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The Space of Discrete Surfaces without Intersections

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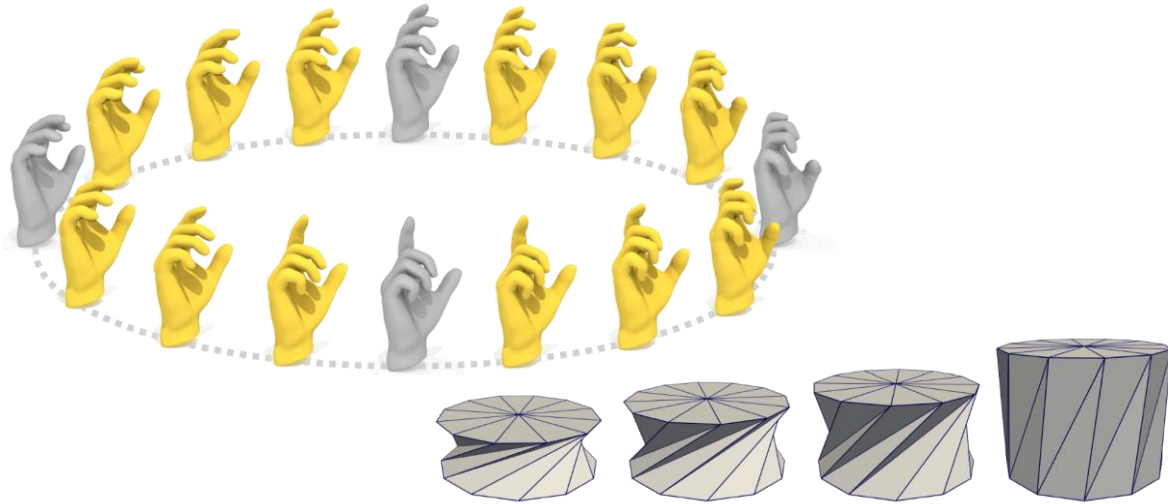
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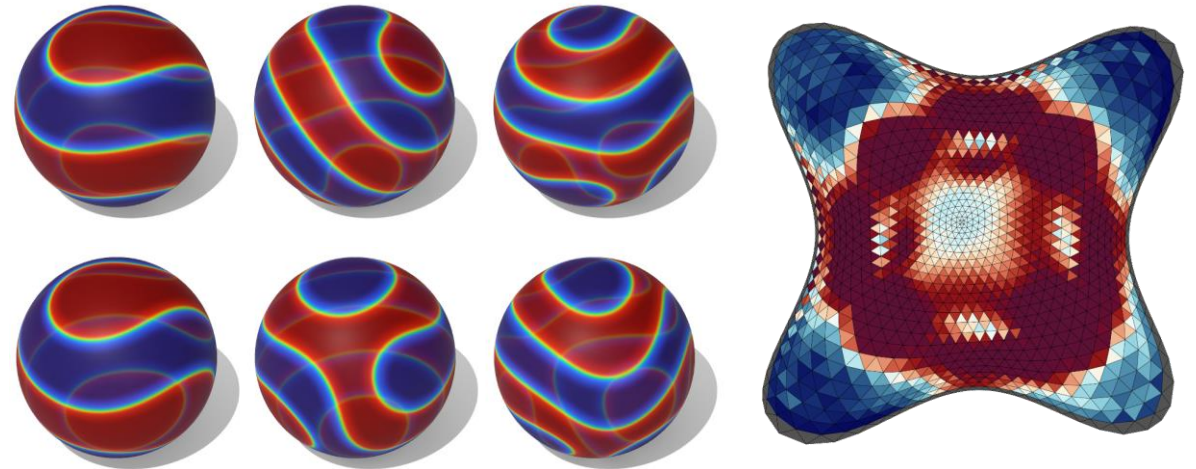


My Research

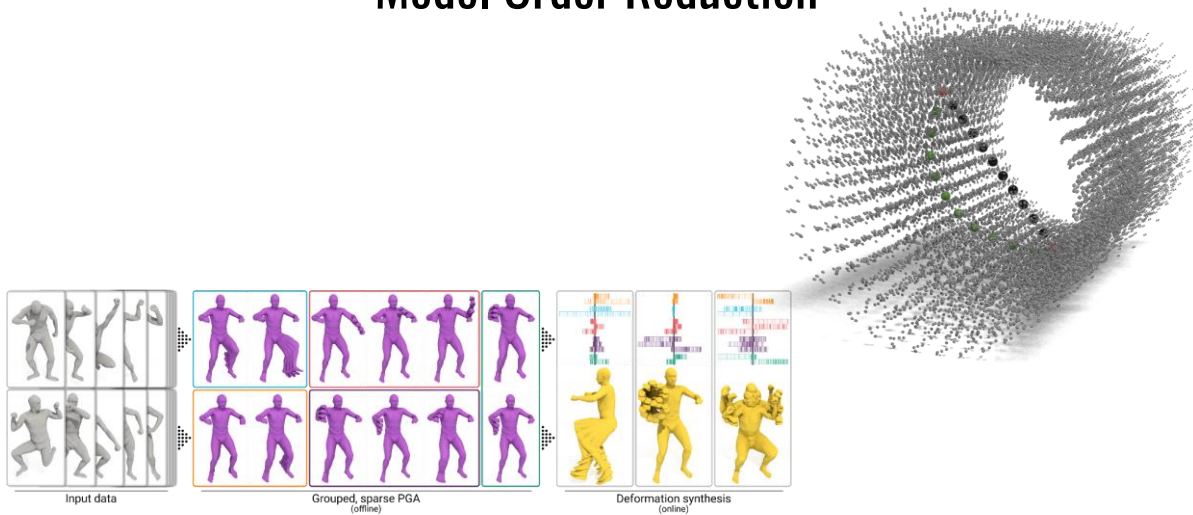
Shape Spaces



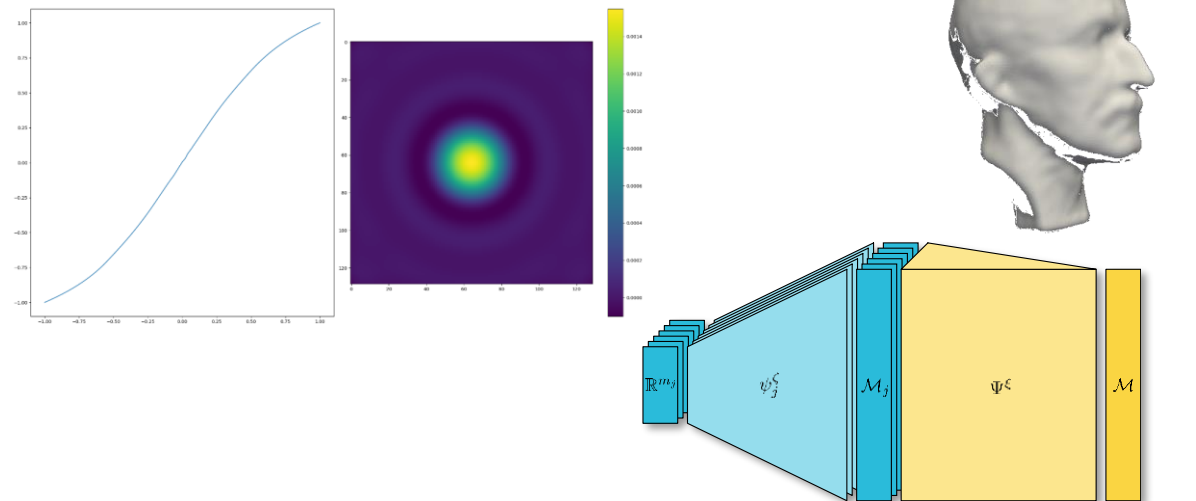
Shape Optimization



Model Order Reduction



Scientific Machine Learning

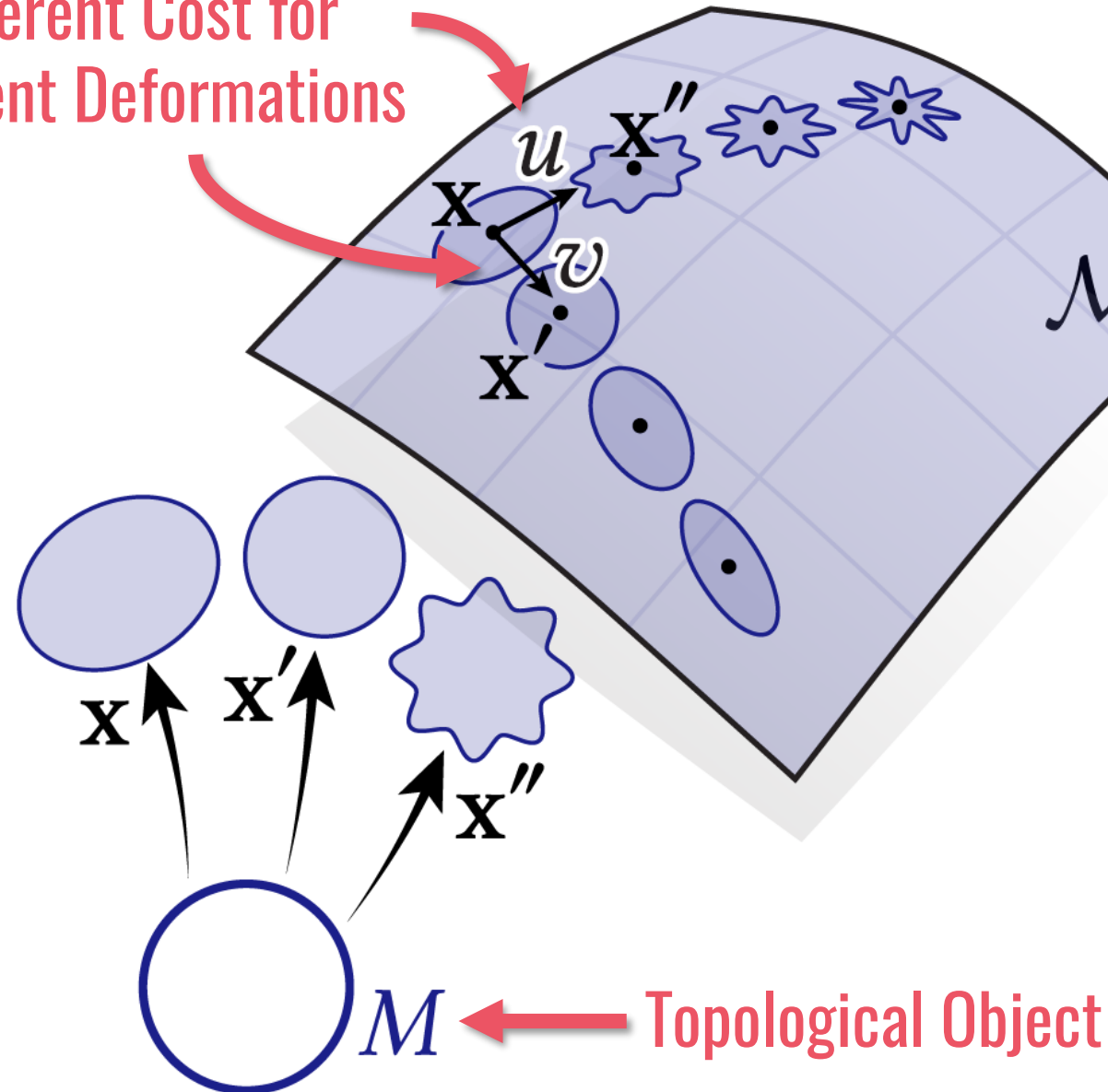


Shape Spaces

General Idea

Different Cost for
Different Deformations

Space of Realizations



Topological Object



Spaces and manifolds of shapes in computer vision: An overview ☆☆☆

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ABSTRACT

We provide an overview of several shape space models that have been proposed in the past few years, focusing, in particular on models involving Riemannian manifolds of shapes. The discussion is organized in three stages, starting with a review of some shape representation methods, followed by shape space structures, from metric spaces to manifolds, and concluding with a short description of some of the applications that resulted from such models.

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1. Introduction

Manifold methods naturally emerge from vision problems that involve shapes, first because a single shape can often be considered as a differential manifold (a curve, a surface, ...), but also because the space of shapes can be represented, as a mathematical object, by various models of finite- or infinite-dimensional spaces on which a manifold structure can be placed. The present discussion will focus on this last issue. Even within this limited scope, the related theory and literature are vast enough that the ideas and references that are listed in this survey cannot claim to provide an exhaustive account of the subject. Like with most surveys, the choices that have been made are subjective and mostly rely on the author's view of the field and on the scope of his knowledge.

The discussion will be organized in three parts, each corresponding to progressive levels in the modeling and application stages. Each part has been the subject of important research and contributions, within and beyond computer vision. The first level is about shape representation, and is about finding a suitable mathematical and computational description of shapes.¹ There is an immense literature on this subject, but we will only mention here some of the approaches that led to specific constructions of shape spaces. These constructions will constitute

the second, and most theoretical part, of this discussion. In the third part, we will describe some of the applications that directly resulted by the theoretical developments, for the definition of algorithms that work in shape spaces, and for the statistical analysis of shape datasets.

The next step is about defining, and structuring a shape space. This is generally the step in which the heaviest mathematical input is required. Starting from the representation, this construction generally includes several important elements, like defining a structure of differential manifold, which allows for well-posed infinitesimal calculus on shapes and associated variational formulations, or defining a (possibly Riemannian) metric between shapes, which allows for comparison and recognition, and inducing invariance by considering equivalence classes under specific groups of transformations. The third component in our discussion is the design of algorithms and data analysis methods that work in shape spaces.

2. Shape representation

We do not intend, here, to provide any account of the immense literature that have been devoted to shape representation, shape descriptors, or shape signatures in computer vision. However, one cannot disregard the fact that, when building a shape space, one necessarily needs to start with a proper choice for such a representation.

Let's define, to simplify, that shape to be the boundary of a two- or three-dimensional object. (Some situations can be more complex, like with complex structures, consisting of several objects, or parts, each with different properties.) A shape representation is a function that assigns to a given shape a well-defined mathematical feature that will simplify further algorithms and analysis. For example, one can describe a shape using a finite number of points, often called landmarks. This is probably the simplest representation from a mathematical viewpoint, and shape spaces of landmarks are still active subjects of studies, and are used in a large number of applications.

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¹ Note that, although the common meaning of the word shape refers to the boundary, or outline, of an object in two or three dimensions, which corresponds, mathematically, to closed curve or surface, the literature has extended this notion to various concepts, including point sets, images, measures, or geometric currents.

Geometric Modeling in Shape Space

Martin Kilian

Nikolaj J. Mitra
Vienna University of Technology

Helmut Pottmann

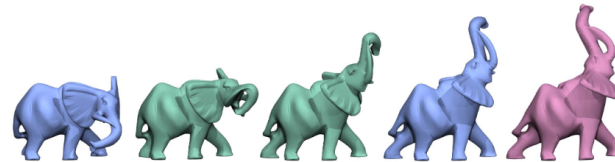


Figure 1: Geodesic interpolation and extrapolation. The blue input poses of the elephant are geodesically interpolated in an as-isometric-as-possible fashion (shown in green), and the resulting path is geodesically continued (shown in purple) to naturally extend the sequence. No semantic information, segmentation, or knowledge of articulated components is used.

Abstract

We present a novel framework to treat shapes in the setting of Riemannian geometry. Shapes – triangular meshes or more generally straight line graphs in Euclidean space – are treated as points in a shape space. We introduce useful Riemannian metrics in this space to aid the user in design and modeling tasks, especially to explore the space of (approximately) isometric deformations of a given shape. Much of the work relies on an efficient algorithm to compute geodesics in shape spaces; to this end, we present a multi-resolution framework to solve the interpolation problem – which amounts to solving a boundary value problem – as well as the extrapolation problem – an initial value problem – in shape space. Based on these two operations, several classical concepts like parallel transport and the exponential map can be used in shape space to solve various geometric modeling and geometry processing tasks. Applications include shape morphing, shape deformation, deformation transfer, and intuitive shape exploration.

Keywords: Riemannian geometry, shape space, geodesic, isometric deformation, parallel transport, shape exploration.

1 Introduction

Computing with geometric shapes lies at the core of geometric modeling and processing. Typically a shape is viewed as a set of points and represented according to the available data, and the intended application. Geometry does not necessarily take this perspective: Projective geometry views hyperplanes as points in a dual

space, line geometry interprets straight lines as points on a quadratic surface [Berger 1987], and the various types of sphere geometries model spheres as points in higher dimensional space [Cecil 1992]. Other examples concern kinematic spaces and Lie groups which are convenient for handling congruent shapes and motion design. These examples show that it is often beneficial and insightful to endow the set of objects under consideration with additional structure and to work in more abstract spaces. We will show that many geometry processing tasks can be solved by endowing the set of closed orientable surfaces – called *shapes* henceforth – with a Riemannian structure. Originally pioneered by [Kendall 1984], shape spaces are an active topic of interest in the mathematical research community. We focus our attention on the computational aspects of shape spaces and point to recent work of Michor and Mumford [2006], which provides a theoretical background for our research.

Our modeling and design paradigm is based on *geodesic curves* – locally shortest curves with respect to some metric. During interpolation, extrapolation (see Figure 1), and more general shape deformations (see Figure 10) shapes move along geodesics. Our approach is entirely geometric. Therefore the same method can be applied to a large class of problems with different underlying physical models, without knowing these models. Our algorithm does not need any segmentation of the model or external advice about the mesh structure. Working in a Riemannian manifold gives nice properties. For example geodesics from a shape M to each of a set of other shapes form a tree, thus generating globally consistent morphs. Such properties are harder to enforce with methods that do not consider a global space of deformations.

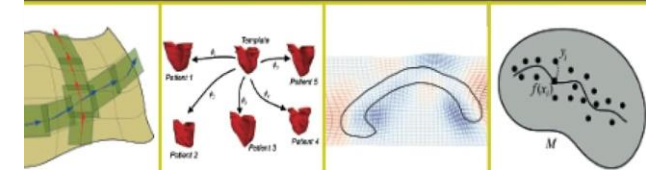
Related Work

To the best of our knowledge, there are only a few contributions treating shape spaces and related topics from a *computational perspective*. Cheng et al. [1998] realized the intimate connection between shape spaces and deformations, but neither discussed the critical choice of a metric, nor investigated essential geometric concepts such as geodesics. A computational approach to spaces of curves was presented in [Klassen et al. 2004] but has no natural extension to surfaces.

The gradient of a function on shape space depends on the met-

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RIEMANNIAN GEOMETRIC STATISTICS IN MEDICAL IMAGE ANALYSIS

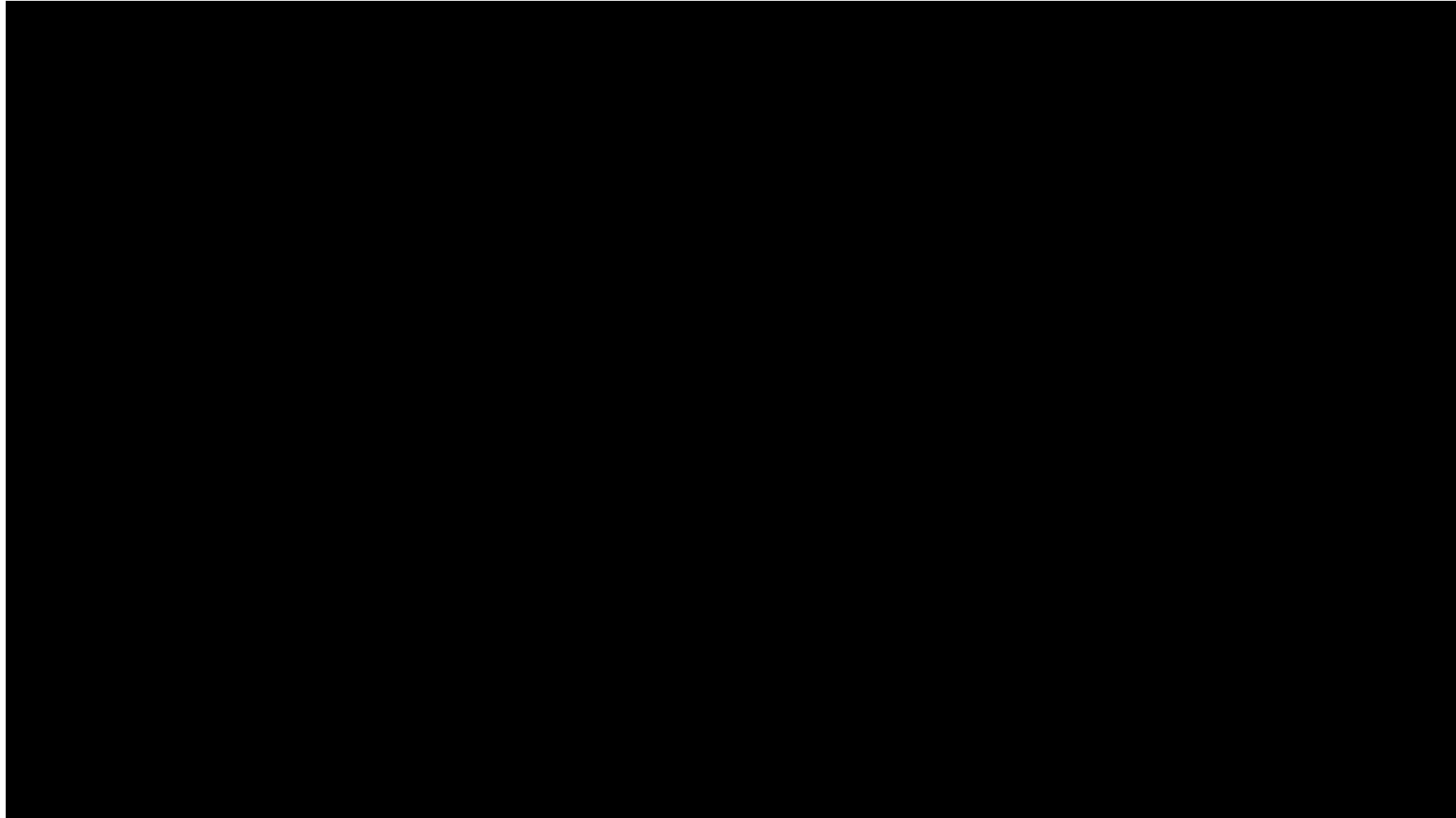


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Stefan Sommer, Tom Fletcher



Applications

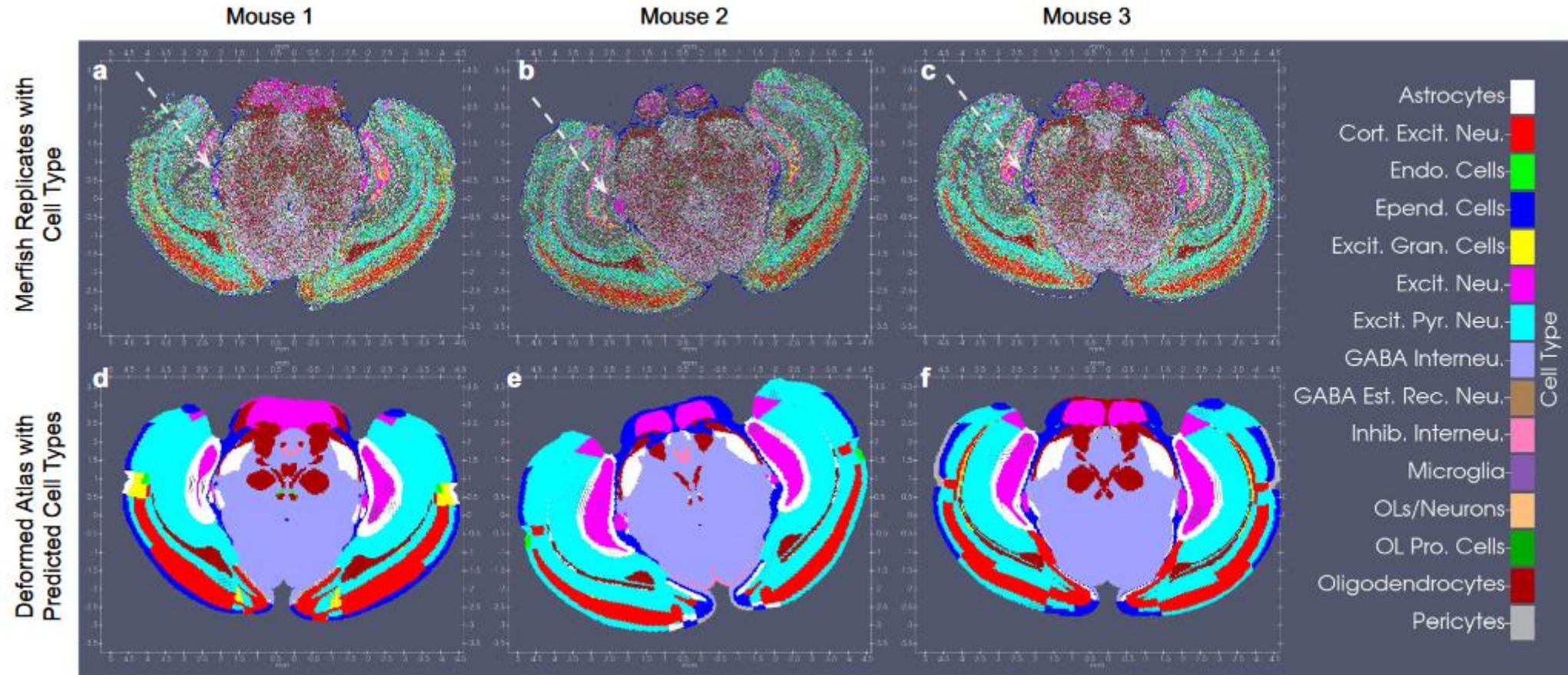
[von Radziewsky et al., Computers & Graphics 2016]



Computer Graphics

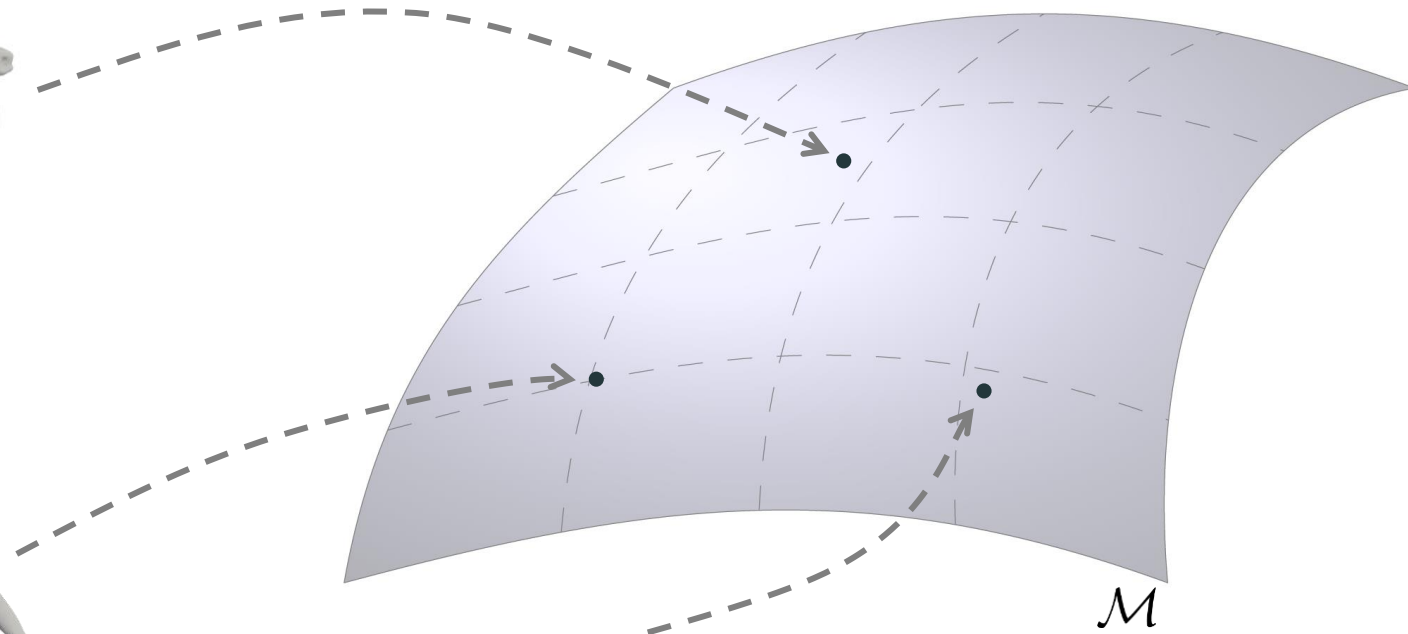
Applications

[Stouffer et al., Nature Communications 2024]



Medical Image Analysis

Space of Immersions



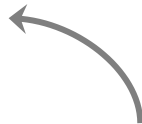
$M =$ topological surface



$\mathcal{M} := \text{Imm}(M) / \text{SE}(3)$

all immersions of the
surface into space
(for meshes: vertex positions)

we ignore rigid
transformations



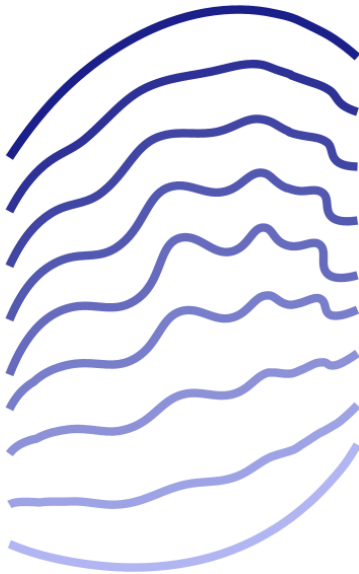
Elasticity-based Metric

Goal

less
expensive



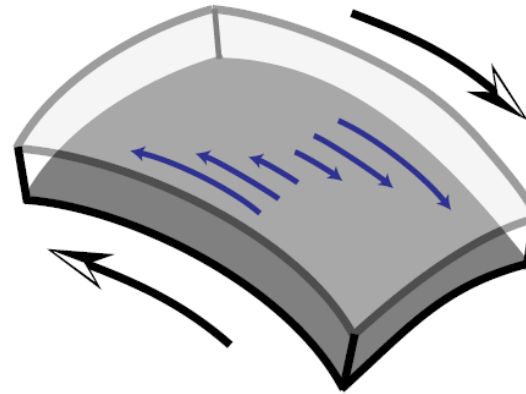
more
expensive



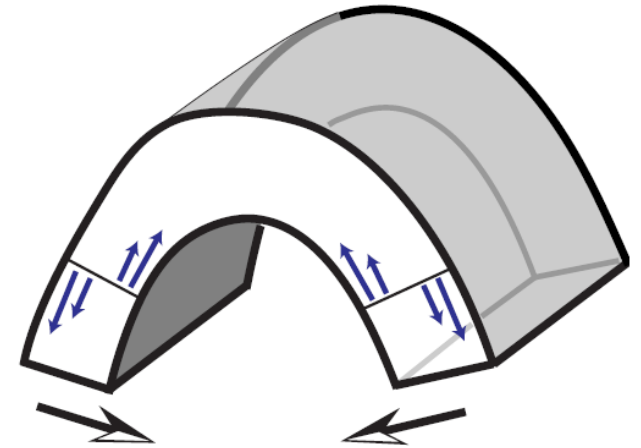
Elastic Shell Energy

$$\mathcal{W}(\mathbf{x}, \tilde{\mathbf{x}}) := \mathcal{W}_{\text{membrane}}(\mathbf{x}, \tilde{\mathbf{x}}) + \mathcal{W}_{\text{bending}}(\mathbf{x}, \tilde{\mathbf{x}})$$

Membrane Energy



Bending Energy



Rayleigh's Paradigm

cost of (small) deformation = heat lost due to internal friction

$$g_{\mathbf{x}}(u, v) := \frac{1}{2} d_{\mathbf{y}}^2 \mathcal{W}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}}(u, v)$$

Path Energy – Continuous & Discrete

Path Energy

$$\mathcal{E}(\mathbf{x}(t)) := \int_0^1 g_{\mathbf{x}(t)}(\dot{\mathbf{x}}(t), \dot{\mathbf{x}}(t)) dt$$

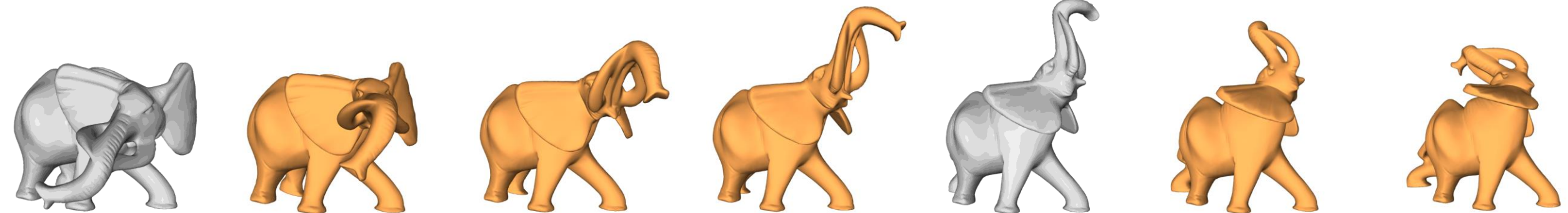
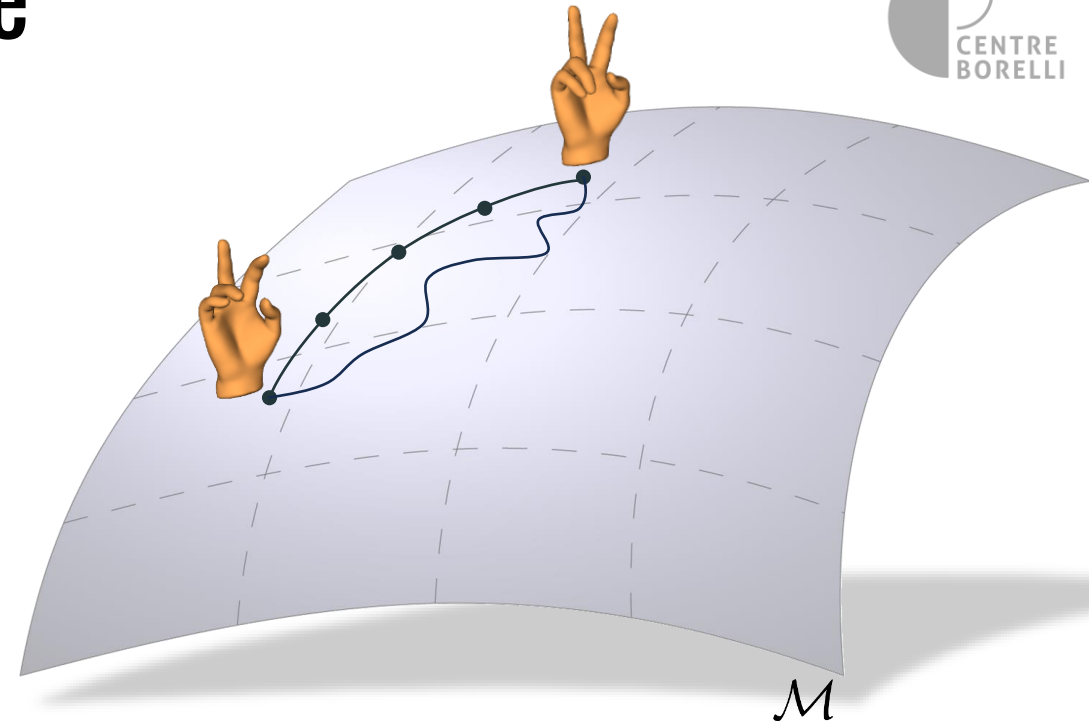
Geodesics

Minimizers of the path energy are geodesics

Time-discrete Geodesics [Heeren et al., 2012; Rumpf and Wirth, 2015]

A discrete geodesic (x_0, \dots, x_K) is a minimizer of

$$E(\mathbf{x}_0, \dots, \mathbf{x}_K) = K \sum_{k=1}^K \mathcal{W}(\mathbf{x}_{k-1}, \mathbf{x}_k)$$





Question:

How can we augment our shape space to obtain self-avoiding surfaces?

Intersection-Free Shape Space

Ingredient: Repulsive Energy

Definition: Repulsive Energy

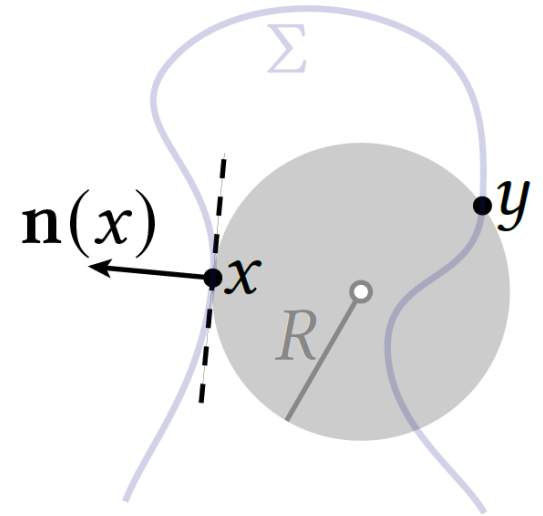
A surface energy is called repulsive if it explodes (i.e. goes to infinity)
if the surface approaches self-intersection.

Tangent-Point Energy [Buck and Orloff, 1995]

$$\Phi(\mathbf{x}) := \int_{\Sigma} \int_{\Sigma} \frac{1}{R(\mathbf{n}(x), x, y)^{\alpha}} dx dy$$

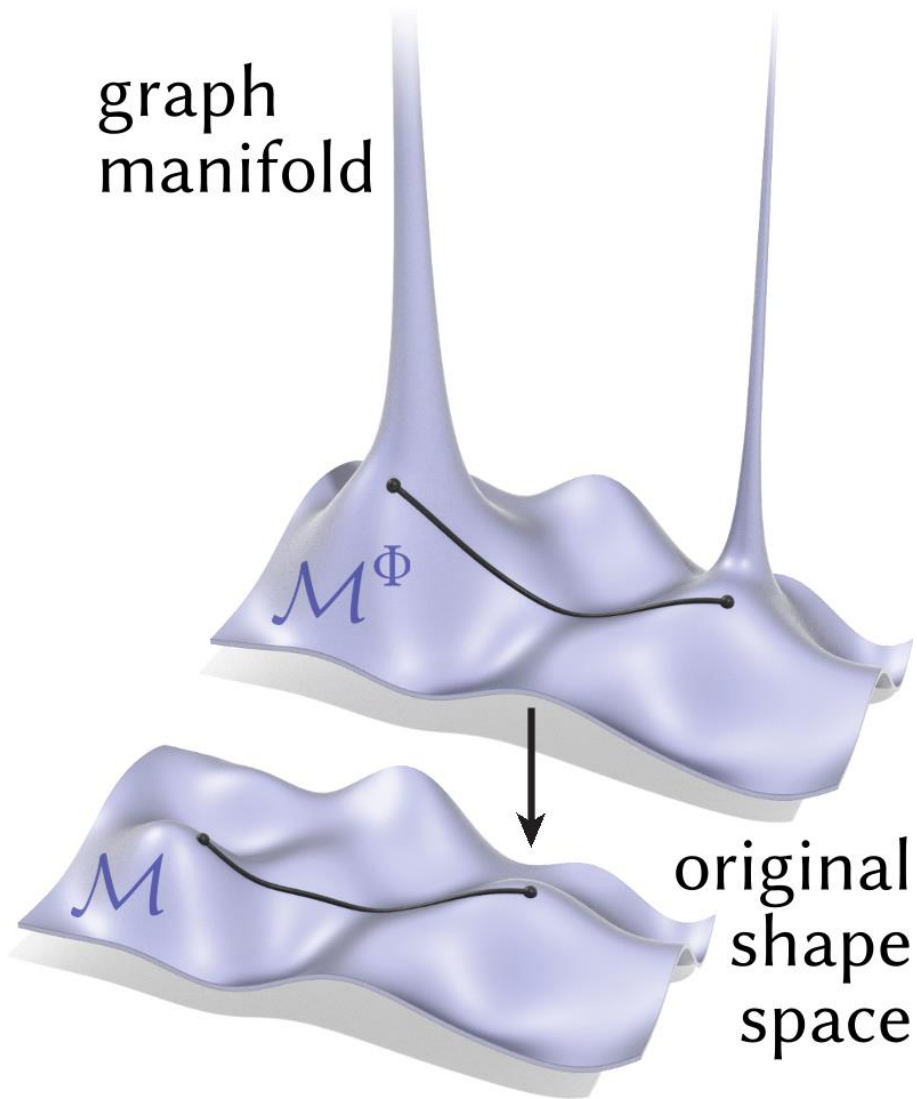
with

R = radius of smallest sphere tangent to x and passing through y



Repulsive for $\alpha > 4$

Graph Manifold



$$\mathcal{M}^\Phi := \{(\mathbf{x}, \Phi(\mathbf{x})) \mid \mathbf{x} \in \mathcal{M}\}$$

new metric = original metric + local change in potential

$$g_{\mathbf{x}}^\Phi(u, v) := g_{\mathbf{x}}(u, v) + d_{\mathbf{x}}\Phi(u) d_{\mathbf{x}}\Phi(v)$$

Crucial Observation

$$\text{dist}_{g^\Phi}(\mathbf{x}, \mathbf{y}) = \infty \text{ if } \Phi(\mathbf{y}) = \infty$$

Repulsive Path Energy – Continuous & Discrete

Continuous Path Energy

$$\mathcal{E}(\mathbf{x}(t)) := \int_0^1 g_{\mathbf{x}(t)}(\dot{\mathbf{x}}(t), \dot{\mathbf{x}}(t)) dt + \int_0^1 d_{\mathbf{x}(t)} \Phi(\dot{\mathbf{x}}(t))^2 dt$$

Discrete Path Energy

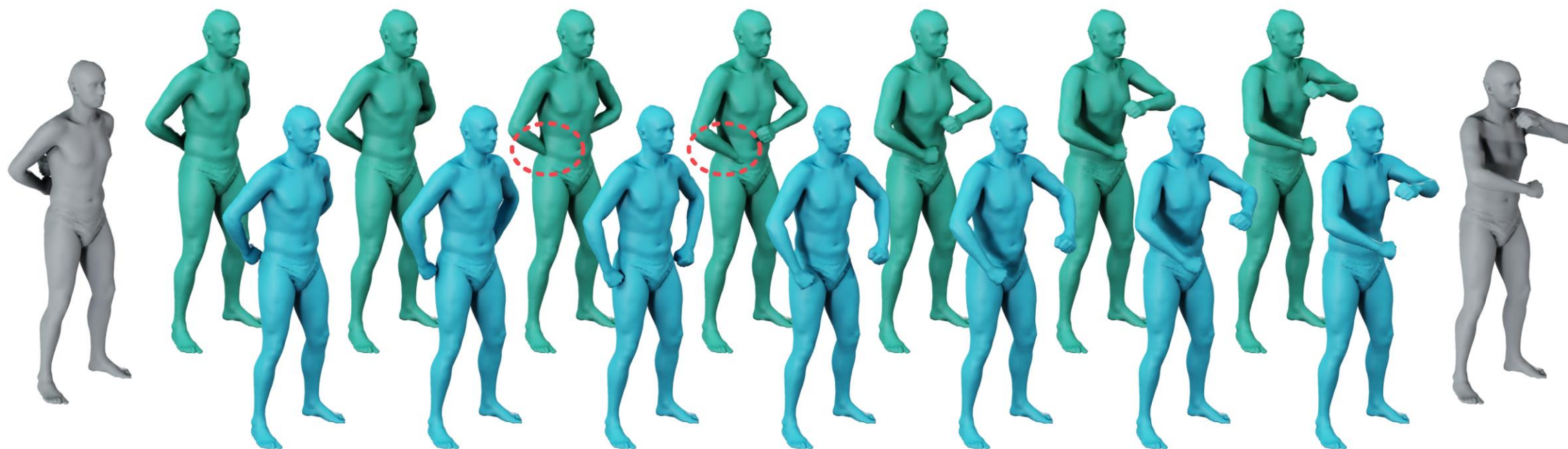
$$E(\mathbf{x}_0, \dots, \mathbf{x}_K) = K \sum_{k=1}^K \mathcal{W}(\mathbf{x}_{k-1}, \mathbf{x}_k) + (\Phi(\mathbf{x}^{k-1}) - \Phi(\mathbf{x}^k))^2$$

Repulsive Path Energy – Continuous & Discrete

Discrete Path Energy

$$E(\mathbf{x}_0, \dots, \mathbf{x}_K) = K \sum_{k=1}^K \mathcal{W}(\mathbf{x}_{k-1}, \mathbf{x}_k) + (\Phi(\mathbf{x}^{k-1}) - \Phi(\mathbf{x}^k))^2$$

Elastic Interpolation



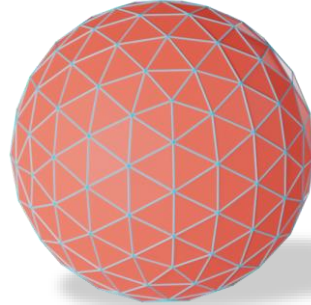
Combined Interpolation

Discretization & Numerics

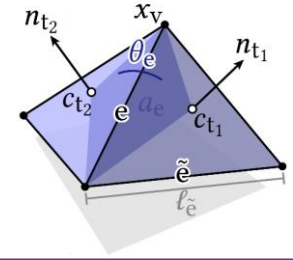
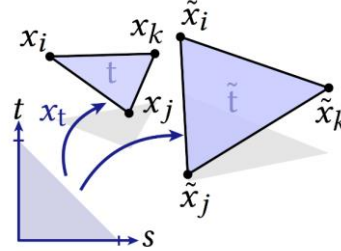
Discrete Surfaces

Triangle Mesh $M = (V, E, T)$

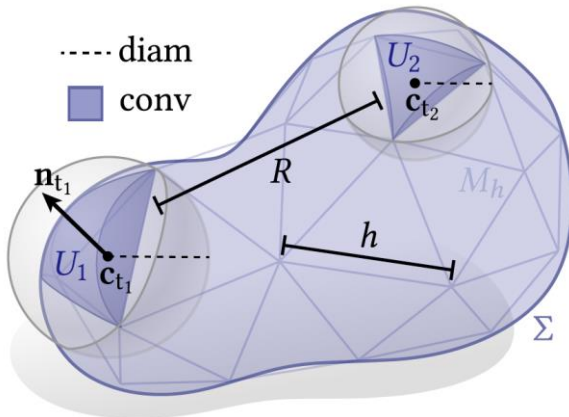
Piece-wise Affine Immersion $\mathbf{x}: M \rightarrow \mathbb{R}^3$



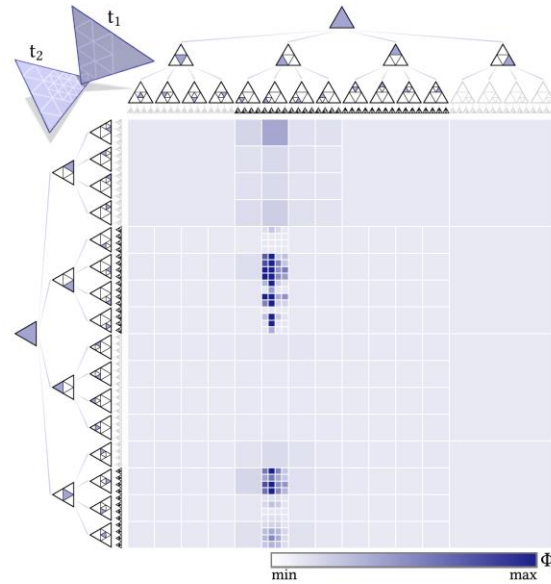
Discrete Shells Energy [Heeren et al., 2014]



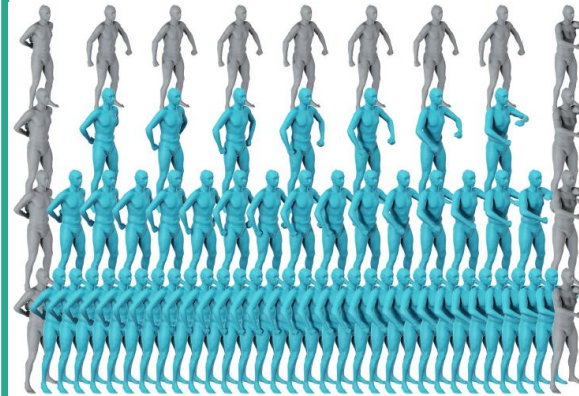
Multipole Approximation



Adaptive Refinement



Temporal Refinement



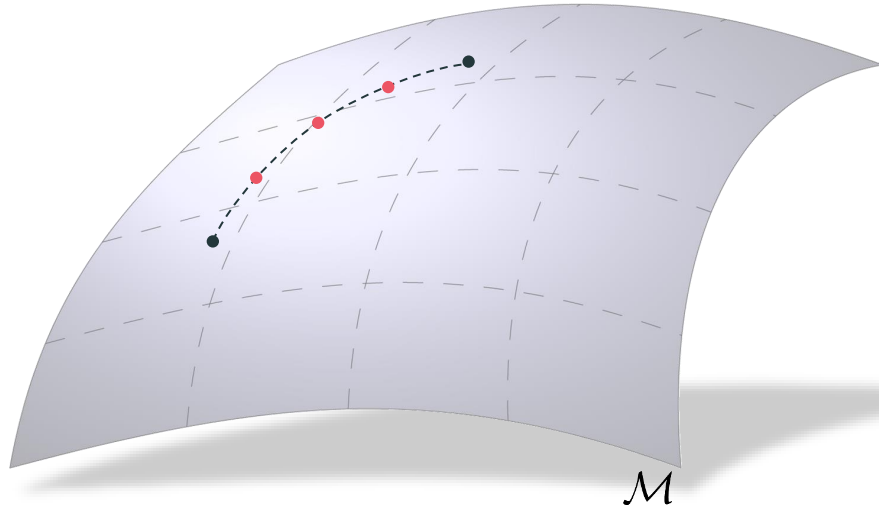
Spatial Prolongation



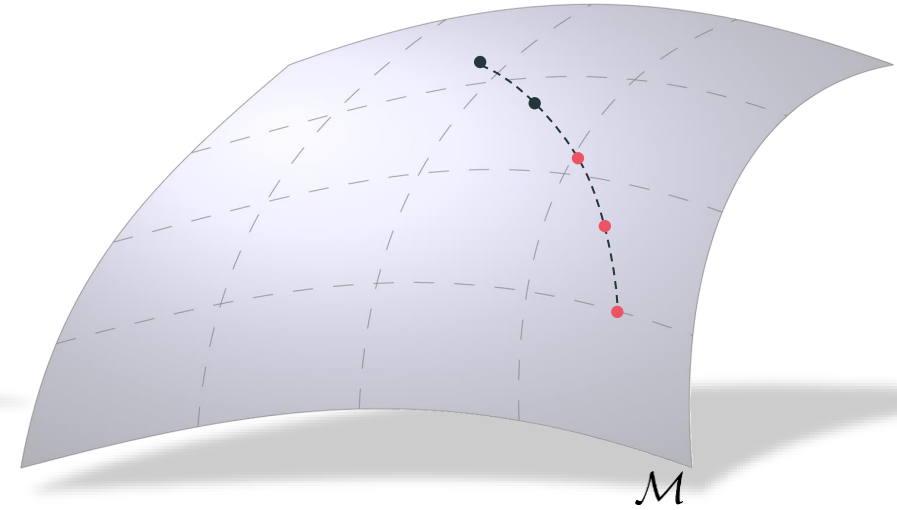
Trust-Region Based Optimization

Examples

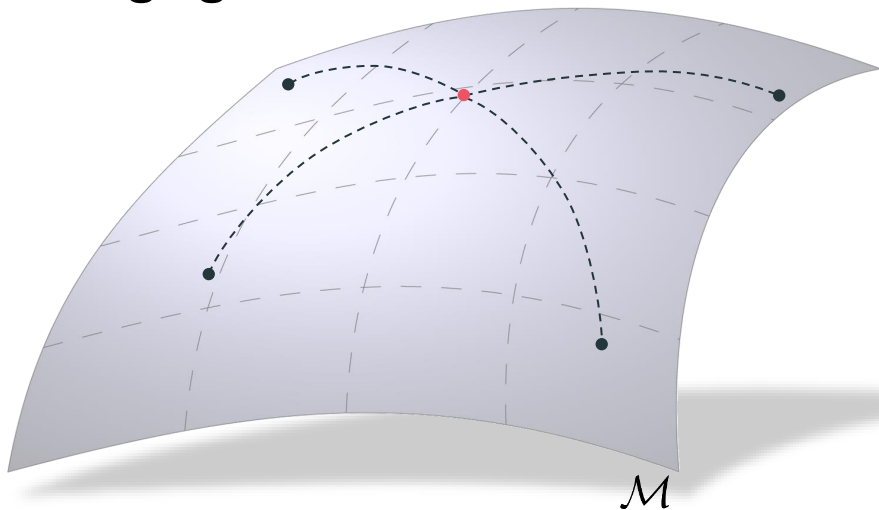
Interpolation



Extrapolation



Averaging



Bonus

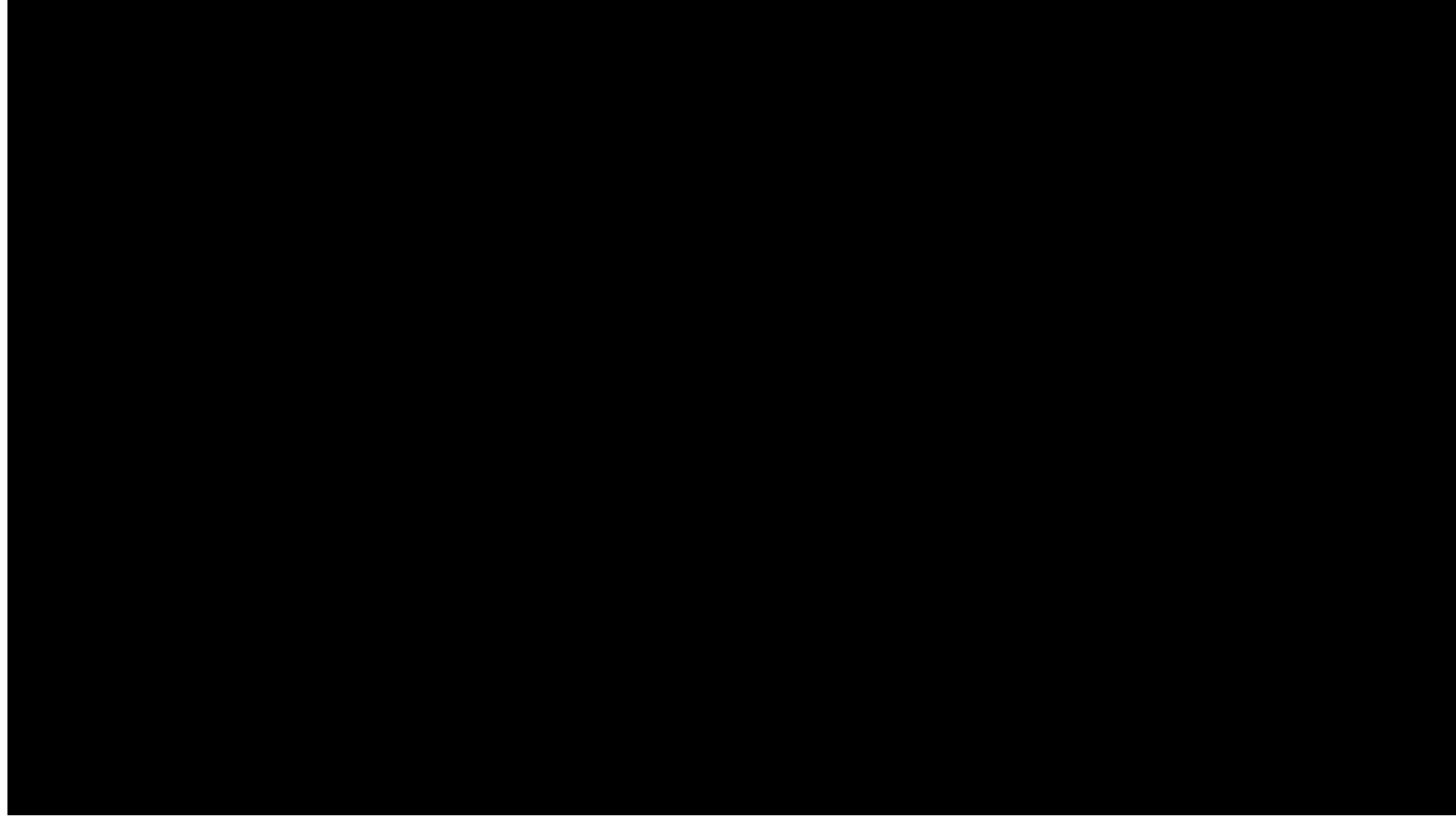


Excellent



Avoiding Obstacles

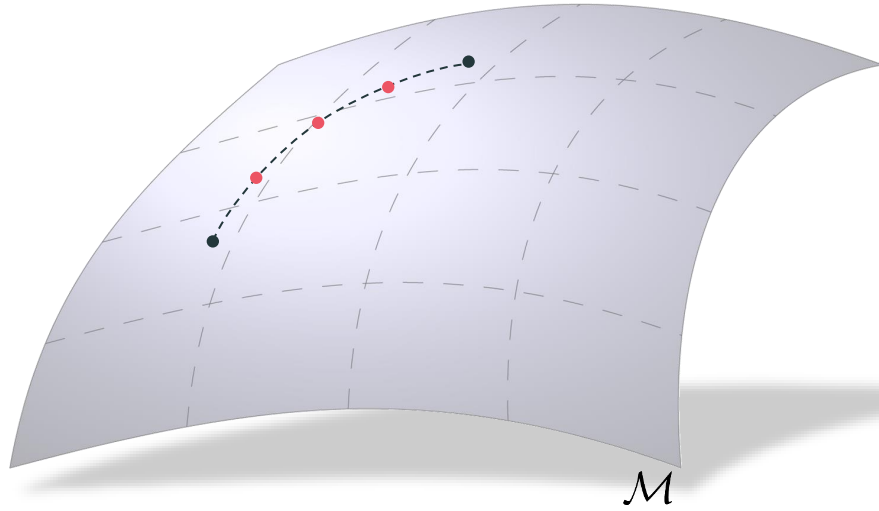
Obstacle



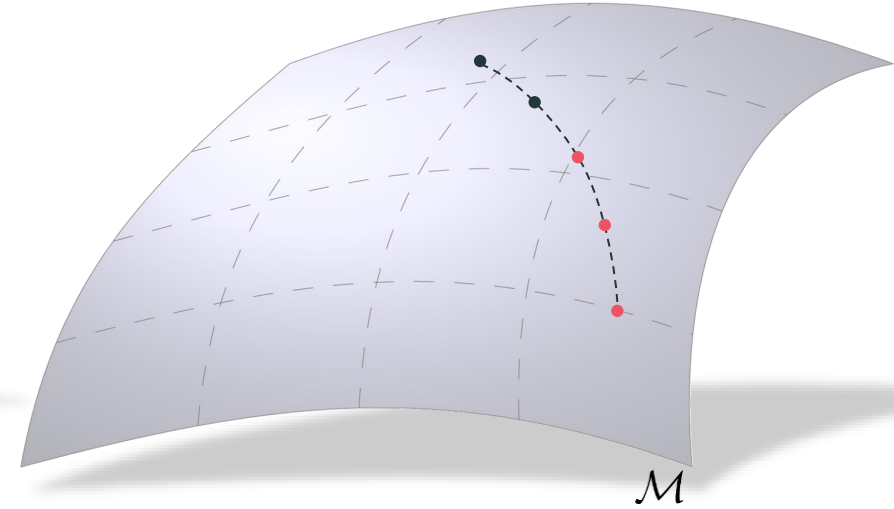
Simplifying Tedious Work



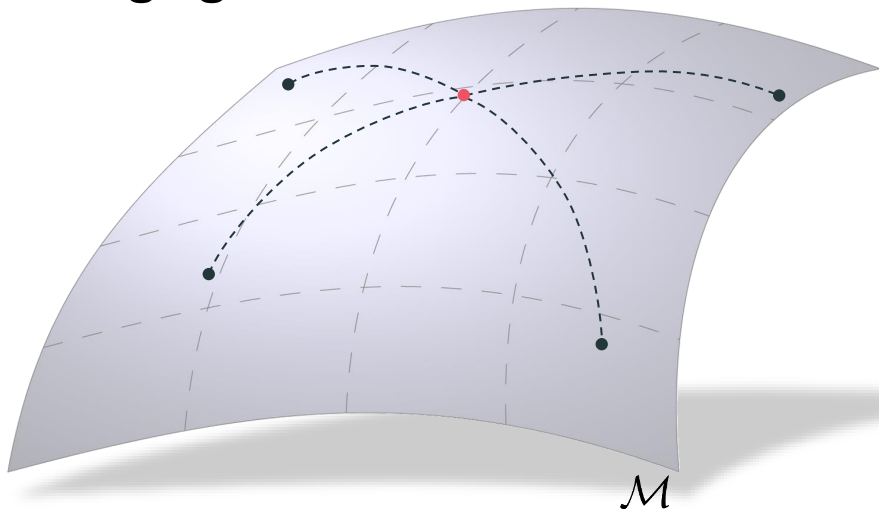
Interpolation



Extrapolation



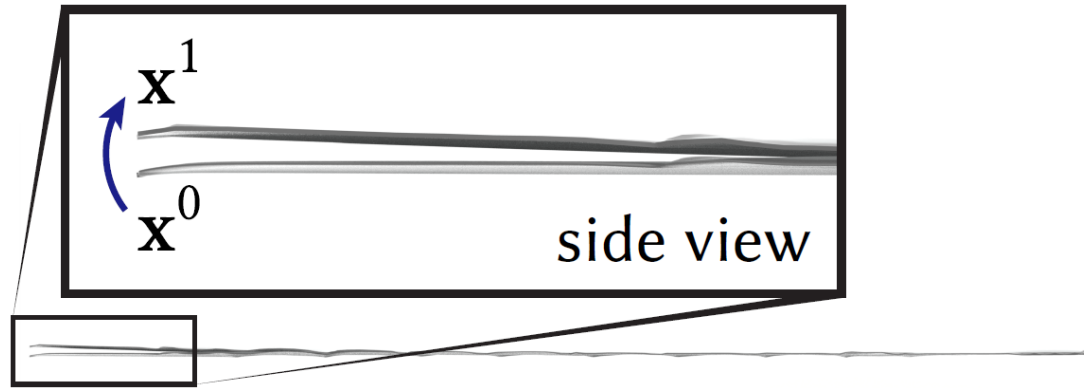
Averaging



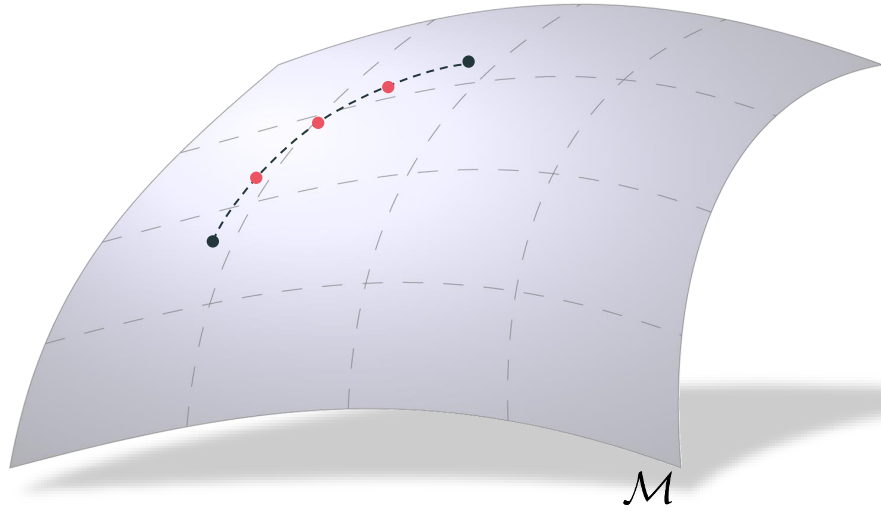
Bonus



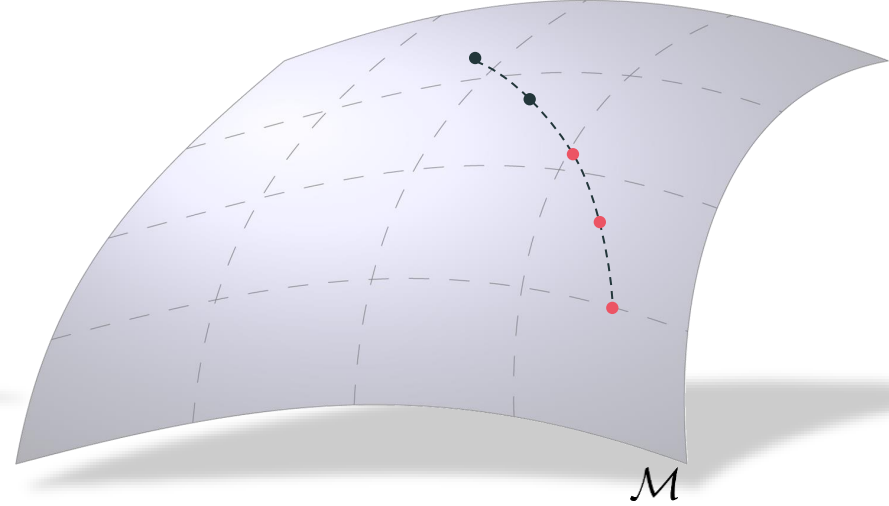
Curling a Leaf



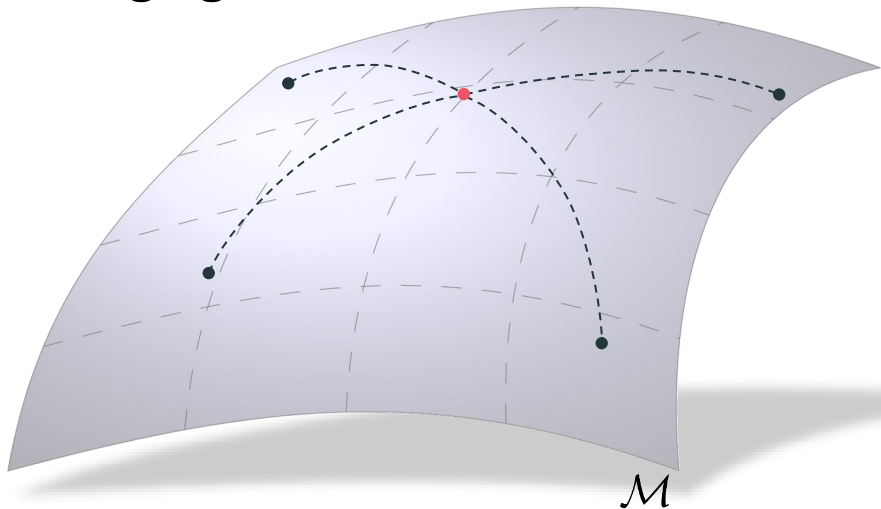
Interpolation



Extrapolation



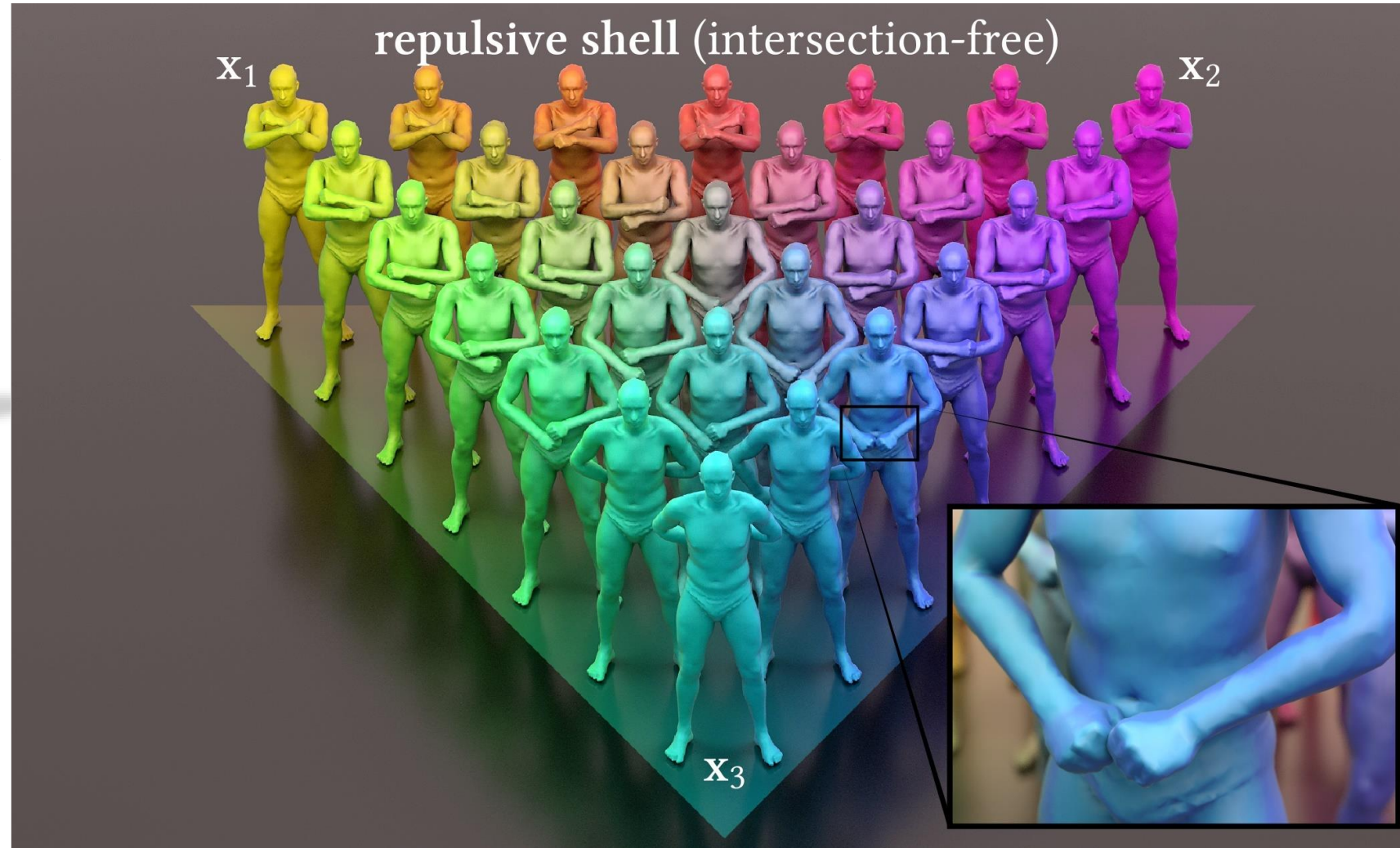
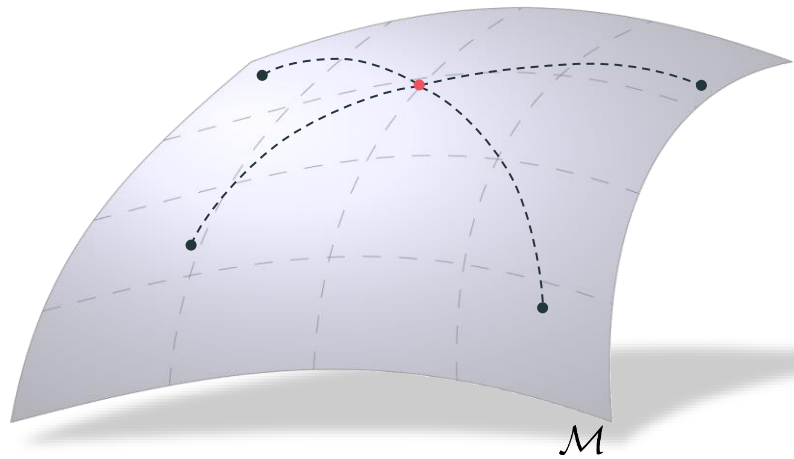
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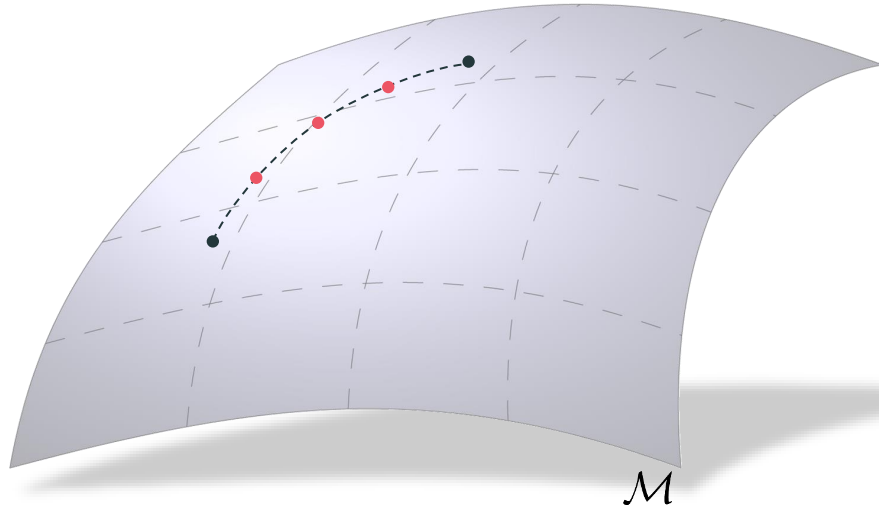
Bonus



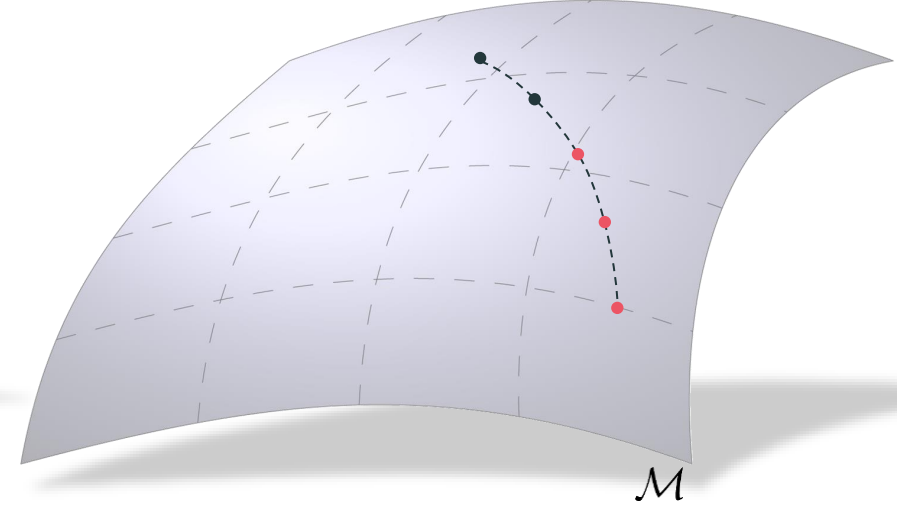
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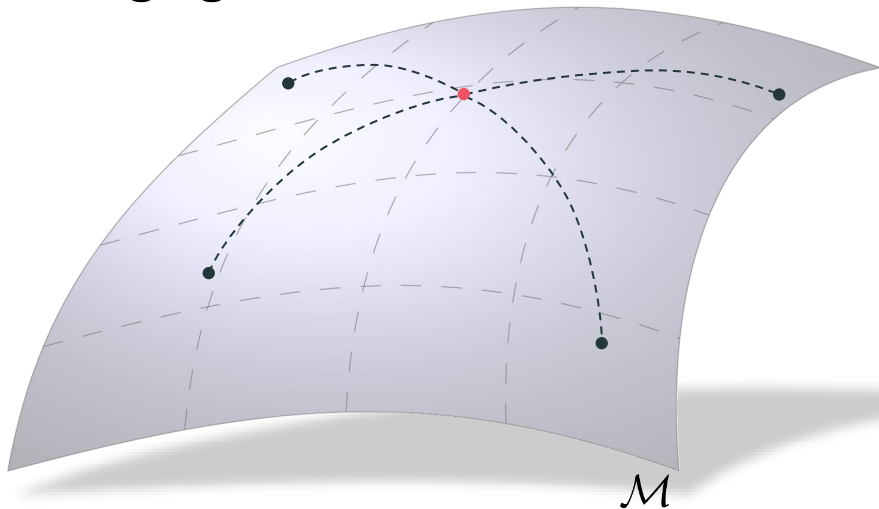
Interpolation



Extrapolation



Averaging

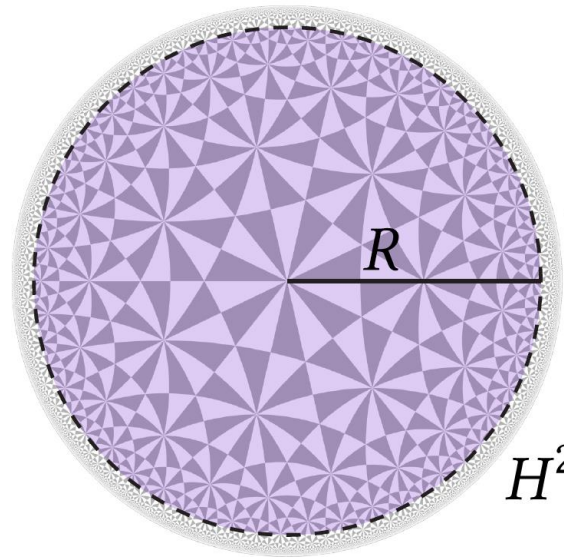


Bonus

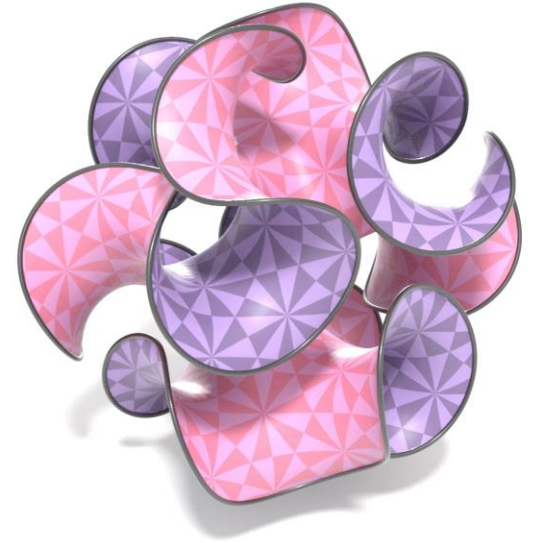


Isometric Embeddings

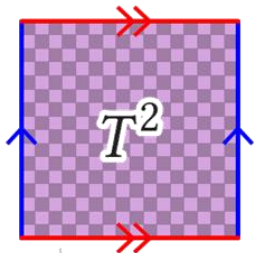
Hyperbolic Disk



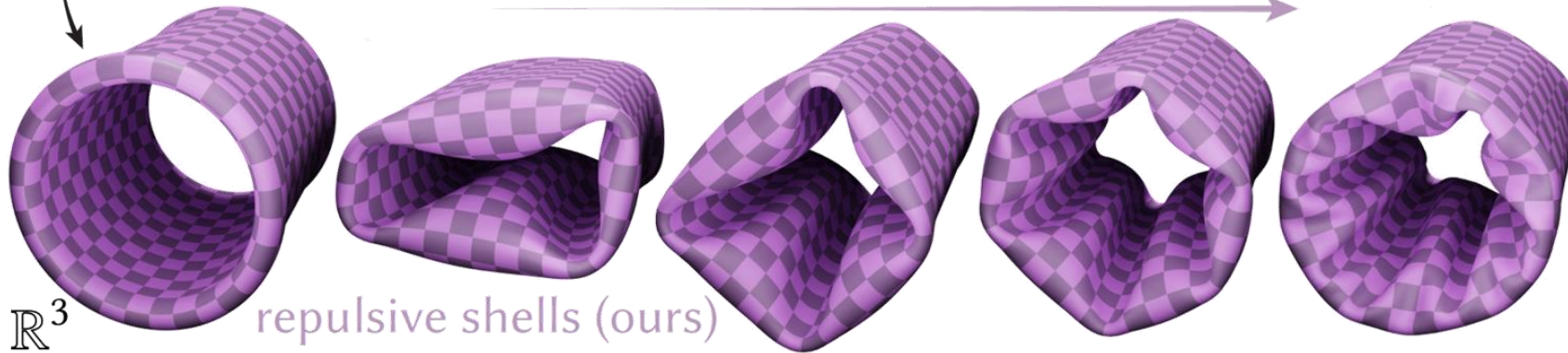
isometric
embedding



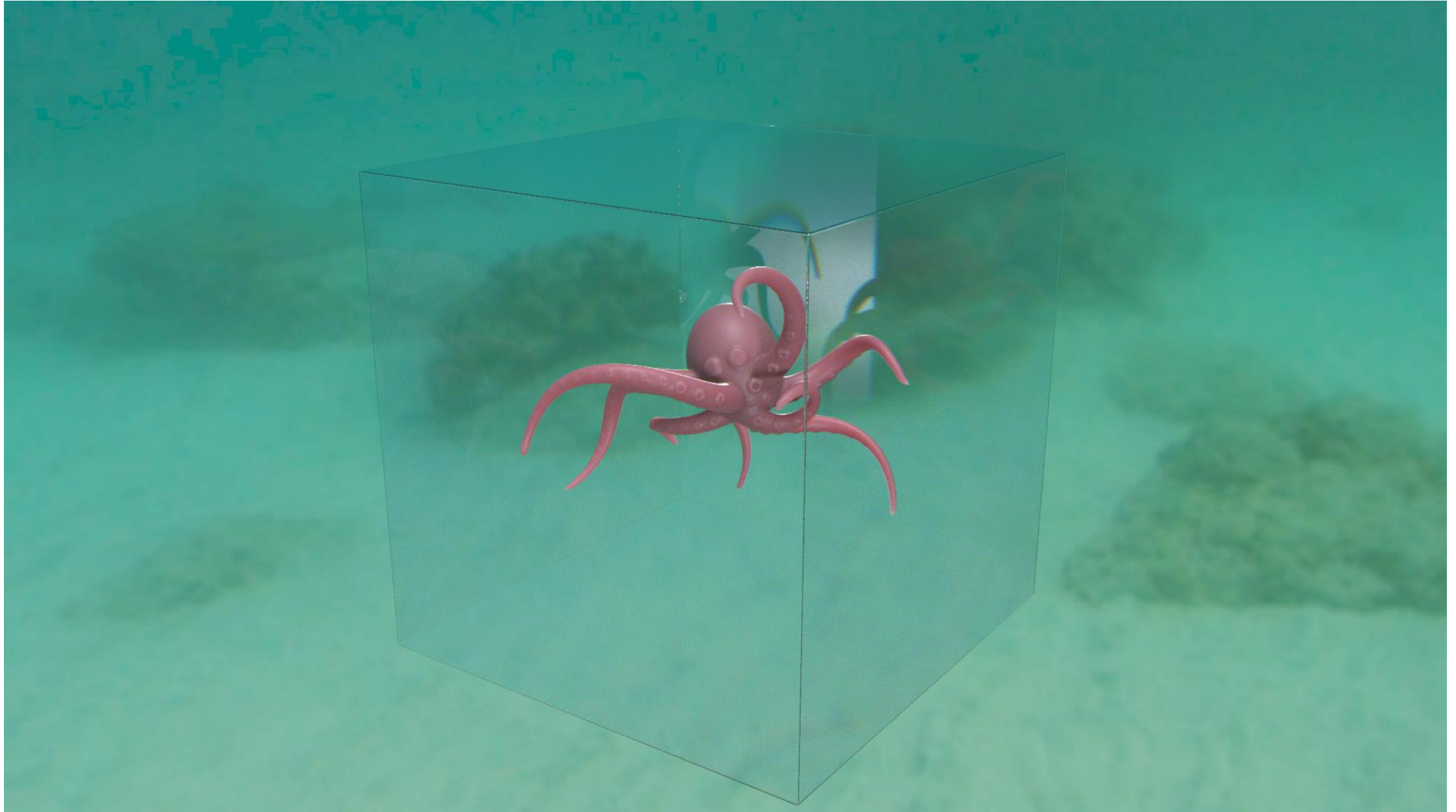
Flat Torus



higher membrane stiffness
(closer to isometry)

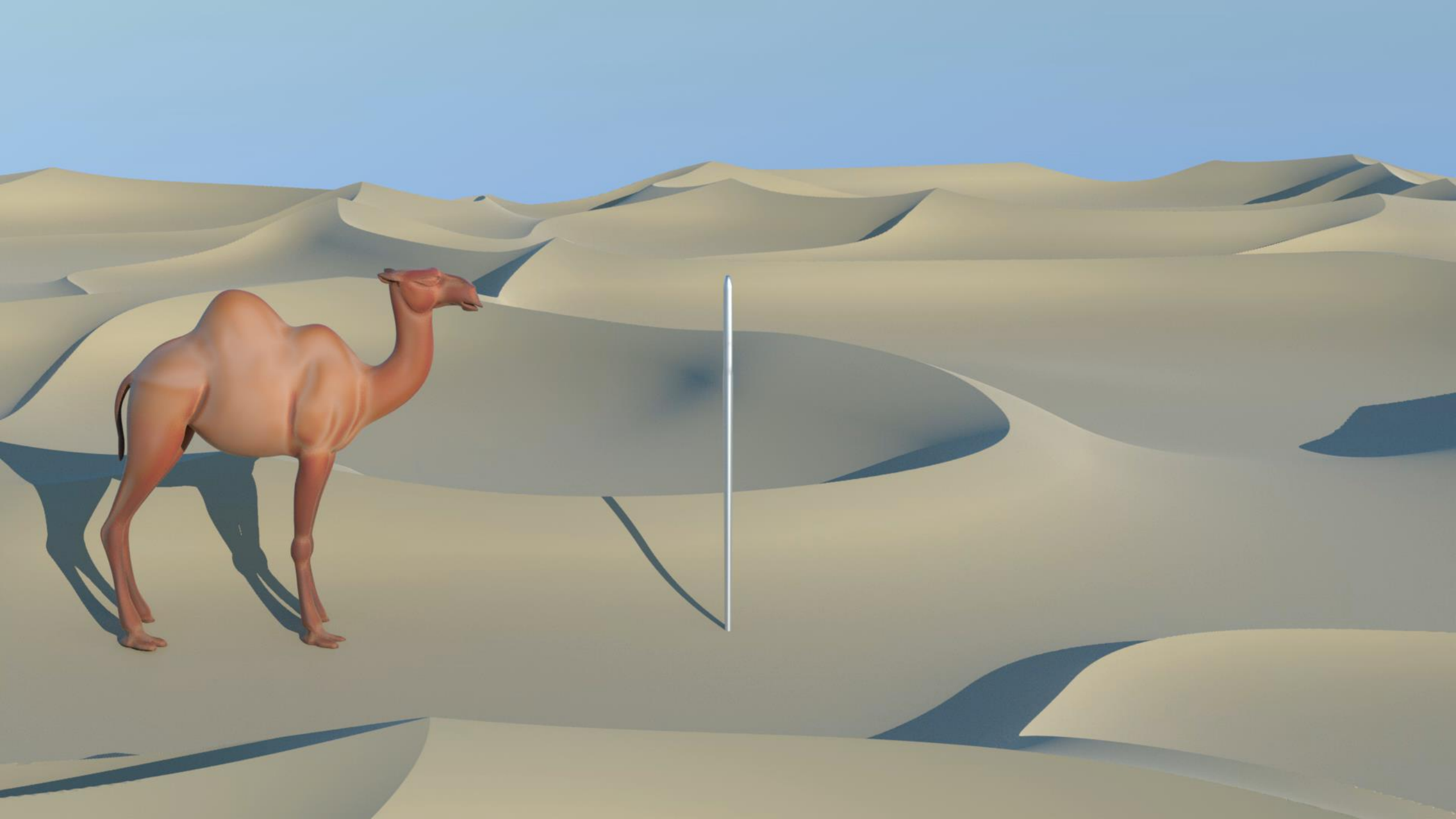


Octopus in a Box



***“It is easier for a camel to go through the eye of a needle,
than for a rich man to enter into the kingdom of God.”***

Matthew 19:24





Thank you!

JOSUA SASSEN, HENRIK SCHUMACHER, MARTIN RUMPF, KEENAN CRANE.
"Repulsive Shells". *ACM Transaction on Graphics* (2024).

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