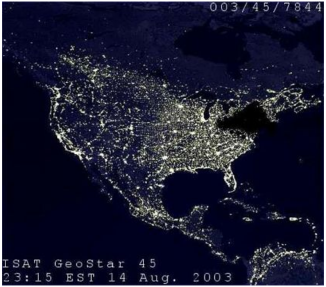
Valid Inequalities for Optimal Transmission Switching



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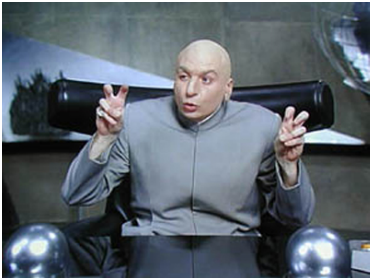
This is a "Power Systems" Talk



- But I don't know much about power systems
- I'm mostly here to evalgalize about structured, mathematical approaches to discrete optimization problems

Economic Dispatch

- Focus today is on a simple problem of meeting demand for power at minimum cost

Power Grid Networks Look Weird

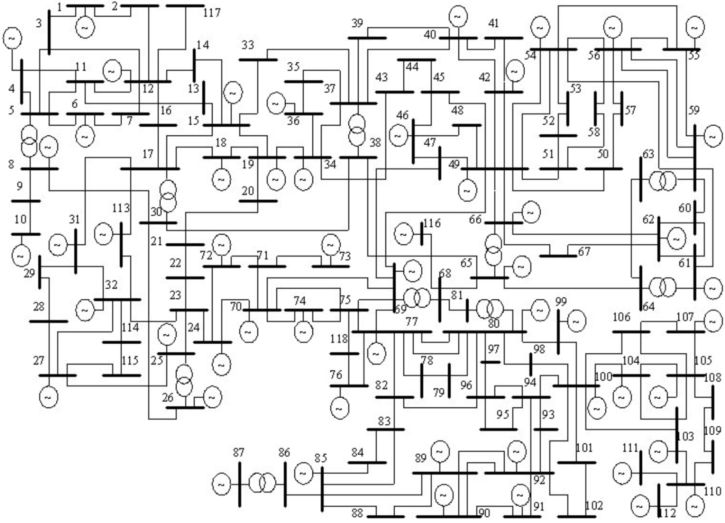
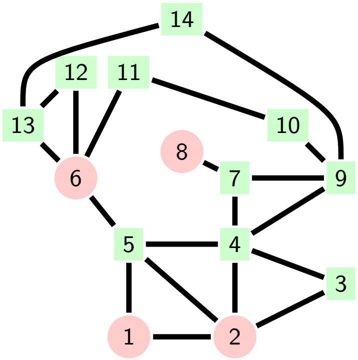


Рис.1. IEEE тестовая схема, состоящая из 118 узлов

It's Just a Network

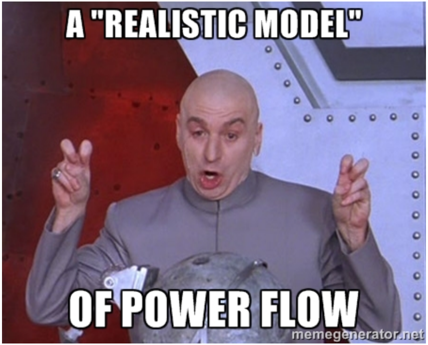


- Power Network: (N, A) with...
- $G \subset N$: generation nodes
- $D \subset N$: demand nodes
- Load forecasts (MW) b_i for $i \in D$
- Generation cost (\$/MW) c_i and capability $\overline{p}_i$ (MW) for $i \in G$
- Peak load rating (MW) u_{ij} for $(i, j) \in A$

Economic Dispatch Problem

- Determine power generation levels for $i \in G$ and power transmission levels for $(i, j) \in A$ to meet demands $b_i, i \in D$, at minimum cost

Power Flow



- Electric power grids follow the laws of physics, characterized by nonlinear, nonconvex equations
- Direct control is difficult—We cannot dictate how power will flow.
- In Alternating Current (AC) circuits, key physical quantities (voltage V_i, power P_{km}) are complex numbers

$$V_k = U_k e^{j\theta_k}$$
$$P_{km} = p_{km} + jq_{km}$$

- Power flow on a line is given by the AC Power Flow Equations:

$$p_{km} = g_{km}U_k^2 - g_{km}U_kU_m\cos(\theta_k - \theta_m) - b_{km}U_kU_m\sin(\theta_k - \theta_m)$$
$$q_{km} = -(b_{mk} + b^s_{km})U_k^2 + b_{km}U_kU_m\cos(\theta_k - \theta_m) - g_{km}U_kU_m\sin(\theta_k - \theta_m)$$

Yeah for Engineers!

- Assume all voltage magnitudes are very close to 1:

$$U_k = 1 \quad \forall k \in N$$

- Assume that imaginary (reactive) power is negligible ($q_{km} \approx 0$)
- Assume voltage angle differences between adjacent buses $(\theta_k - \theta_m)$ are "small", so that

$$\sin(\theta_k - \theta_m) \approx (\theta_k - \theta_m)$$
$$\cos(\theta_k - \theta_m) \approx 1$$

- Then we can model power flow as a set of linear equations

ELL—Engineers Love to Linearize

Variables

- p_i: (Real) power inject at generator $i \in G$
- x_{ij}: (Real) power flow on line $(i, j) \in A$
- θ_i: Voltage angle at node $i \in N$

DC Power Flow Assumption

- The (real) power transmit over line $(i, j) \in A$ is proportional to angle differences at the endpoint nodes $i \in N$ and $j \in N$.

$$x_{ij} = \alpha_{ij}(\theta_i - \theta_j)$$

Linear Program for (DC) Economic Dispatch

- Minimize cost of producing and delivering electricity to meet demands

$$\min_{x,p,\theta} \sum_{i\in G} c_i p_i$$

$$\text{s.t.} \quad \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ji} = \begin{cases} p_i & \forall i \in G \\ d_i & \forall i \in D \\ 0 & \forall i \in N \setminus G \setminus D \end{cases}$$

$$-u_{ij} \le x_{ij} \le u_{ij} \quad \forall (i,j) \in E$$

$$\underline{p}_i \le p_i \le \overline{p}_i \quad \forall i \in G$$

$$x_{ij} = \alpha_{ij}(\theta_i - \theta_j) \quad \forall (i,j) \in E$$

$$x_{ij} \in \mathbb{R} \quad \forall (i,j) \in E$$

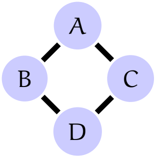
$$p_i \in \mathbb{R}_+ \quad \forall i \in G$$

$$\theta_i \in \mathbb{R} \quad \forall i \in N$$

- x, θ need not be ≥ 0

MCNF++

- This economic dispatch problem is just a min cost network flow problem with some additional “potential” constraints
- The potential drop $(\theta_A - \theta_D)$ must be the same aloing the paths: $A \to B \to D$ and $A \to C \to D$



“Braess Paradox”

- If line (C, D) didn’t exist, I wouldn’t have to enforce this potential balance constraint.
- Thus, removing lines of the transmission network may actually increase the efficiency of delivery.

Transmission Switching

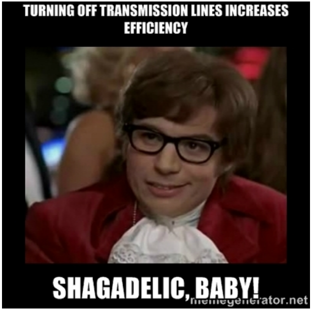
Tradeoff

- Having Lines Allows You to Send Flow:

$$-U_{ij} \leq x_{ij} \leq U_{ij} \ \forall (i,j) \in E$$

- Having Lines Induces Constraints in the Network:

$$x_{ij} = \alpha_{ij}(\theta_i - \theta_j) \ \forall (i,j) \in E$$



- Fisher, O'Neill & Ferris ('08) show that efficiency improved by switching off transmission lines

Lines Off	% Improvement
1	6.3%
2	12.4%
3	19.9%
4	20.5%
∞	24.9%

The \$64(M?) Question

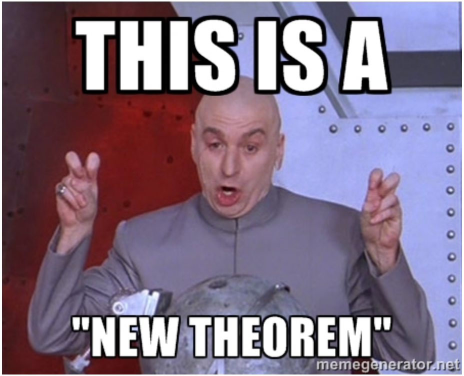
Very Good Questions

1. Which lines should we turn off to maximize efficiency?
2. Is it easy or hard to determine an optimal set of lines?

DC Transmission Switching

- **Given:** A network $G = (N, A)$ with arc capacities and susceptances $(u_{ij}, \alpha_{ij})\ \forall (i,j) \in A$, generation levels $p_i\ \forall i \in G$, demand levels $b_i \forall i \in D$.
- **Question:** Does there exist a subset of arcs $S \subseteq A$ such that deactivating arcs in S leads to a *feasible* DC power flow?

“New” Result



Theorem

DC Transmission Switching is NP-Complete

- This (and other) complexity results appear in the recent paper by Lehmann, Grastien, and Van Hentenryck ('14).
- Dan Bienstock told us he proved this a while ago, but never wrote it up

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- Reduction from subset-sum
- The problem remains hard...
  - Even if there are a polynomial number of cycles in the network
  - Even on a series-parallel graph with only one supply/demand pair

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- So the problem is “hard...”

My Most Favorited Tweet Ever (Besides Human Pyramid Pictures)

- Please do not give up on a problem and resort to a heuristic[1] just because a problem is NP-Complete
- Of course, the best way to attack every NP-Complete problem is to write an integer programming formulation

[1] or even worse a meta-heuristic

Switching Off Lines

- Regular Flow Constraints

$$x_{ij} = \alpha_{ij}(\theta_i - \theta_j) \quad \forall (i,j) \in E$$
$$-U_{ij} \leq x_{ij} \leq U_{ij} \quad \forall (i,j) \in E$$

- Let $z_{ij} \in \{0,1\} \; \forall (i,j) \in A$, Switched Flow Constraints

$$x_{ij} = \alpha_{ij} z_{ij} (\theta_i - \theta_j) \quad \forall (i,j) \in E$$

- If (and only if) $\theta_i - \theta_j$ is bounded, one can write an MILP formulation
- $z_{ij} = 1 \Leftrightarrow$ line $(i,j) \in A$ is used

MILP Formulation

- This is the "big-M" formulation of Fisher, O'Neil, and Ferris '08:

$$
\begin{aligned}
&\min_{x,p,\theta,z} \quad \sum_{i \in G} c_i p_i \\
\text{s.t.} \quad & \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ij} = \begin{cases} p_i & \forall i \in G \\ d_i & \forall i \in D \\ 0 & \forall i \in N \setminus G \setminus D \end{cases} \\
& -U_{ij} z_{ij} \le x_{ij} \le U_{ij} z_{ij} \quad \forall (i,j) \in E \\
& \alpha_{ij}(\theta_i - \theta_j) - x_{ij} + M(1 - z_{ij}) \ge 0 \quad \forall (i,j) \in E \\
& \alpha_{ij}(\theta_i - \theta_j) - x_{ij} - M(1 - z_{ij}) \le 0 \quad \forall (i,j) \in E \\
& -L_i \le \theta_i \le L_i \quad \forall i \in N \\
& \underline{p}_i \le p_i \le \overline{p}_i \quad \forall i \in G \\
& z_{ij} \in \{0, 1\} \quad \forall (i,j) \in E
\end{aligned}
$$

Throwing Down the Gauntlet

- Hedman, Ferris, O'Neill, Fisher, Oren, (2010) state

  > *"When solving the transmission switching problem, ... the techniques for closing the optimality gap, specifically improving the lower bound, are largely ineffective."*

- So they resort to a variety of heuristic, ad-hoc techniques to get good solutions to the MILP they propose

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- My good colleague and continuous optimizer Michael Ferris ignores my previous plea to not resort to heuristics
- You will later see that CPLEX v12 is already orders of magnitude better than CPLEX v9 on DC transmission switching instances
- But still it's not good enough for large-scale networks...
- Thus we have...

Ferris's Challenge to Integer Programmers

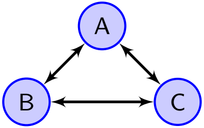
Solve realistically-sized DC transmission switching instances to provable optimality

- As integer programmers, we would like to rise to the challenge, and improve these "ineffective" lower bound techniques.

The IP Way

- We study the mathematical structure of the problem, create a useful relaxation of the problem, and improve our description of the relaxation through cutting planes (facets)

Key (Simple) Insight?!



- Assume (WLOG) that $\alpha_{ij} = 1$
  - We can just set $x_{ij} = \alpha_{ij} x'_{ij}$ and scale u_{ij} by α_{ij}
- Then we have...

$$
\begin{array}{r}
x_{AB} = \theta_A - \theta_B \\
x_{BC} = \theta_B - \theta_C \\
x_{CA} = \theta_C - \theta_A \\
\hline
x_{AB} + x_{BC} + x_{CA} = 0
\end{array}
$$

- The potential constraints essentially (only) enforce that flow around a cycle is zero.
  - If you didn't forget everything from your introductory electrical engineering class (like I did), then you will recognize this as Kirchoff's Voltage Law.

Insight

- We should focus on what goes on around a cycle and try to model this in a better way

The "IP" Way

Simple IP People (like me) Like Simple Sets

- Directed cycle $G = (V, C)$, with $V = [n]$, $C = \{(i, i+1) \mid \forall i \in [n-1]\} \cup \{(n, 1)\}$:

$$\mathcal{C} = \Big\{ (x, \theta, z) \in \mathbb{R}^{2n} \times \{0,1\}^n \mid -u_{ij} \leq x_{ij} \leq u_{ij} \ \forall (i,j) \in C \\ z_{ij}(\theta_i - \theta_j) = x_{ij} \ \forall (i,j) \in C \Big\}$$

- The inequalities in this set model the potential drop across each arc in a cycle
- This is a relaxation
  1. Flow balance is ignored
  2. Everything outside the cycle is ignored
- Even though $\mathcal{C}$ has the "nonlinear" equations $z_{ij}(\theta_i - \theta_j) = x_{ij}$, it is the union of 2^n polyhedra, so $\operatorname{cl}\operatorname{conv}(\mathcal{C})$ is a polyhedron.

The IP Way—Structure, Structure, Structure!

- Even though $\mathcal{C}$ is just a relaxation of the true problem, we hope that my generating valid inequalities for $\mathcal{C}$, we can improve performance of IP approaches

Now We Do Math

$$\mathcal{C} = \Big\{ (x, \theta, z) \in \mathbb{R}^{2n} \times \{0,1\}^n \mid -u_{ij} \le x_{ij} \le u_{ij} \;\; \forall (i,j) \in C$$
$$z_{ij}(\theta_i - \theta_j) = x_{ij} \;\; \forall (i,j) \in C \Big\}$$

Theorem

For $S \subseteq C$ such that $u(S) > u(C \backslash S)$, the shagadelic-cycle inequalities (SCI)

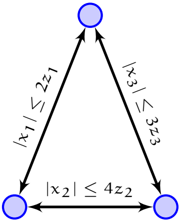
$$x(S) + \sum_{a \in C} \beta_a^S z_a \le b^S \quad (1)$$

$$-x(S) + \sum_{a \in C} \beta_a^S z_a \le b^S \quad (2)$$

are valid for $\mathcal{C}$, where

$$\beta_a^S = u(S \setminus a) - u(C \setminus S) \quad \forall a \in C$$
$$b^S = (n-1)(2u(S) - u(C))$$

Shagadelic-Cycle Inequalities, Example



$$
\begin{aligned}
x_1 + x_2 + z_1 - z_2 + 3z_3 &\le 6 \quad S = \{1,2\}\\
x_1 + x_3 - z_1 + z_2 - 2z_3 &\le 2 \quad S = \{1,3\}\\
x_2 + x_3 + 5z_1 + z_2 + 2z_3 &\le 10 \quad S = \{2,3\}\\
x_1 + x_2 + x_3 + 7z_1 + 5z_2 + 6z_3 &\le 18 \quad S = \{1,2,3\}
\end{aligned}
$$

Logic Enforced

- For $S = \{1,2\}$, if $z_1 = z_2 = 1$, then

$$x_1 + x_2 \le \begin{cases} 6 & z_3 = 0 \\ 3 & z_3 = 1 \end{cases}$$

- For $S = \{1,3\}$, if $z_1 = z_3 = 1$, then

$$x_1 + x_3 \le \begin{cases} 5 & z_2 = 0 \\ 4 & z_2 = 1 \end{cases}$$

Proofs! Yeah, Baby!

Theorem

If $S \subseteq C$, and $u(C \setminus S) < u(S)$, then the shagadelic-cycle inequalities ($\textsc{sci}$) are facet-defining for $\operatorname{cl}\operatorname{conv}(\mathcal{C})$.

- Thus, all 2^n inequalities are necessary in the description of $\operatorname{cl}\operatorname{conv}(\mathcal{C})$

Even More Proofs

- Along with some other trivial inequalities, the shagadelic cycle inequalities are sufficient to describe the convex hull of $\mathcal{C}$

$$
\begin{aligned}
\operatorname{cl}\operatorname{conv}(\mathcal{C}) = \Big\{ (x, \theta, z) \in \mathbb{R}^{3n} \mid \\
-u_{ij} z_{ij} \le x_{ij} \le u_{ij} z_{ij} \ \forall (i,j) \in C \\
z_{ij} \le 1 \qquad \forall (i,j) \in C \\
x(S) + \sum_{a \in C} \beta_a^S z_a \le b^S \ \forall S \subseteq C : u(S) > u(C \setminus S) \\
-x(S) + \sum_{a \in C} \beta_a^S z_a \le b^S \ \forall S \subseteq C : u(S) > u(C \setminus S) \Big\}
\end{aligned}
$$

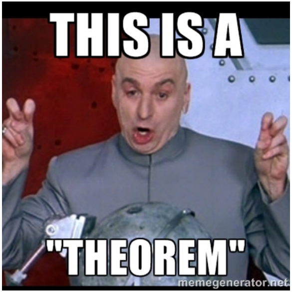
Can We Use the SCI?

- Given solution $\hat{x} \in \mathbb{R}^n_+, \hat{z} \in [0,1]^n$, the separation problem for (SCI) is

$$\max_{C \subseteq A : C \text{ is a cycle}} \max_{S \subseteq C : 2u(S) \geq u(C)} \{\hat{x}(S) + (\beta^S)^\top \hat{z} - b^S\},$$

where

$$\beta^S_a = u(S \setminus a) - u(C \setminus S) \quad \forall a \in C$$
$$b^S = (n-1)(2u(S) - u(C))$$



"Jeffrem"

The separation problem for (SCI) is NP-Hard

- "Jeffrem"—Something that seems like it must be true, but Jeff can't prove it.

Simple Observations

- Observation: If $\sum_{a \in C} \hat{z}_a \leq |C| - 1$, then $(\hat{x}, \hat{z})$ cannot be violated by any $(\textsc{sci})$
- This suggests a two-phase separation heuristic.

Separation Heuristic

1. Find a "necessary cycle" C such that $\sum_{a \in C} \hat{z}_a > |C| - 1$
2. Find $S \subset C$ in the given cycle

- Do (1) by (truncated) enumeration
- Given C, algebra shows that (2) is a knapsack problem:
  - $\hat{\lambda} = |C| - 1 - \sum_{a \in C} \hat{z}_a$
  - $\hat{v}_a = \hat{x}_a + u_a \hat{z}_a - 2u_a (\sum_{e \in C \setminus a} (1 - \hat{z}_e))$

$$\nu = \max_{y \in \{0,1\}^n} \left\{ \sum_{a \in C} \hat{v}_a y_a \mid \sum_{a \in C} u_a y_a \geq \frac{1}{2} u(C) \right\}$$

  - If $\nu + u(C)\hat{\lambda} > 0$, then $(\textsc{sci})$ is violated by $(\hat{x}, \hat{z})$

P=NP

- I Can Solve the Knapsack Problem in Polynomial Time!
- Since I have "proved" that $P = NP$, the Clay Mathematics Institute should pay me...

- Not really, it is just that this specific knapsack problem is easy
- Take the items:

$$S_C^* = \{a \in C \mid \hat{x}_a - u_a \hat{z}_a + 2u_a K_C > 0\}$$

Standard IEEE Benchmark Instances

- Optimal Switching can make some difference in generation cost

	Generation Cost	
Instance	No Switching	With Switching
case3Les	831.63	378.00
case6ww	2959.00	2912.33
case9	1699.21	1552.80
case14	6948.34	6424.00
case_ieee30	6479.51	6373.86
case30	343.15	308.40
case39	1878.27	1878.27
case57	28270.98	25016.00
case118B	1895.11	1505.77
case118	96638.81	91180.00
case300	472068.32	470517.00

But These Are (Now) Too Easy

	no cuts		with cuts	
instance	time	nodes	time	nodes
case3Les	0.161	0	0.160	0
case6ww	0.109	198	0.120	198
case9	0.018	0	0.018	0
case14	0.044	40	0.13	6
case_ieee30	0.088	338	0.110	309
case30	0.012	0	0.023	0
case39	0.006	0	0.019	0
case57	0.325	100	0.523	679
case118B	34.235	39900	13.928	8991
case118	1.960	1171	1.098	699
case300	2.230	510	3.604	820

- Solving the MIP model using CPLEX v12.5
- Case118 was the instance that Ferris *et al.* report not being able to solve with CPLEX (version 9)

Creating More Instances

- Modify the 118B instance by modifying the demands randomly.
- Create 15 new instances

Comparing on Many 118B Instances

	Avg. Time	Avg. Nodes
No Cuts	542.8	102382
With Cuts	40.9	28218

- Cuts show some promise
- We continue to work on pure transmission switching on larger instances (> 2000) nodes.
- These problems are still way too hard for CPLEX with and without cuts
- There are many alternative optimal solutions to the linear programming relaxation—Which one(s)? should we cut off?

Design Instances

- Power grid network design problem.
- One (expensive) generator can supply power to n nodes
- Possibility to "plug in" up to $n/5$ cheaper generators, with fixed cost of constructing new lines
- Also can do transmission switching

- Ten instances (each) of size $n = 30$, $n = 50$.
- Run CPLEX for one hour, record, initial LP Gap, Final LP Gap, and Final Gap
- Report (arithmetic) averages
- All Gaps taken w.r.t. best feasible solution found

Computational Results

CPLEX Cuts Turned On—Gap %

		No (SCI)		With (SCI)	
n	LP	Root	Final	Root	Final
30	10.46	9.52	9.16	9.09	8.90
50	11.88	11.46	11.37	11.14	11.10

	No (SCI)	With (SCI)	
n	#node	#node	# cuts
30	67928.2	1525.5	2074.8
50	6202.3	223.0	759.6

Accomplishments

- Prove that transmission switching problem is NP-Complete
- Understand a "cycle" relaxation derived from the structure of the problem
  - Give a complete description of the convex hull of the set with 2^n inequalities
  - Also have an extended formulation in dimension $6n + 1$
- Even with initial implementation, we can significantly improve default CPLEX behavior

Still To Do

- Working on effective mechanisms for using these inequalities for larger instances
- A special challenge for (pure) transmission switching is the extreme dual generacy of LP solutions—so engineering effective cutting plane mechanisms is important

Up Next

- Study more complicated structures besides cycles—Try to include demands at nodes, for Flow-($\textsc{sci}$)
- Extend to potential preserved, but nonlinear relationship between potential and flow—Gas and Water Network design

The Real Conclusion