Formulations and solution approaches for the Euclidean Steiner Problem in n-space

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- F. K. Hwang, D. S. Richards and P. Winter, *The Steiner Tree Problem*, volume 53 of Annals of Discrete Mathematics. North-Holland, Amsterdam, Netherlands, 1992.
- W. D. Smith, *How to find Steiner minimal trees in Euclidean d-space*, Algorithmica, vol. 7, pp. 137-177,1992.
- M. Fampa and Kurt M. Anstreicher, *An improved algorithm for computing Steiner minimal trees in Euclidean d-space*, Discrete Optimization, pp. 530-540, 2008.
- N. Maculan, P. Michelon and A. E. Xavier, *The Euclidean Steiner problem in Rn : A mathematical programming formulation*, Annals of Operations Research, vol. 96, pp. 209-220, 2000.
- M. Fampa and N. Maculan, *Using a conic formulation for finding Steiner minimal trees*, Numerical Algorithms, vol. 35, pp. 315-330, 2004.

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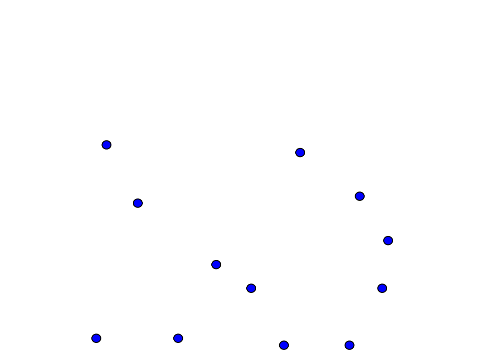
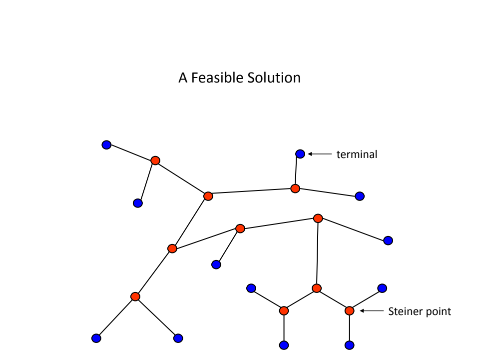
Given p points in R^n.
Find a tree with minimal Euclidean length that spans these points using or not extra points, which are called Steiner points.



The Euclidean Steiner Problem Definition

Given p points in R^n.
Find a tree with minimal Euclidean length that spans these points using or not extra points, which are called Steiner points.



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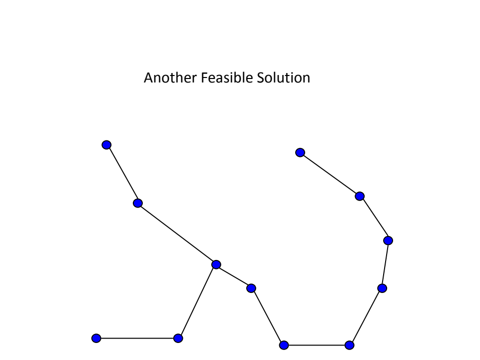
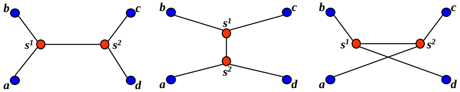


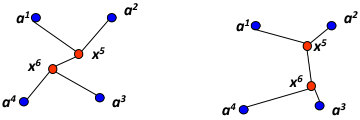
Determine:

- The number of Steiner points to be used on the minimal tree.
- The arcs of the tree.
- Geometrical position of the Steiner points.

- Topologies are graphs that show connections between Steiner points and terminals



Three different topologies for a graph with 4 terminals and 2 Steiner points

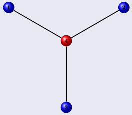


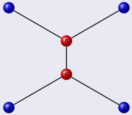
Two different solutions for a given topology

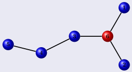
To obtain the best solution:

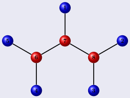
$$\text{Minimize } ||a^1 - x^5|| + ||a^2 - x^5|| + ||x^5 - x^6|| + ||a^3 - x^5|| + ||a^4 - x^6||$$
$$\text{subject to: } x^5, x^6 \in \mathbb{R}^n$$

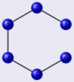
Steiner Minimal Trees (SMT)









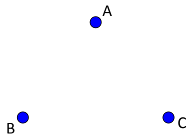


Historical Background

The problem traces back to an ancient problem studied by Pierre de Fermat

Challenge of Fermat in the 17th century

Given three points in the plane, find a fourth point such that the sum of its distance to the three given points is at minimum.



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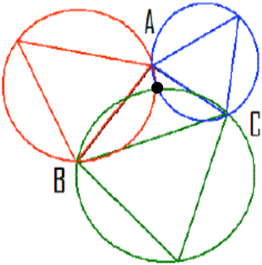
Evangelista Torricelli's geometric solution (1640)

Three circles circumscribing the equilateral triangles constructed on the sides of and outside the triangle ABC intersect in the point that is sought (the so called Torricelli point).



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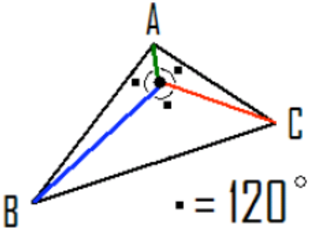
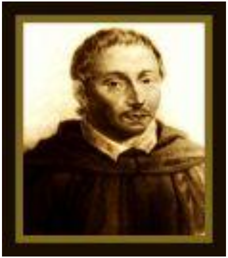
Bonaventura Cavalieri (1647)

The line segments from the three given points to the Torricelli point make 120° with each other.





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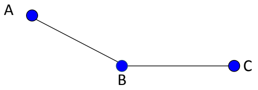
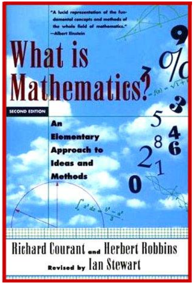
Franz Heinen (1834)

If the triangle ABC has one angle greater than or equal to 120°, then the minimizing point that solves Fermat problem is the vertex of the obtuse angle.



- Find a point such that the sum of p distances from the point to p given points achieves minimal – still called Fermat Problem.
- Find a shortest network interconnecting p given points on the Euclidean plane – called Steiner Problem in the famous book:
  - Richard Courant and Herbert Robbins, *What is Mathematics?*, Oxford University Press, 1941.



Angle condition

No two edges on a SMT can meet at a point with angle less than $120°$.

Node's degree

Each node has degree between 1 and 3.

Angle condition

No two edges on a SMT can meet at a point with angle less than 120°.

Terminal's degree

Each terminal has degree between 1 and 3.

Steiner point's degree

Each Steiner point has degree equal to 3.

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Terminal's degree

Each terminal has degree between 1 and 3.

Steiner point's degree

Each Steiner point has degree equal to 3.

Number of Steiner Points

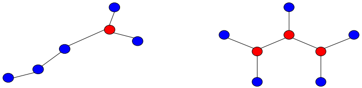
Given p points $x^i \in \mathbb{R}^n$, $i = 1, 2, \ldots, p$, the *maximum number of Steiner points* is $p - 2$.

A Steiner Tree (ST) is a tree that contains the p given terminals and possible k additional Steiner points, such that:

- No two edges meet at a point with angle less than 120°.
- Each terminal point has degree between 1 and 3.
- Each Steiner point has degree equal to 3.
- $k \leq p - 2$.

A Full Steiner Tree (FST) is an ST with the maximum $p - 2$ Steiner points. Each terminal is of degree one in an FST.

A Steiner Topology (Full Steiner Topology) is a topology that meets the degree requirements of an ST (FST).



The Euclidean Steiner problem (ESP)

- The optimization problem
  - GIVEN: A set P of terminals in Euclidean plane.
  - FIND: A Steiner tree of shortest length spanning P.

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  - GIVEN: A set P of terminals in Euclidean plane and an integer B.
  - DECIDE: Is there a Steiner tree T that spans P such that $|T| \leq B$?

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- The discrete decision problem
  - GIVEN: A set P of terminals with integer coordinates in the Euclidean plane and integer B.
  - DECIDE: Is there a Steiner tree T that spans P such that , such that all Steiner points have integer coordinates, and the discrete length of T is less than or equal to B, where the discrete length of each edge of T is the smallest integer not less than the length of that edge ?

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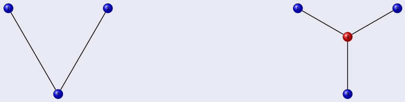
Garey, Grahan and Jonhnson (1977)

- The ESP has been shown to be NP-Hard.

Arora (1998)

- A polynomial time approximation scheme (PTAS) for the ESP exists.

- Let SMT(P) be the length of the Steiner Minimal Tree on the set of terminals P.
- Let MST(P) be the length of the Minimum Spanning Tree on the set of terminals P.
- Let
$$\rho(P) := \frac{\mathrm{SMT}(P)}{\mathrm{MST}(P)}.$$
- Clearly $\rho(P) \leq 1$, for all P.
- If P is the set of the three corners of an equilateral triangle,



then
$$\rho(P) = \frac{\sqrt{3}}{2} \approx 0.866.$$

Gilbert-Pollak Conjecture (for $\mathbb{R}^2$)

$$\rho = \inf_P \rho(P) = \sqrt{3}/2.$$

For small p

- Gilbert and Pollak (1968) - Proof for $p = 3$.
- Pollak (1978) - Proof for $p = 4$.
- Du, Yao annd Hwang(1982); Booth(1991); Freld and Widmayer (1989); Rubinstein - Proof for $p = 5$.

For general p

- F. R. K. Chung and R. L. Graham: A new bound for the Steiner minimal trees, Ann. N.Y. Acad. Sci. 440, 325-346, 1985. ($\rho \geq .834$).
- D-.Z Du and F.K. Hwang: The Steiner ratio conjecture of Gilbert–Pollak is true. Proc. Natl. Acad. Sci. USA 87, 9464–9466, 1990.
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- N. Innami, B. H. Kim, Y. Mashiko, K. Shiohama: The Steiner Ratio Conjecture of Gilbert-Pollak May Still Be Open. Algorithmica, 57(4):869-872, 2010.
- A.O. Ivanov · A.A. Tuzhilin: Algorithmica (2012) 62: The Steiner Ratio Gilbert–Pollak Conjecture Is Still Open. Algorithmica, 62:630–632, 2012.

The minimum Steiner ratio is achieved at the corners of the n-dimensional regular simplex.

Smith (1992) computed the Steiner Ratio for the n-dimensional regular simplex and for the n-dimensional regular octahedron for all $n = 3, \ldots, 9$. The former is always larger than the latter. Disproof of the conjecture for those n.

Extensive literature elucidating properties of SMTs in the plane that do not extend to $n > 2$.

1961 Melzak.

1985 Winter – GeoSteiner Algorithm

2001 Warme, Winter and Zacharisen - version 3.1 of the GeoSteiner (10000 terminals solved).

GeoSteiner Algorithm

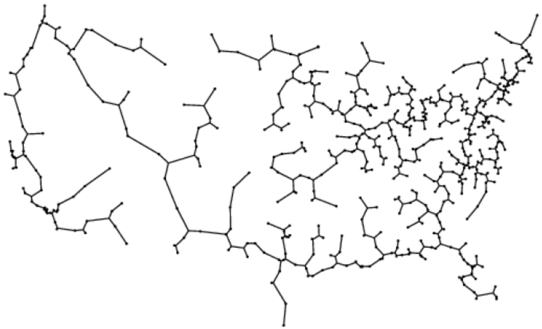

Figure 6: 532 cities in the United States

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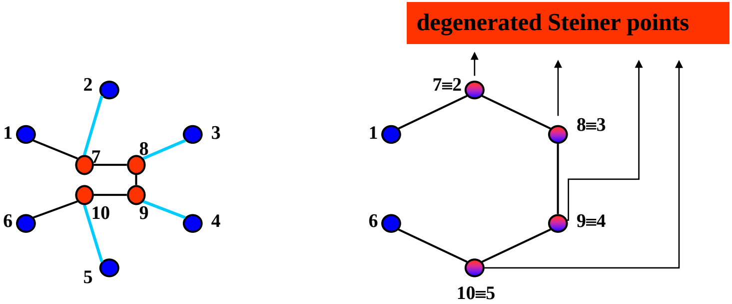
Gilbert and Pollak (1968)

- Find all Steiner topologies on the p given terminals and k Steiner points, with $k \leq p - 2$.
- For each topology optimize the coordinates of the Steiner points.
- Output: the shortest tree found.

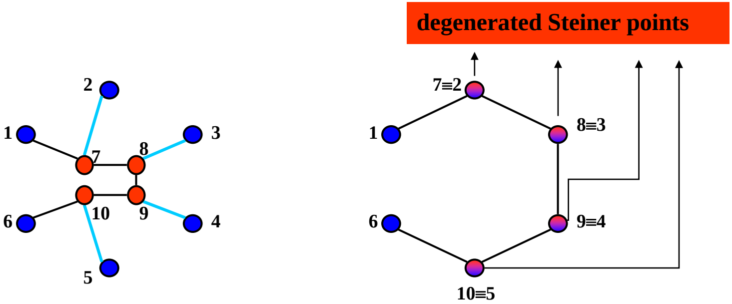
A topology is called a degeneracy of another if the former can be obtained from the latter by shrinking edges.



Degenerate Steiner Topologies

A topology is called a degeneracy of another if the former can be obtained from the latter by shrinking edges.

Fact: each Steiner topology is either a full Steiner topology or a degeneracy of a full Steiner topology.



Gilbert and Pollak

- Find all the *FULL* Steiner topologies on the p given terminals and k Steiner points, with $k = p - 2$.
- For each topology optimize the coordinates of the Steiner points.
- Output: the shortest tree found.

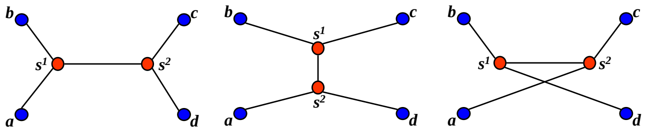
Number of Topologies (Gilbert and Pollak)

If we consider p given points in $\mathbb{R}^n$ and $k \in \{0, 2, \ldots, p-2\}$ Steiner points, the total number of different topologies with k Steiner points is

$$C_{p,k+2}\frac{(p+k-2)!}{k!2^k}.$$

When $k = p - 2$ (full Steiner topologies) the above relation will bi written as $t(p) := 1 \cdot 3 \cdot 5 \cdot 7 \ldots (2p-5) = (2p-5)!!$.

$t(2) = 1,\ t(4) = 3,\ t(6) = 105,\ t(8) = 10395,\ t(10) = 2{,}027{,}025,\ t(12) = 654{,}729{,}075$

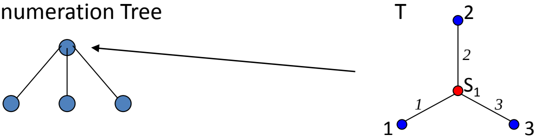


Three different topologies for a graph with 4 terminals and 2 Steiner points

Smith (1992)

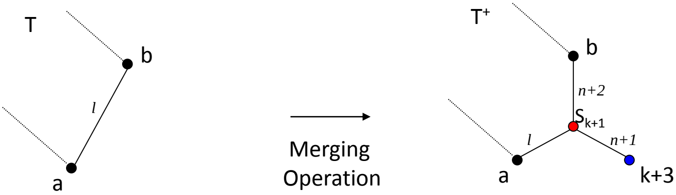
An implicit enumeration scheme to generate full Steiner topologies and a numerical algorithm to solve the ESP for a given topology.

Enumeration Tree



Tree with k Steiner points,
k+2 terminals and n edges (n=2k+1)

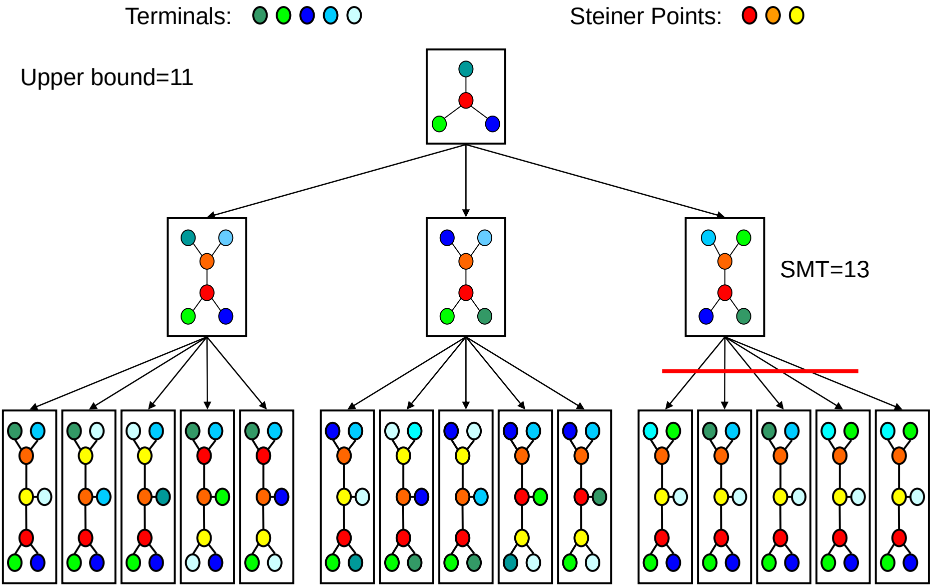
Tree with k+1 Steiner points,
k+3 terminals and n+2 edges



Enumeration Tree - Smith (1992)

- Nodes at level k of tree enumerate full Steiner topologies with k Steiner points and $k+2$ terminals, $k = 1, \ldots, p-2$.
- Children of a given node are obtained by merging a new terminal node with each arc in current FST.

- **Good**: Merging operation cannot decrease minimum length of FST - allows pruning!
- **Bad**: No easy way to account for effect of missing terminal nodes.
- **Ugly**: Growth of tree is super-exponential with depth, and problems get larger at deeper levels.


Terminals:
Steiner Points:
Upper bound=11
SMT=13

Topology T with k Steiner points,
k+2 terminals and n edges (n=2k+1)

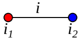
$$(P) \quad \begin{array}{ll} \text{Min} & \sum_{i=1}^n ||s_i|| \\ \text{s.t.} & A_i^T y + s_i = c_i, \quad i = 1, \ldots, n \end{array}$$

$$(D) \quad \begin{array}{lll} \text{Max} & \sum_{i=1}^n c_i^T x_i & \\ \text{s.t.} & \sum_{i=1}^n A_i x_i = 0 & \\ & ||x_i|| \leq 1, & i = 1, \ldots, n \end{array}$$

where:

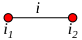
$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix}$$

Case 1:

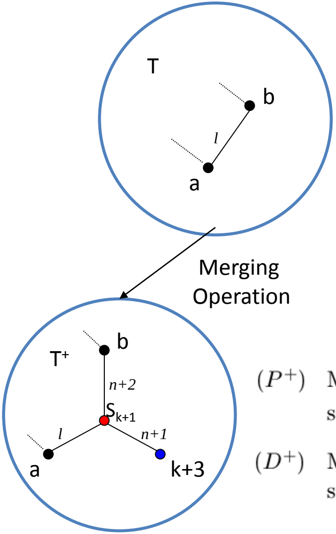


$$c_i = t_i \qquad A_i^T = \begin{array}{c} i_2 \\ \begin{pmatrix} \quad & I_d & \quad \end{pmatrix} \end{array}$$

Case 2:



$$c_i = 0_d \qquad A_i^T = \begin{array}{cc} i_1 & i_2 \\ \multicolumn{2}{c}{\begin{pmatrix} -I_d & I_d \end{pmatrix}} \end{array}$$



$$
\begin{array}{lll}
(P) & \text{Min} & \sum_{i=1}^{n} \|s_i\| \\
 & \text{s.t.} & A_i^T y + s_i = c_i, \quad i = 1, \ldots, n
\end{array}
$$

$$
\begin{array}{lll}
(D) & \text{Max} & \sum_{i=1}^{n} c_i^T x_i \\
 & \text{s.t.} & \sum_{i=1}^{n} A_i x_i = 0 \\
 & & \|x_i\| \leq 1, \qquad i = 1, \ldots, n
\end{array}
$$

SMT(T)

$$
\begin{array}{lll}
(P^+) & \text{Min} & \sum_{i=1}^{n+2} \|s_i\| \\
 & \text{s.t.} & A_i^{+T} y^+ + s_i = c_i^+, \quad i = 1, \ldots, n+2
\end{array}
$$

$$
\begin{array}{lll}
(D^+) & \text{Max} & \sum_{i=1}^{n+2} c_i^{+T} x_i \\
 & \text{s.t.} & \sum_{i=1}^{n+2} A_i^+ x_i = 0 \\
 & & \|x_i\| \leq 1, \qquad i = 1, \ldots, n+2
\end{array}
$$

SMT(T$^+$)

Let:

$$x_i = \bar{x}_i, \quad i \in \bar{E} \subset \{1,\ldots,n\}$$

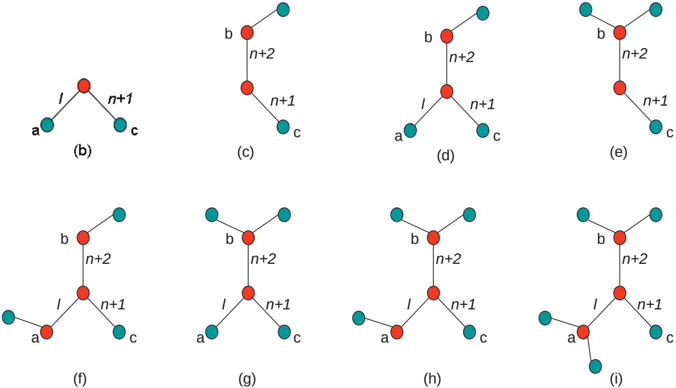
$$\begin{array}{lll} (\bar{D}^+) & \text{Max} & \bar{v} + \sum_{i\in E^+} {c_i^+}^T x_i \\ & \text{s.t.} & \sum_{i\in E^+} A_i^+ x_i = b^+ \\ & & ||x_i|| \leq 1, \qquad\qquad i \in E^+ \end{array}$$

where:

$$E^+ = \{1,\ldots,n+2\} \setminus \bar{E}, \quad b = -\sum_{i\in\bar{E}} A_i^+ \bar{x}_i, \quad \bar{v} = \sum_{i\in\bar{E}} {c_i^+}^T \bar{x}_i$$

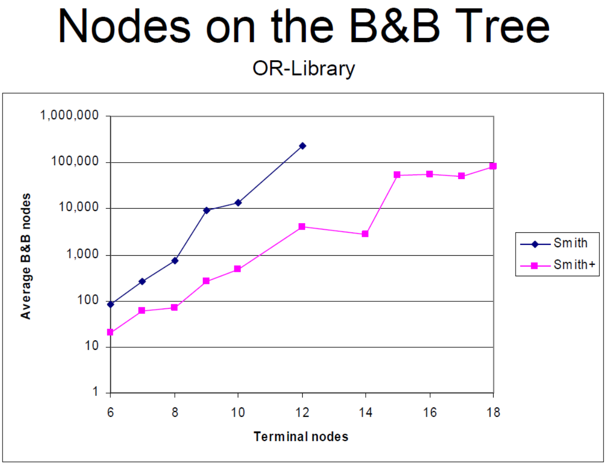
$$z^*(\bar{D}^+) \leq z^*(D^+) = SMT(T^+)$$

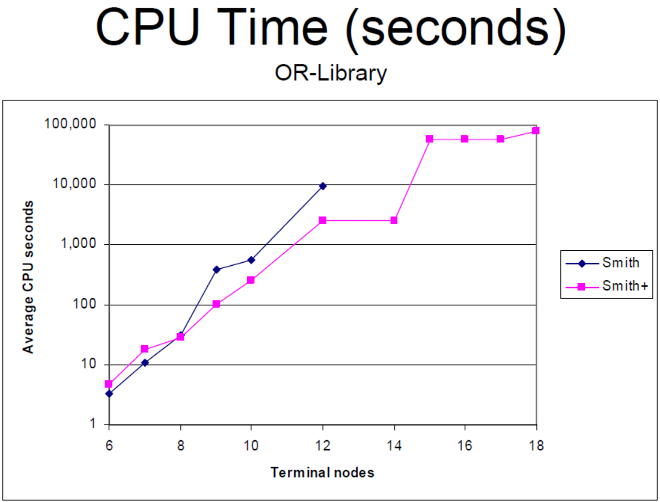
Topologies for the Subproblems



F&A uses conic interior-point code (MOSEK) to obtain bounds on minimum length tree for given topology. Also use MOSEK to solve subproblems with fixed dual variables.

Choose next terminal node to add so as to minimize number of children created/maximize sum of child bounds (strong branching). Smith enumeration argument was extended to allow for varying order in which terminals are added.


Nodes on the B&B Tree
OR-Library
1,000,000
100,000
10,000
1,000
100
10
1
Average B&B nodes
6
8
10
12
14
16
18
Terminal nodes
Smith
Smith+


CPU Time (seconds)
OR-Library
100,000
10,000
1,000
100
10
1
Average CPU seconds
6
8
10
12
14
16
18
Terminal nodes
Smith
Smith+

Effect of dimension *d*

Dimension	Average B&B nodes			Average CPU seconds		
d	Smith	Smith+	Factor	Smith	Smith+	Factor
2	16,821.6	105.0	160.2	717.8	68.4	10.5
3	55,222.2	1,652.4	33.4	2,334.1	753.5	3.1
4	368,762.8	13,685.6	26.9	16,153.0	5,735.6	2.8
5	470,321.8	9,250.0	50.8	20,805.6	4,680.5	4.4

Average nodes/time for 5 randomly-generated instances with 10 terminals in R^d (d=2,3,4).

Conclusions

Conic formulation provides rigorous bounds.

Fixing dual variables allows for estimate of effect of next merge via solution of smaller problem.

New setting for strong branching; effective in reducing size of the tree.

Key problem with use of Smith's enumeration scheme is approximating the effect of terminals that are not present in partial Steiner trees.

Maculan, Michelon, Xavier (2000)

Given p points in $\mathbb{R}^n$, let $G = (V, E)$ be a special graph.
Define

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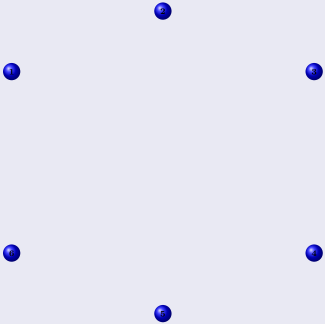
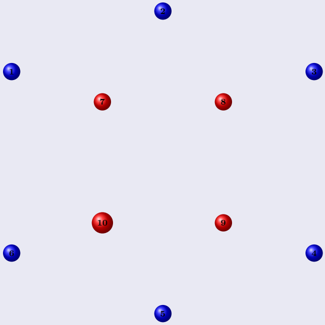
Maculan, Michelon, Xavier (2000): an example with $p = 6$

- 6 given points;



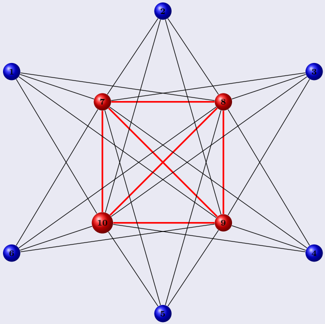
Maculan, Michelon, Xavier (2000): an example with $p = 6$

- 6 given points;
- 4 Steiner points;



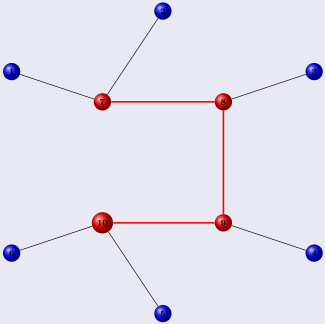
Maculan, Michelon, Xavier (2000): an example with $p = 6$

- 6 given points;
- 4 Steiner points;
- all possible edges;



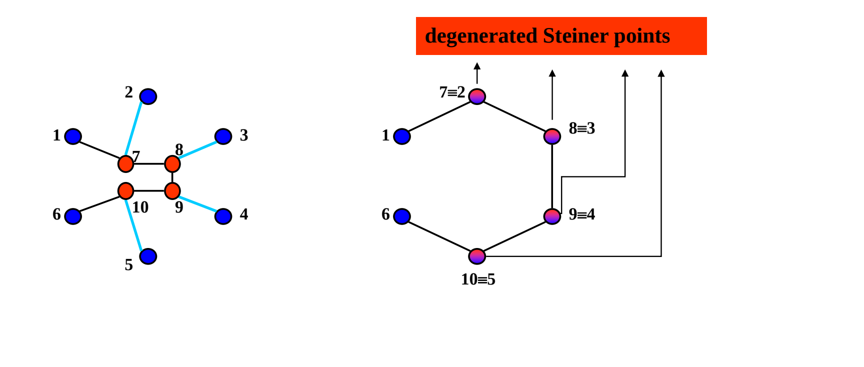
Maculan, Michelon, Xavier (2000): an example with $p = 6$

- 6 given points;
- 4 Steiner points;
- all possible edges;
- a feasible solution;



Maculan, Michelon, Xavier (2000): an example with $p = 6$

- 6 given points;
- 4 Steiner points;
- all possible edges;
- a feasible solution;
- the optimal solution;



Maculan, Michelon, Xavier (2000)

(P): Minimize $\sum_{[i,j]\in E} ||x^i - x^j|| y_{ij}$
subject to

$$\sum_{j\in S} y_{ij} = 1, \quad i \in P, \tag{1}$$

$$\sum_{i\in P} y_{ij} + \sum_{k<j, k\in S} y_{kj} + \sum_{k>j, k\in S} y_{jk} = 3, \quad j \in S, \tag{2}$$

$$\sum_{k<j, k\in S} y_{kj} = 1, \quad j \in S - \{p+1\}, \tag{3}$$

$$x^i \in \mathbb{R}^n, \quad i \in S, \tag{4}$$

$$y_{ij} \in \{0,1\}, \quad [i,j] \in E, \tag{5}$$

where $||x^i - x^j|| = \sqrt{\sum_{l=1}^n (x_l^i - x_l^j)^2}$ is the Euclidean distance between x^i and x^j.

Instituto Alberto Luiz Coimbra de Pós-Graduação e Pesquisa de Engenharia UFRJ

Fampa and Maculan (2004)

Given a^i terminals in $\mathbb{R}^n$, $i = 1, \ldots, p$.

- Steiner vertices x^i, $i = p+1, \ldots, 2p-2$, are in the **convex hull** of the p terminals.
- Let $M = maximum\{||a^i - a^j|| \text{ for } 1 \leqslant i \leqslant j \leqslant p\}$.
- Thus, we have $||a^i - x^j|| \leqslant M$, $[i,j] \in E_1$.
- And $||x^i - x^j|| \leqslant M$, $[i,j] \in E_2$.
- We considerer that $(a^i)_k \geqslant 0$, for $i = 1, \ldots, p$ and $k = 1, \ldots, n$.
- We also have $(x^i)_k \geqslant 0$, for $i = p+1, \ldots, 2p-2$ and $k = 1, \ldots, n$.

Fampa and Maculan (2004)

$$(\mathbf{P}): \quad \textbf{Minimize} \sum_{[\mathbf{i},\mathbf{j}]\in \mathbf{E}} \mathbf{d_{ij}} \textbf{ subject to} \tag{6}$$

$$d_{ij} \geqslant ||a^i - x^j|| - M(1-y_{ij}),\ [i,j] \in E_1, \tag{7}$$

$$d_{ij} \geqslant ||x^i - x^j|| - M(1-y_{ij}),\ [i,j] \in E_2, \tag{8}$$

$$d_{ij} \geqslant 0,\ [i,j] \in E \tag{9}$$

$$\sum_{j\in S} y_{ij} = 1,\quad i \in P, \tag{10}$$

$$\sum_{i\in P} y_{ij} + \sum_{k<j,k\in S} y_{kj} + \sum_{k>j,k\in S} y_{jk} = 3,\quad j \in S, \tag{11}$$

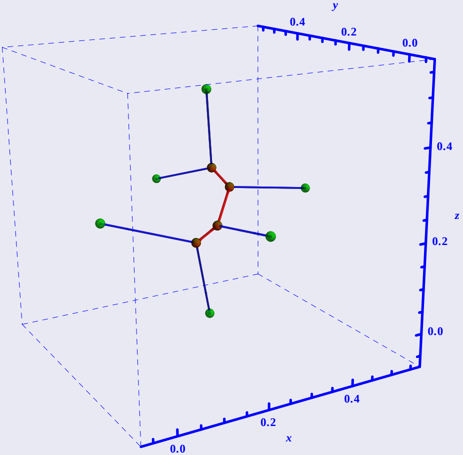
$$\sum_{i<j,i\in S} y_{kj} = 1,\quad j \in S - \{p+1\}, \tag{12}$$

$$x^i \in \mathbb{R}^n,\ i \in S, \tag{13}$$

$$y_{ij} \in \{0,1\},\quad [i,j] \in E, \tag{14}$$

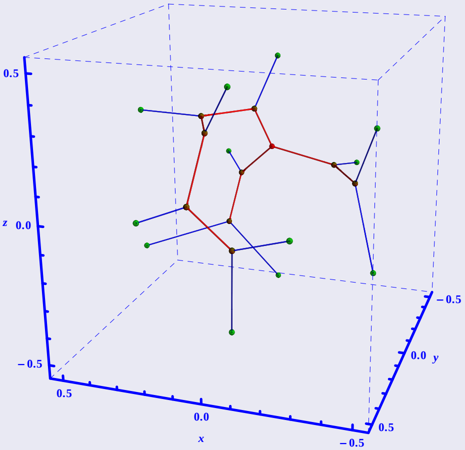
$$d_{ij} \in R. \tag{15}$$

Fampa and Maculan (2004): One Solution for an Octahedron



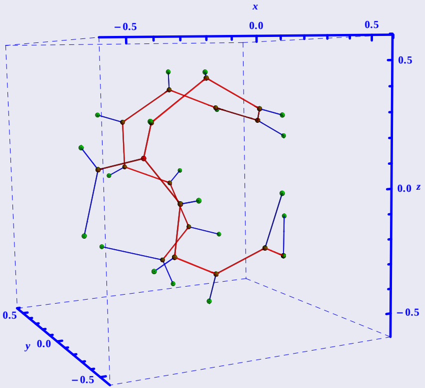
- Number of Points (Green): 6
- Number of Steiner Points (Red): 4
- Objective Function: 1.4341
- Execution Time: 53.53 s

Fampa and Maculan (2004): One Solution for an Icosahedron



- Number of Points (Green): 12
- Number of Steiner Points (Red): 10
- Objective Function: 5.0351
- Execution Time: 1 day.

Fampa and Maculan (2004): One Solution for a Dodecahedron



- Number of Points (Green): 20
- Number of Steiner Points (Red): 18
- Objective Function: 6.7796
- Execution Time: 10 s (*)

Thank you!

fampa@cos.ufrj.br