Chaotic dynamics in celestial mechanics

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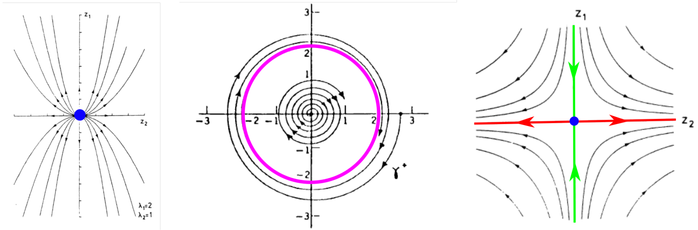
Dynamical systems

A **dynamical system** consists of a phase space whose coordinates describe the state at any instant, and a dynamical rule that specifies the immediate future of all state variables.

- **Continuous dynamical system**: described by a system of differential equations.
- Fundamental problem: understand how different **invariant objects** (fixed points, periodic orbits, invariant tori...) **structure the global dynamics**.

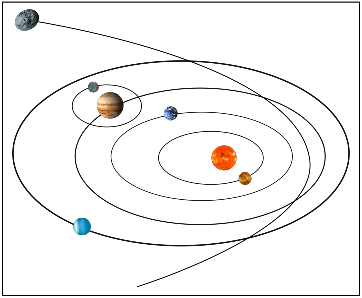


Source: F. Verhulst

The N-Body Problem

One of the classical problems of the discipline is the **N-Body Problem**.

It considers N point masses m_i for $i = 1, .., N$ moving under their mutual gravitational attraction:



$$m_1 \frac{d^2\mathbf{r}_1}{dt^2} = \sum_{j\neq 1} G m_1 m_j \frac{\mathbf{r}_j - \mathbf{r}_1}{\left\|\mathbf{r}_j - \mathbf{r}_1\right\|^3},$$

$$m_2 \frac{d^2\mathbf{r}_2}{dt^2} = \sum_{j\neq 2} G m_2 m_j \frac{\mathbf{r}_j - \mathbf{r}_2}{\left\|\mathbf{r}_j - \mathbf{r}_2\right\|^3},$$

$$\cdots$$

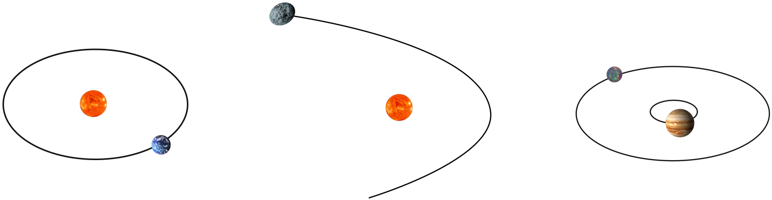
$$m_N \frac{d^2\mathbf{r}_N}{dt^2} = \sum_{j\neq N} G m_N m_j \frac{\mathbf{r}_j - \mathbf{r}_N}{\left\|\mathbf{r}_j - \mathbf{r}_N\right\|^3}.$$

It consists on 6N first-order differential equations.

2-Body Problem

The **2-Body Problem**, also known as the classical Kepler problem:

$$m_1 \frac{d^2\mathbf{r}_1}{dt^2} = G m_1 m_2 \frac{\mathbf{r}_2 - \mathbf{r}_1}{\|\mathbf{r}_2 - \mathbf{r}_1\|^3}, \qquad m_2 \frac{d^2\mathbf{r}_2}{dt^2} = \sum_{j\neq 2} G m_2 m_1 \frac{\mathbf{r}_1 - \mathbf{r}_2}{\|\mathbf{r}_1 - \mathbf{r}_2\|^3},$$



The motion of the 2-Body Problem are given by **planar conic sections** (circles, ellipses, parabolas or hyperbolas on a plane).

- **Integrable system:** systems with enough constant of motions (conserved quantities) to define all the trajectories.

3-Body Problem

Is it the 3-Body Problem integrable?

Source: K. Long. https://botwiki.org/bot/random-three-body-problem

A little bit of history

- In 1887, Oscar II King of Sweden, advised by Gösta Mittag-Leffler, established a prize for anyone who could find the solution to the 3-BP.

> *Given a system of arbitrarily many mass points that attract each according to Newton's law, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly.*

- Poincaré won but later he withdrew his candidature. There were serious errors in his proof.
- In 1890 it was published in a revised form.
- The prize was finally awarded to Poincaré, even though he did not solve the original problem.

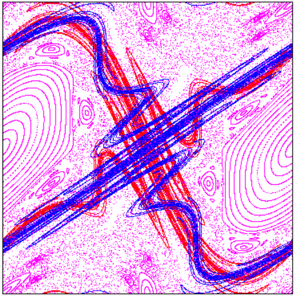
> *"This work cannot indeed be considered as furnishing the complete solution of the question proposed, but that it is nevertheless of such importance that its publication will inaugurate a new era in the history of celestial mechanics". K. Weierstrass.*

Source: Wikipedia.

N-Body Problem

For $N \geq 3$, the N-Body Problem is **not integrable**.

One of the obstructions for integrability is given by the creation of "complex tangles" between certain invariant objects.



Source: C. Simó.

Nowadays, a dynamical system exhibiting this type of behavior is said to display **chaotic dynamics**.

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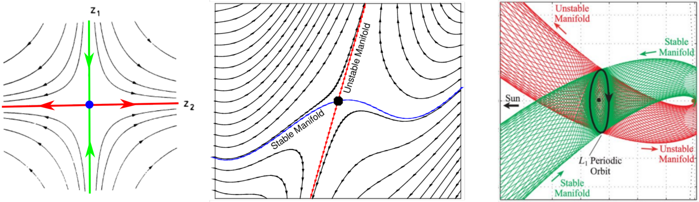
Stable and unstable invariant manifolds

The study of the stable and unstable manifolds of invariant objects reveals a great deal about the dynamics.

Let P be an invariant object (fixed point, periodic orbit,...).

Its **stable set**, $W^{\mathrm{s}}(P)$, is the set of points that tend to P when time $\to +\infty$.

Its **unstable set**, $W^{\mathrm{u}}(P)$, is the set of points that tend to P when time $\to -\infty$.



Source: F. Verhulst, D. Zhigunov, S. Ross

Homoclinic tangle

One of the obstructions for integrability is given by the possibility of **transversal intersections** between the unstable and stable manifolds of periodic orbits.

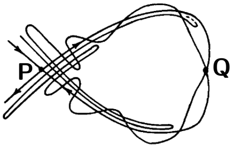
These intersections create a **complex tangle** between the manifolds.

Smale-Birkhoff homoclinic Theorem (1967):

Let $f : U \to \mathbb{R}^2$ be a diffeomorphism with a hyperbolic point **P** and a transverse homoclinic point **Q**.

On a small neighborhood of **P** one can build an **Smale's horseshoe map**: there exists a hyperbolic invariant set $X \subset U$ such that $f|_X$ is conjugated to the shift map

$$\begin{aligned} \sigma : \{0,1\}^{\mathbb{Z}} &\to \{0,1\}^{\mathbb{Z}} \\ (\omega_k)_{k\in\mathbb{Z}} &\mapsto (\omega_{k+1})_{k\in\mathbb{Z}}. \end{aligned}$$



Source: Guckenheimer-Holmes 1983.

As a consequence, f exhibits **chaotic behavior**.

Chaotic dynamics

A **chaotic dynamical system** (R. L. Devaney definition) must satisfy:

- **Sensible dependence with respect to initial conditions**:

  Let $\Phi_t : X \to X$ be the flow of the system and $x \in X$. Then, there exists $y \in X$ such that $d(x, y) < \varepsilon$ such that $d(\Phi_t(x), \Phi_t(y)) > e^{at} d(x, y)$ for some constant $a > 0$.

- **Topologically transitive**:

  Let $\Phi_t : X \to X$ be the flow of the system and non-empty open sets $A, B \subset X$. Then, there exists $T > 0$ such that $\Phi_T(A) \cap B \neq \emptyset$.

- **The periodic orbits are dense**.

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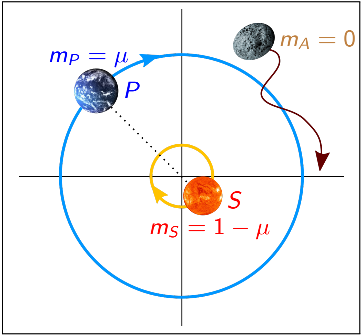
Restricted Planar Circular 3-Body Problem



- **Restricted**: one body is massless, $m_A = 0$.
- **Planar**: the bodies move on the same plane.
- **Circular**: the primaries (m_S and m_P) perform a circular motion.

We study the motion of the **massless body**:

$$(q(t), p(t)) \in \mathbb{R}^4.$$

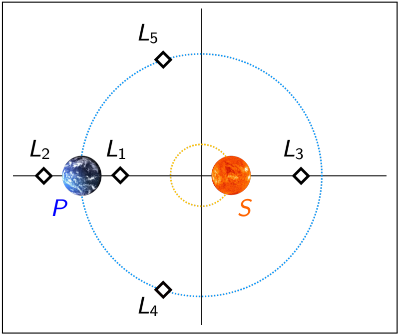
We normalize:

- $m_S = 1 - \mu$ and $m_P = \mu$ with $\mu \in \left(0, \frac{1}{2}\right]$.

Approach: Perturbative study for $0 < \mu \ll 1$.

Rotating framework and Lagrange points

Rotating framework: the primaries are fixed at $(\mu, 0)$ and $(\mu - 1, 0)$.

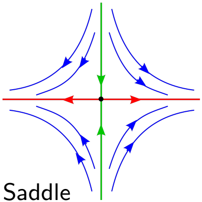


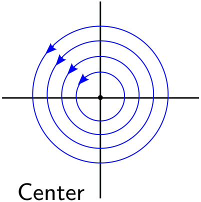
Five critical points: Lagrange points.

For $\mu > 0$ small:

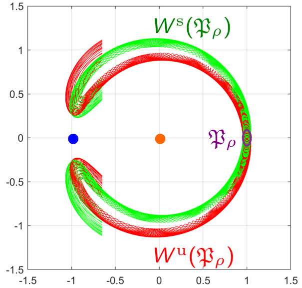
- L_1, L_2 and L_3 saddle-center.
- L_4 and L_5 center-center.

We study chaotic dynamics around L_3





Main result



- Family of **periodic orbits** around L_3:

$$\{\mathfrak{P}_\rho\}_{\rho\in[0,\rho_0]}$$

- 2-dim stable and unstable manifolds, $W^{\mathrm{s}}(\mathfrak{P}_\rho)$ and $W^{\mathrm{u}}(\mathfrak{P}_\rho)$.

Theorem A (Baldomá, G., Guardia, 23'):

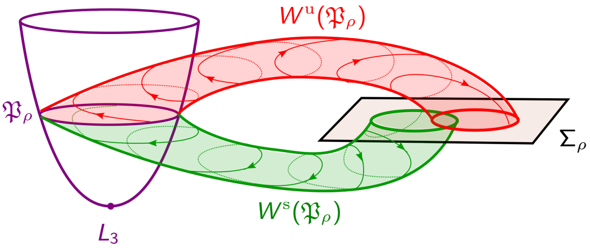
For $0 < \mu \ll 1$, there exist energy levels $\rho_{\text{min}}(\mu), \rho_{\text{max}}(\mu) > 0$ such that for $\rho \in (\rho_{\text{min}}, \rho_{\text{max}}]$ the manifolds $W^{\mathrm{u}}(\mathfrak{P}_\rho)$ and $W^{\mathrm{s}}(\mathfrak{P}_\rho)$ **intersect transversally**.

Thus, there exists a neighborhood around L_3 where one can found **chaotic dynamics**.

Idea of the proof

Restrict to a fixed energy level ρ (3-dim phase space) and take a section transversal to the flow Σ_ρ (2-dim).

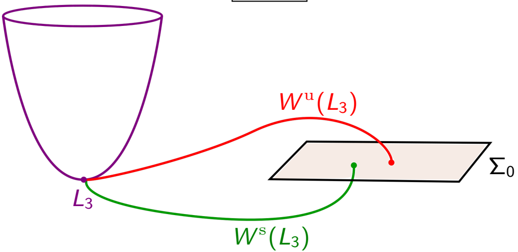
$$\boxed{\rho \in (\rho_{\min}, \rho_{\max}]}$$



Idea of the proof

Restrict to a fixed energy level ρ (3-dim phase space) and take a section transversal to the flow Σ_ρ (2-dim).

$\boxed{\rho = 0}$



Theorem B (Baldomá, Capiński, G., Guardia):

For $0 < \mu \ll 1$, there exists $A > 0$ such that

$$0 \neq \mathrm{dist}_{\Sigma_0}\Big(W^{\mathrm{u}}(L_3), W^{\mathrm{s}}(L_3)\Big) = \mathcal{O}(e^{-\frac{A}{\sqrt{\mu}}}).$$

Beyond all orders phenomenon

Thanks for your attention!