nu_t)_{0\leq t\leq T}\Big)\Big]$$

$\rightsquigarrow I\Big((\nu_t)_{0\leq t\leq T}\Big)$ is the cost!

- If $\eta = 0$ (no common noise), fixed initial condition $\Rightarrow$ Dawson-Gartner

$$I^0((\nu_t)_{0\leq t\leq T}) = \frac{1}{2}\int_0^T \left\|\dot{\nu}_t - \mathcal{L}^*_{t,\nu_t}\nu_t\right\|^2_{\nu_t} dt$$

  - $\mathcal{L}_{t,\nu_t} f = \frac{1}{2}\Delta f - \partial_x \mathcal{U}(t,\cdot,\nu_t)\cdot\nabla f, \quad \|\gamma\|^2_{\nu_t} = \sup_{\langle \nu_t, |\nabla f|^2\rangle \neq 0} \frac{\langle \gamma, f\rangle^2}{\langle \nu_t, |\nabla f|^2\rangle}$

Part VI. Convergence Problem

g. Compactness methods

[Lacker, Djete]

General philosophy

- As before, take simple dynamics (without common noise)

$$dX_t^i = \alpha_t^i dt + dW_t^i$$

∘ but do not use any knowledge about the limiting MFG (e.g. no information about uniqueness/stability)

- Instead address directly convergence of

$$\bar{\mu}_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^{\star,i,N}}, \quad t \in [0,T]$$

with $(X^{\star,1,N}, \cdots, X_t^{\star,N,N})$ a Nash equilibrium

∘ weak convergence of

$$\underbrace{\mathbb{P} \circ \Big(t \in [0,T] \mapsto \bar{\mu}_t^N \Big)^{-1}}_{Q_N} \in \mathcal{P}\Big(\mathcal{C}([0,T]; \mathcal{P}(\mathbb{R}^d)) \Big)?$$

∘ argue by compactness: call Q a limiting point of Q_N

Relaxed controls

- As already mentioned $\rightsquigarrow$ no compactness on feedback functions
- Embed controls into measures $\rightsquigarrow$ $q_t(da)dt = \delta_{\alpha_t}(da)$
  - reformulate the dynamics $\rightsquigarrow$ $dX_t = \int_{\mathbb{R}^d} a q_t(da)dt + dW_t$
  - reformulate the cost in environment $(\mu_t)_{0 \le t \le T}$

$$\mathbb{E}\left[g(X_T, \mu_T) + \int_0^T \left[f(X_t, \mu_t) + \frac{1}{2} \int_{\mathbb{R}^d} |a|^2 q_t(da) \right] dt \right]$$

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  - reformulate the cost in environment $(\mu_t)_{0 \le t \le T}$

$$\mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T \Big[f(X_t, \mu_t) + \frac{1}{2} \int_{\mathbb{R}^d} |a|^2 q_t(da)\Big]dt\Big]$$

- $\boxed{\text{1st Step}}$: prove that, under limiting point Q, for a.e. $(\mu_t)_{0 \le t \le T}$,

$$\mu_t = \mathcal{L}\Big(X_t^\star | (\mu_s)_{0 \le s \le t}\Big)$$

with

$$dX_t^\star = \int_{\mathbb{R}^d} a q^\star(t, X_t^\star, (\mu_s)_{0 \le s \le t})(da)dt + dW_t$$

where $W \perp\!\!\!\perp \mu$

Relaxed controls

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- 1st Step : prove that, under limiting point Q, for a.e. $(\mu_t)_{0\le t\le T}$,

$$\mu_t = \mathcal{L}\Big(X_t^\star | (\mu_s)_{0\le s\le t}\Big)$$

with

$$dX_t^\star = \int_{\mathbb{R}^d} a q^\star(t, X_t^\star, (\mu_s)_{0\le s\le t})(da) dt + dW_t$$

- 2st Step : prove that, under limiting point Q, for a.e. $(\mu_t)_{0\le t\le T}$,

$$\mathbb{E}^{Q}\Big[g(X_T^\star, \mu_T) + \int_0^T \Big[f(X_t^\star, \mu_t) + \frac{1}{2}\int_{\mathbb{R}^d} |a|^2 q^\star(t, X_t^\star, (\mu_s)_{0\le s\le t})(da)\Big]dt\Big]$$

$$= \lim_{n\to\infty} \frac{1}{N_n}\sum_{i=1}^{N_n} \mathbb{E}\Big[g(X_T^{\star,i,N_n}, \bar{\mu}_T^{N_n}) + \int_0^T \Big[f(X_t^{\star,i,N_n}, \bar{\mu}_t^{N_n}) + \frac{1}{2}\big|\alpha_t^{\star,i,N_n}\big|^2\Big]dt\Big]$$

with $\alpha_t^{\star,i,N_n} = \alpha^{\star,i,N_n}\Big(t, (X_t^{\star,j,N_n})_{j=1,\dots,N_n}\Big)$

Identification of the limit

• It remains to prove that weak limits satisfy equilibrium condition!
Most challenging step in the proof

• 3rd Step: prove that for any other relaxed control q (in feedback form) and with

$$dX_t = \int_{\mathbb{R}^d} aq(t, X_t, (\mu_s)_{0\le s\le t})(da)dt + dW_t$$

it holds that

$$\mathbb{E}^Q\Big[g(X_T^\star, \mu_T) + \int_0^T \Big[f(X_t^\star, \mu_t) + \frac{1}{2}\int_{\mathbb{R}^d} |a|^2 q^\star(t, X_t^\star, (\mu_s)_{0\le s\le t})(da)\Big]dt\Big]$$
$$\le \mathbb{E}^Q\Big[g(X_T, \mu_T) + \int_0^T \Big[f(X_t^\star, \mu_t) + \frac{1}{2}\int_{\mathbb{R}^d} |a|^2 q(t, X_t, (\mu_s)_{0\le s\le t})(da)\Big]dt\Big]$$

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$$\le \mathbb{E}^Q\Big[g(X_T, \mu_T) + \int_0^T \Big[f(X_t^\star, \mu_t) + \frac{1}{2}\int_{\mathbb{R}^d} |a|^2 q(t, X_t, \mu_t)(da)\Big]dt\Big]$$

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$$\le \mathbb{E}^{Q}\Big[g(X_T, \mu_T) + \int_0^T \Big[f(X_t^\star, \mu_t) + \frac{1}{2}\int_{\mathbb{R}^d} |a|^2 q(t, X_t, \mu_t)(da)\Big]dt\Big]$$

$$\underbrace{=}_{?} \lim_{n\to\infty} \frac{1}{N_n} \sum_{i=1}^{N_n} \mathbb{E}\Big[g(X_T^{i,N_n}, \bar{\mu}_T^{-i,N_n})$$

$$+ \int_0^T \Big[f(X_t^{i,N_n}, \bar{\mu}_t^{-i,N_n}) + \frac{1}{2}\big|q(t, X_t^{i,N_n}, \bar{\mu}_t^{-i,N_n})\big|^2\Big]dt\Big]$$

◦ with X^{i,N_n} playing $q(t, X_t^{i,N_n}, \bar{\mu}_t^{-i,N_n})$ and the others in $\bar{\mu}_t^{-i,N_n}$ playing $\alpha^{\star,j,N_n}$

□ Pass to the limit into

$$J^i(\alpha^1,\dots,\alpha^N) = \mathbb{E}\Big[g(X_T^i,\bar\mu_T^N) + \int_0^T \Big(f(X_s^i,\bar\mu_s^N) + \frac{1}{2}|\partial_x u^{\mu}(s,X_s^i)|^2\Big)ds\Big]$$

⇝ take the coefficients to be locally Lipschitz

$$\big|f(x',\mu') - f(x,\mu)\big| \le C\Big(1 + |x| + |x'| + M_2(\mu) + M_2(\mu')\Big)(|x'-x| + W_2(\mu',\mu))$$

⇝ $M_2(\mu)^2 = \int_{\mathbb{R}^d} |y|^2 d\mu(y)$

□ Compare with X^i with $\bar{X}^{\star,i}$ and $\bar\mu^N$ with μ where

$$dX_t^{\star,i} = -\partial_x u^{\mu}(t, X_t^{\star,i})dt + dW_t^i$$

□ Uniform controls on L^2 norms ⇒ error is mostly driven by

$$\sup_{0\le t\le T}\Big(\mathbb{E}[|X_t^i - X_t^{\star,i}|^2]^{1/2} + \mathbb{E}[W_2(\bar\mu_t^N,\mu_t)^2]^{1/2}\Big)$$

⇝ analysis ⇒ forget t and focus on $\mathbb{E}[W_2(\bar\mu_t^{\star,N},\mu_t)^2]^{1/2}$, where $\bar\mu_t^{\star,N}$ is empirical distribution of $X_t^{\star,1},\cdots,X_t^{\star,N}$

$$X_0 \in L^q,\ q > 4 \Rightarrow N^{-2/\max(d,4)}(1 + \ln(N)1_{d=4})$$

(Dereich, Scheutzow, Schottstedt, 2013, Fournier, Guillin, 2015)

□ Pass to the limit into

$$J^1(\beta,\dots,\alpha^N) = \mathbb{E}\Big[g(X_T^\beta, \bar{\mu}_T^{N,\beta}) + \int_0^T \Big(f(X_s^\beta, \bar{\mu}_s^{N,\beta}) + \frac{1}{2}|\partial_x u^{\mu}(s, X_s^\beta)|^2\Big)ds\Big]$$

⇝ where

$$dX_t^\beta = dt + dW_t, \quad \bar{\mu}_s^{N,\beta} = \frac{1}{N}\delta_{X_t^\beta} + \frac{1}{N}\sum_{i=2}^N \delta_{X_t^i}$$

⇝ up to $O(1/N)$, suffices to handle

$$\mathbb{E}\Big[g(X_T^\beta, \bar{\mu}_T^{N}) + \int_0^T \Big(f(X_s^\beta, \bar{\mu}_s^{N}) + \frac{1}{2}|\partial_x u^{\mu}(s, X_s^\beta)|^2\Big)ds\Big]$$

⇝ up to $O(N^{-1/\max(d,4)}(1 + \ln(N)1_{d=4})^{1/2})$, coincides with

$$\mathbb{E}\Big[g(X_T^\beta, \mu_T) + \int_0^T \Big(f(X_s^\beta, \bar{\mu}_s) + \frac{1}{2}|\partial_x u^{\mu}(s, X_s^\beta)|^2\Big)ds\Big]$$

□ Take a specific label $i \in \{1, \cdots, N\}$

$$dX_t^i = \alpha(t, X_t^1, \cdots, X_t^{i-1}, X_t^i, X_t^{i+1}, \cdots)dt + dW_t^i$$

∘ $\alpha : [0, T] \times (\mathbb{R}^d)^N \to \mathbb{R}^d$ arbitrary bounded feedback function

□ For $j \neq i$

$$dX_t^j = -\partial_{x_j} v^j(t, X_t^1, \cdots, X_t^{i-1}, X_t^i, X_t^{i+1}, \cdots)dt + dW_t^j$$

□ Then (see Friedman (70's), Bensoussan and Frehse (80's))

$$v^{N,i}(t, \boldsymbol{x}) = \inf_{\alpha} \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^{N,\alpha}) + \int_t^T \Big(f(X_s^i, \bar{\mu}_T^{N,\alpha}) + \tfrac{1}{2} \big| \alpha(t, X_t^1, \cdots, X_t^{i-1}, X_t^i, X_t^{i+1}, \cdots) \big|^2 \Big) dt \Big]$$

⇝ $\bar{\mu}_t^{N,\alpha} = \frac{1}{N} \sum_{j=1}^{N} \delta_{X_t^j}$

⇝ optimal feedback is $(t, \boldsymbol{x}) \mapsto -\partial_{x_i} v^{N,i}(t, \boldsymbol{x})$

□ Master equation when $\eta = 0$

$$\partial_t \mathcal{U}(t,x,\mu) - \int_{\mathbb{R}^d} \partial_x \mathcal{U}(t,v,\mu) \cdot \partial_\mu \mathcal{U}(t,x,\mu,v) d\mu(v) - \frac{1}{2}|\partial_x \mathcal{U}(t,x,\mu)|^2 + f(x,\mu) + \frac{1}{2}\mathrm{Trace}\Big(\partial_x^2 \mathcal{U}(t,x,\mu)\Big) + \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{Trace}\Big(\partial_v \partial_\mu \mathcal{U}(t,x,\mu)(v)\Big) d\mu(v) = 0$$

□ Take $x = x_i$ and $\mu = \mu_{\boldsymbol{x}}^N = \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$ for some $\boldsymbol{x} = (x_1, \cdots, x_N)$,

$$\partial_t \mathcal{U}(t,x_i,\bar{\mu}_{\boldsymbol{x}}^N) - \int_{\mathbb{R}^d} \partial_x \mathcal{U}(t,v,\bar{\mu}_{\boldsymbol{x}}^N) \cdot \partial_\mu \mathcal{U}(t,x_i,\bar{\mu}_{\boldsymbol{x}}^N,v) d\bar{\mu}_{\boldsymbol{x}}^N(v) - \frac{1}{2}|\partial_x \mathcal{U}(t,x_i,\bar{\mu}_{\boldsymbol{x}}^N)|^2 + f(x_i,\bar{\mu}_{\boldsymbol{x}}^N) + \frac{1}{2}\mathrm{Trace}\Big(\partial_x^2 \mathcal{U}(t,x_i,\bar{\mu}_{\boldsymbol{x}}^N)\Big) + \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{Trace}\Big(\partial_v \partial_\mu \mathcal{U}(t,x_i,\bar{\mu}_{\boldsymbol{x}}^N)(v)\Big) d\bar{\mu}_{\boldsymbol{x}}^N(v) = 0$$

□ Expand $\phi(\mathcal{L}(X_t^\star))$ as

$$d[\phi(\mathcal{L}(X_t^\star))] = \mathbb{E}\Big[b(X_t^\star, \mathcal{L}(X_t^\star)) \cdot \partial_\mu \phi(\mathcal{L}(X_t^\star))\Big]dt + \frac{1}{2}\mathbb{E}\Big[\mathrm{Trace}(\partial_v \partial_\mu \phi(X_t^\star, \mathcal{L}(X_t^\star)))\Big]dt$$

$\rightsquigarrow$ get

$$\frac{d}{dt}_{|t=0} \mathcal{P}_t \phi(\mu) = \int_{\mathbb{R}^d} b(v,\mu) \cdot \partial_\mu \phi(\mu)(v) d\mu(v) + \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{Trace}(\partial_v \partial_\mu \phi(v,\mu)) d\mu(v)$$

□ Recall

$$(\mathcal{P}_t\phi)(\mathcal{L}(X_0)) = \phi(\mathcal{L}(X_t^\star))$$

⇝ therefore

$$(\mathcal{P}_{T-t}\phi)(\mathcal{L}(X_t^\star)) = \phi(\mathcal{L}(X_{T-t}^{\mathcal{L}(X_t^\star),\star})) = \phi(\mathcal{L}(X_T^\star))$$

⇒ test how far

$$\left(\mathcal{P}_{T-t}\phi(\bar{\mu}_t^N\right)_{0\le t\le T}$$

is from being constant?

□ Expand

$$
\begin{aligned}
d[\mathcal{P}_{T-t}\phi(\bar{\mu}_t^N)] = & -\partial_t \mathcal{P}_{T-t}\phi(\bar{\mu}_t^N)dt \\
& + \frac{1}{N}\sum_{i=1}^{N} \partial_\mu \mathcal{P}_{T-t}\phi(\bar{\mu}_t^N)(X_t^i) \cdot b(X_t^i, \mathcal{L}(X_t^\star))dt \\
& + \frac{1}{N}\sum_{i=1}^{N} \mathrm{Trace}(\partial_v \partial_\mu \mathcal{P}_{T-t}\phi(\bar{\mu}_t^N)(X_t^i))dt \\
& + \frac{1}{N}\sum_{i=1}^{N} \partial_\mu \mathcal{P}_{T-t}\phi(\bar{\mu}_t^N)(X_t^i) \cdot dW_t^i \\
& + \frac{1}{2N^2}\sum_{i=1}^{N} Trace\Big[\partial_\mu^2 \mathcal{P}_{T-t}\phi(\bar{\mu}_t^N)(X_t^i, X_t^i)\Big]dt
\end{aligned}
$$

□ Conclusion

$$\mathbb{E}\Big[\big|\phi(\bar{\mu}_T^N) - \mathcal{P}_T\phi(\bar{\mu}_0^N)\big|^2\Big] \leq \frac{C}{N}$$

□ Suffices to discuss

$$\mathbb{E}\Big[\big|\mathcal{P}_T\phi(\mu_0) - \mathcal{P}_T\phi(\bar{\mu}_0^N)\big|^2\Big]$$

$\rightsquigarrow$ Here, $\bar{\mu}_0^N$ i.i.d. sample from μ_0!

□ Take backward SDE (say $1d$ to simplify)

$$Y_t = \int_t^T f_s ds + \int_t^T Z_s ds + \int_t^T Z_s dB_s$$

□ apply Itô's formula

$$Y_t^2 + \int_t^T Z_s^2 ds = 2 \int_t^T Y_s(f_s + Z_s) ds + 2 \int_t^T Y_s Z_s dB_s$$

⇝ Take $\mathbb{E}$ and use Young's type inequality

$$\mathbb{E}[Y_t^2] + \mathbb{E} \int_t^T Z_s^2 ds$$
$$\leq C_\epsilon \mathbb{E} \int_t^T |Y_s|^2 ds + \epsilon \mathbb{E} \int_t^T (f_s^2 + Z_s^2) ds$$