The Space of Discrete Surfaces without Intersections

Josua Sassen[1], Henrik Schumacher[2], Martin Rumpf[3], Keenan Crane[4]

[1] **ENS Paris-Saclay** [2] TU Chemnitz
[3] University of Bonn [4] Carnegie Mellon University

JOSUA SASSEN, HENRIK SCHUMACHER, MARTIN RUMPF, KEENAN CRANE. "Repulsive Shells". *ACM Transaction on Graphics* (2024).
Best Paper Award at SIGGRAPH 2024.

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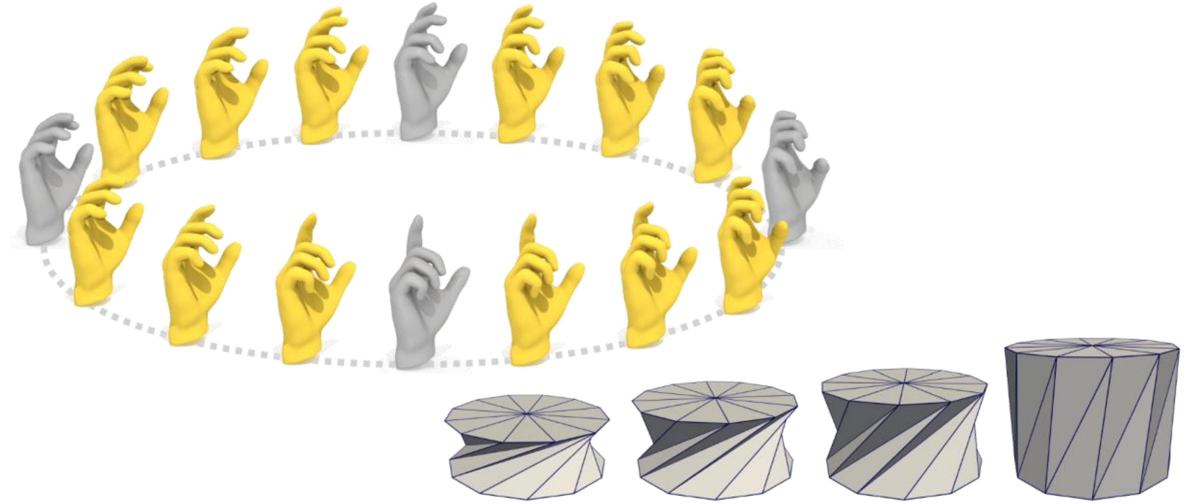
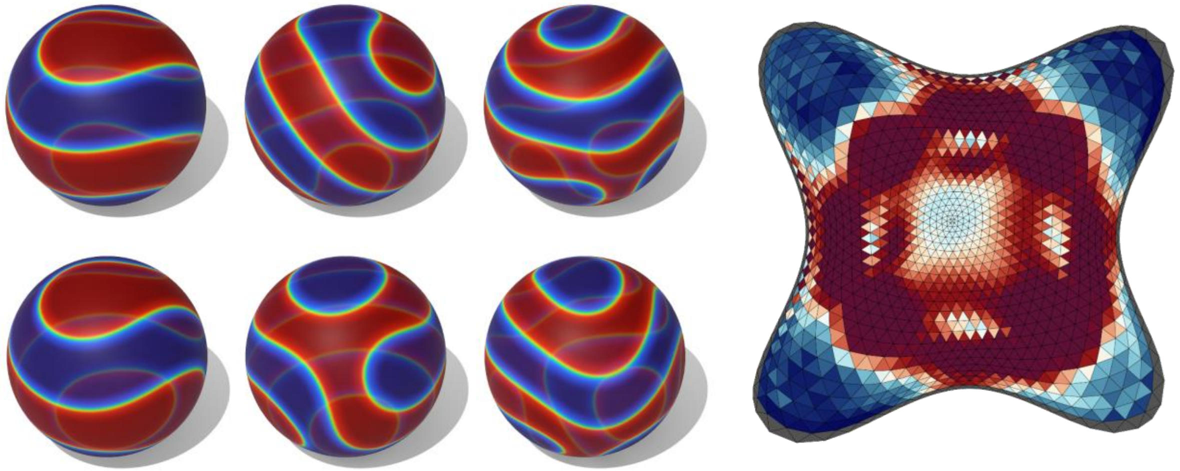
My Research

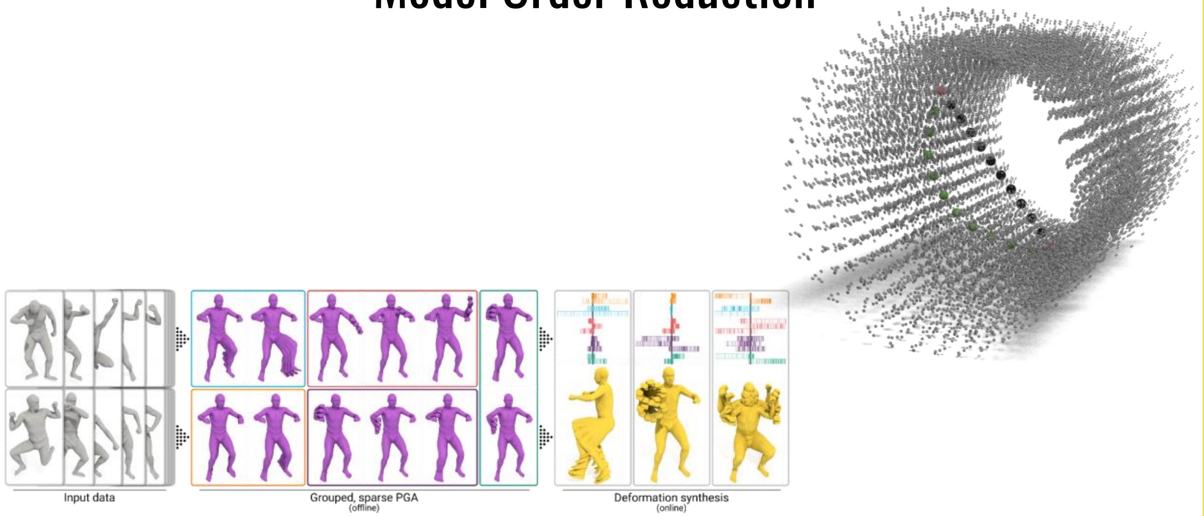
Shape Spaces



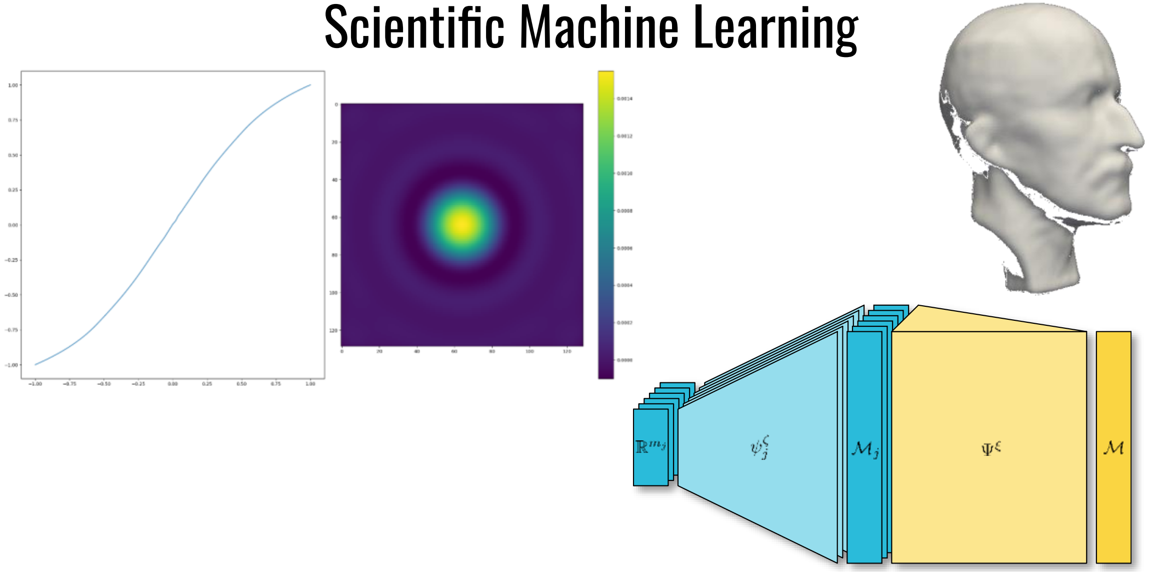
Shape Optimization



Model Order Reduction

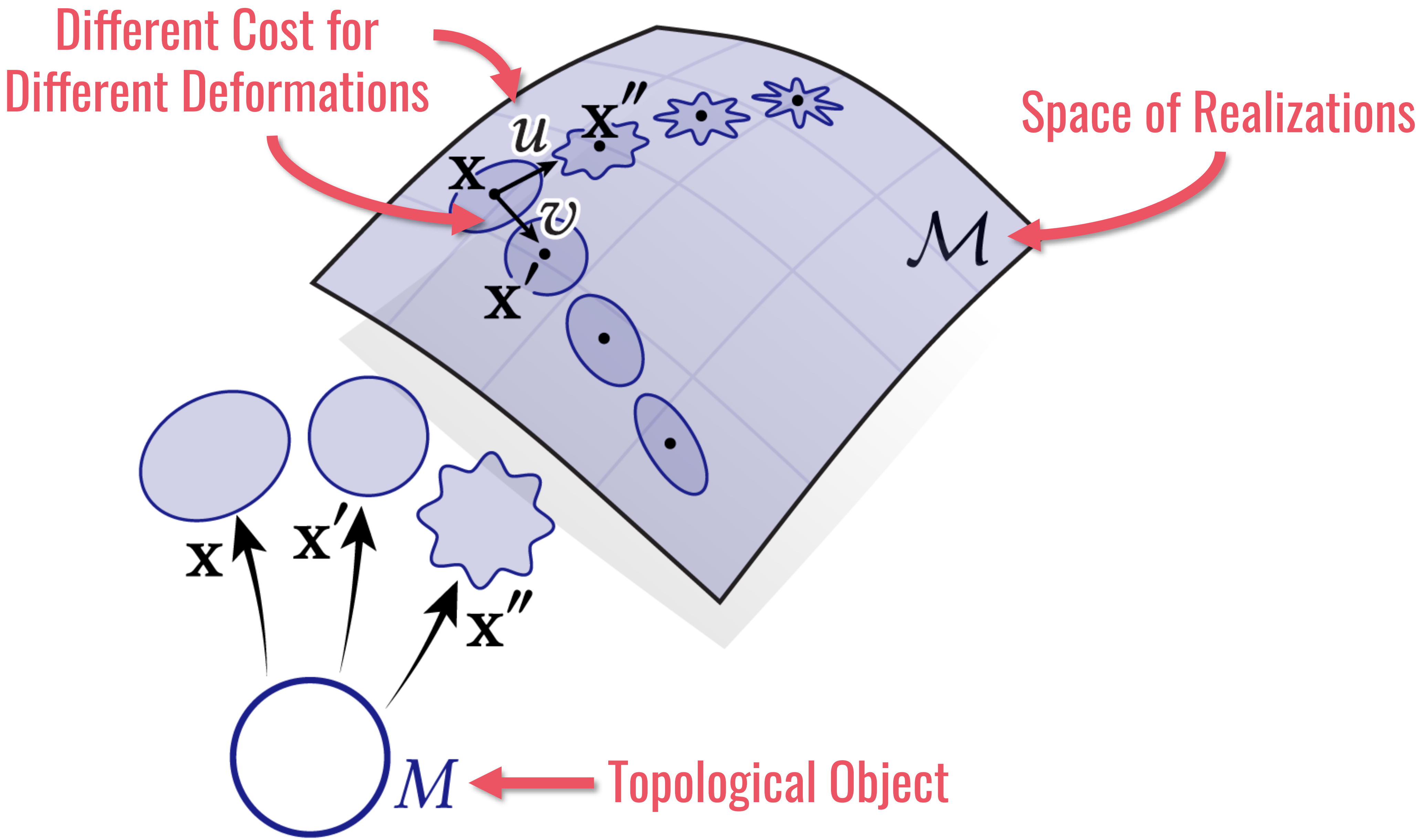


Scientific Machine Learning



Shape Spaces

General Idea



Applications

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Spaces and manifolds of shapes in computer vision: An overview ☆,☆☆

Laurent Younes *

Department of Applied Mathematics and Statistics, Johns Hopkins University, USA

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ABSTRACT

We provide an overview of several shape space models that have been proposed in the past few years, focusing, in particular on models involving Riemannian manifolds of shapes. The discussion is organized in three stages, starting with a review of some shape representation methods, followed by shape space structures, from metric spaces to manifolds, and concluding with a short description of some of the applications that resulted from such models.

1. Introduction

2. Shape representation

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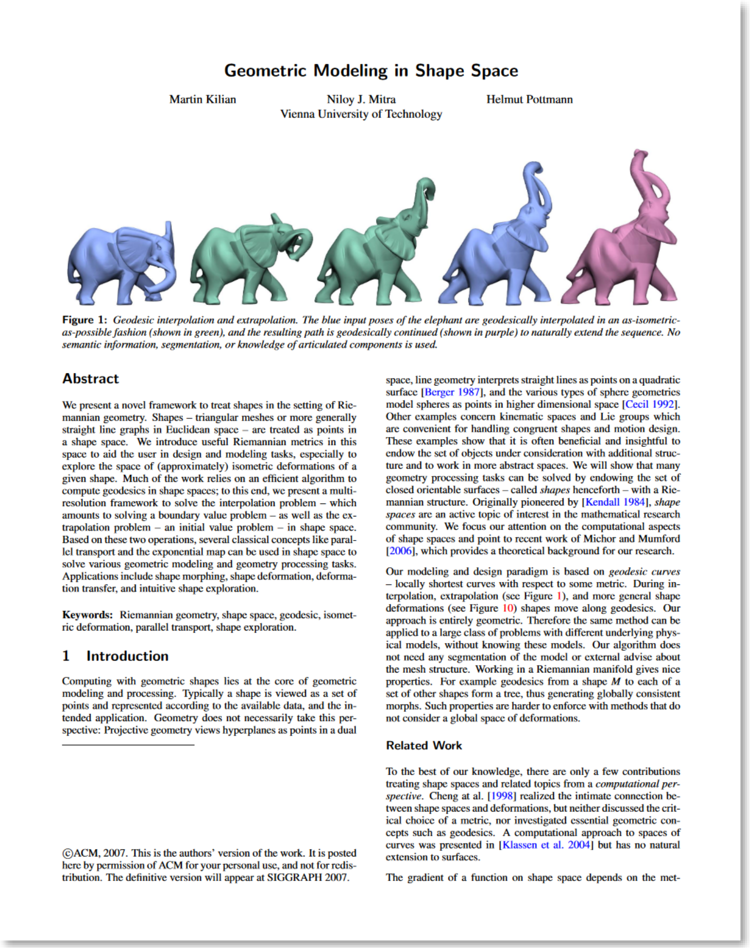
doi:10.1016/j.imavis.2011.09.009

Computer Vision



Geometric Modeling in Shape Space

Martin Kilian Niloy J. Mitra Helmut Pottmann
Vienna University of Technology

Figure 1: *Geodesic interpolation and extrapolation. The blue input poses of the elephant are geodesically interpolated in an as-isometric-as-possible fashion (shown in green), and the resulting path is geodesically continued (shown in purple) to naturally extend the sequence. No semantic information, segmentation, or knowledge of articulated components is used.*

Abstract

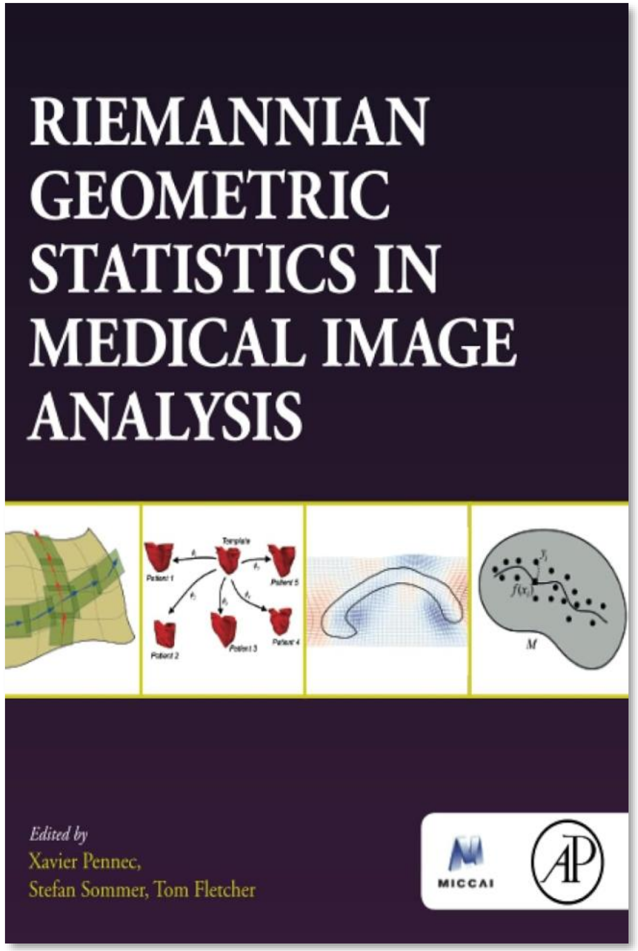
We present a novel framework to treat shapes in the setting of Riemannian geometry. Shapes – triangular meshes or more generally straight line graphs in Euclidean space – are treated as points in a shape space. We introduce useful Riemannian metrics in this space to aid the user in design and modeling tasks, especially to explore the space of (approximately) isometric deformations of a given shape. Much of the work relies on an efficient algorithm to compute geodesics in shape spaces; to this end, we present a multi-resolution framework to solve the interpolation problem – which amounts to solving a boundary value problem – as well as the extrapolation problem – an initial value problem – in shape space. Based on these two operations, several classical concepts like parallel transport and the exponential map can be used in shape space to solve various geometric modeling and geometry processing tasks. Applications include shape morphing, shape deformation, deformation transfer, and intuitive shape exploration.

Keywords: Riemannian geometry, shape space, geodesic, isometric deformation, parallel transport, shape exploration.

1 Introduction

Related Work

Computer Graphics



RIEMANNIAN GEOMETRIC STATISTICS IN MEDICAL IMAGE ANALYSIS

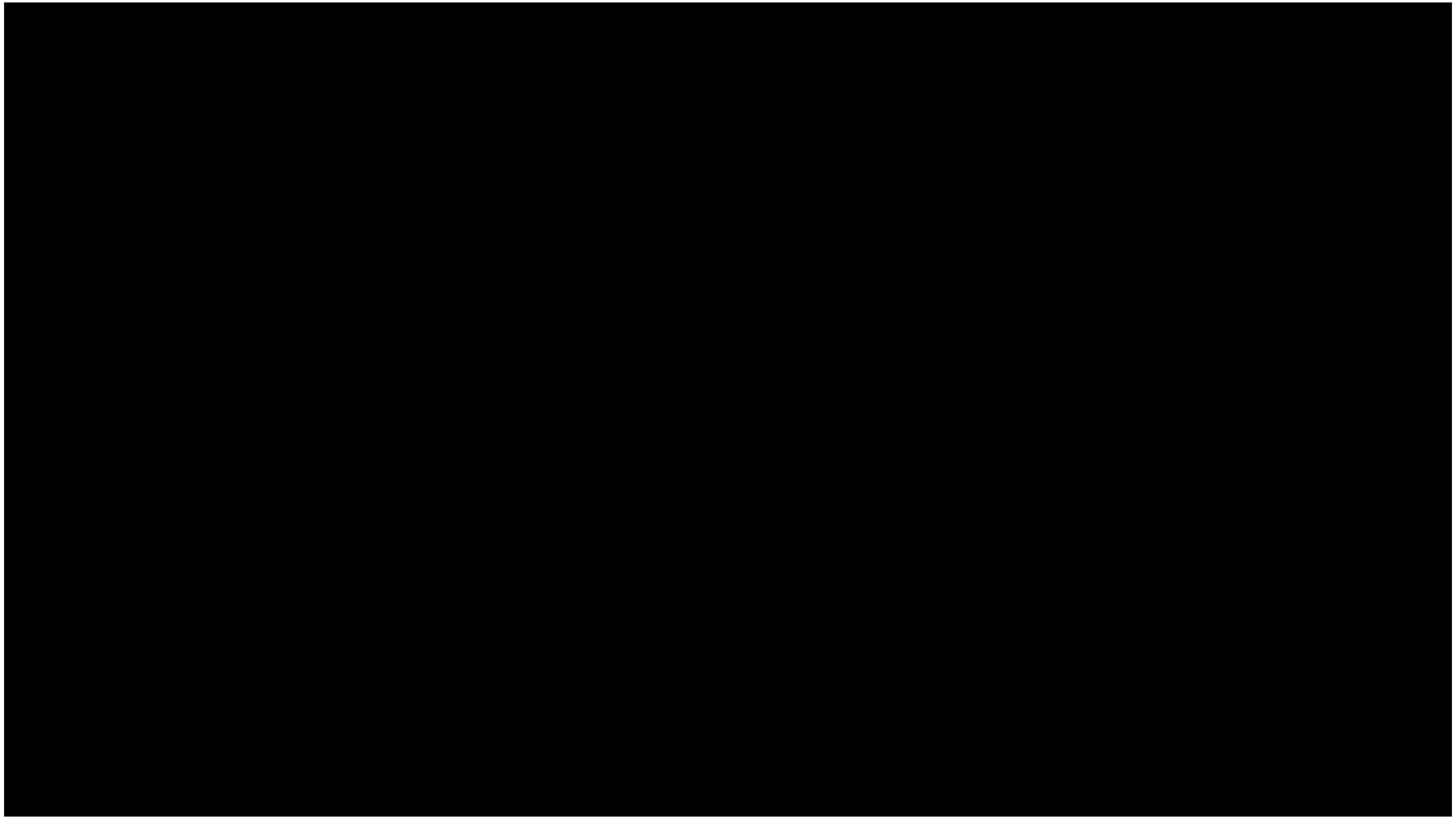
Edited by
Xavier Pennec,
Stefan Sommer, Tom Fletcher

MICCAI

Medical Image Analysis

Applications

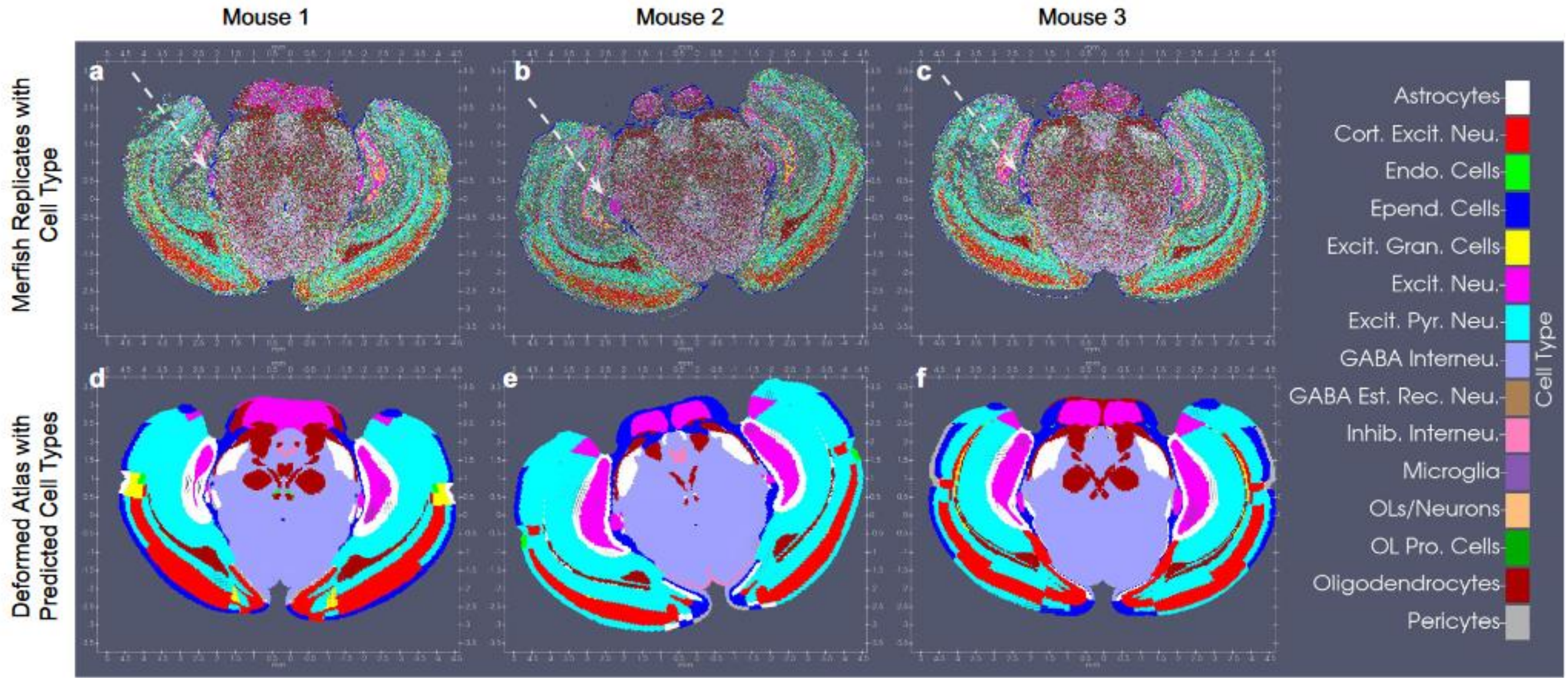
[von Radziewsky et al., Computers & Graphics 2016]



Computer Graphics

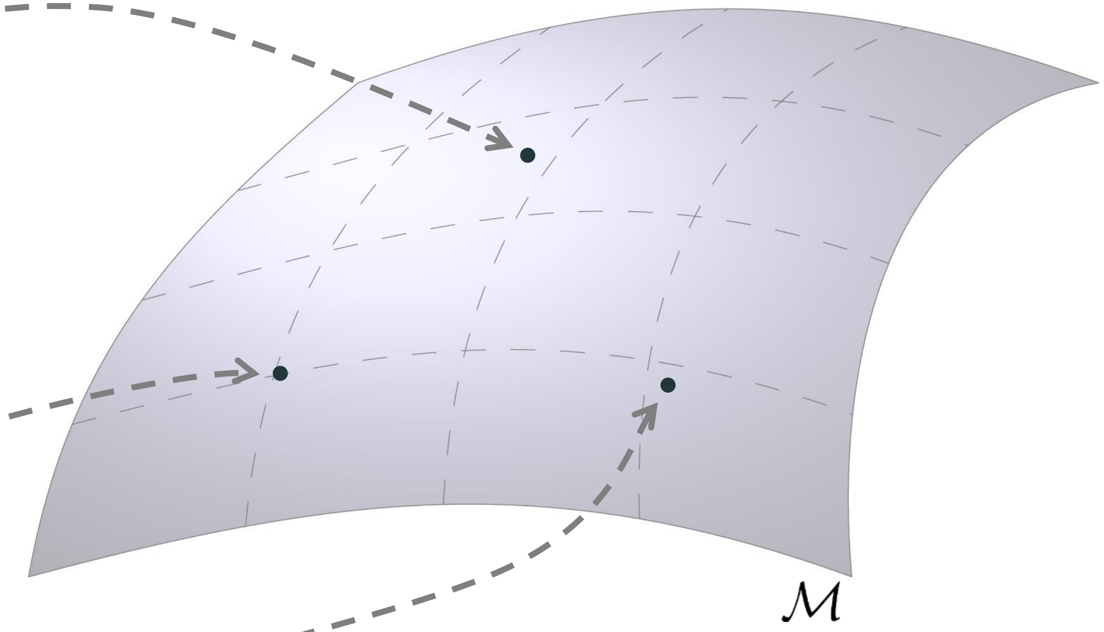
Applications

[Stouffer et al., Nature Communications 2024]



Medical Image Analysis

Space of Immersions



$$M = \text{topological surface}$$

↓

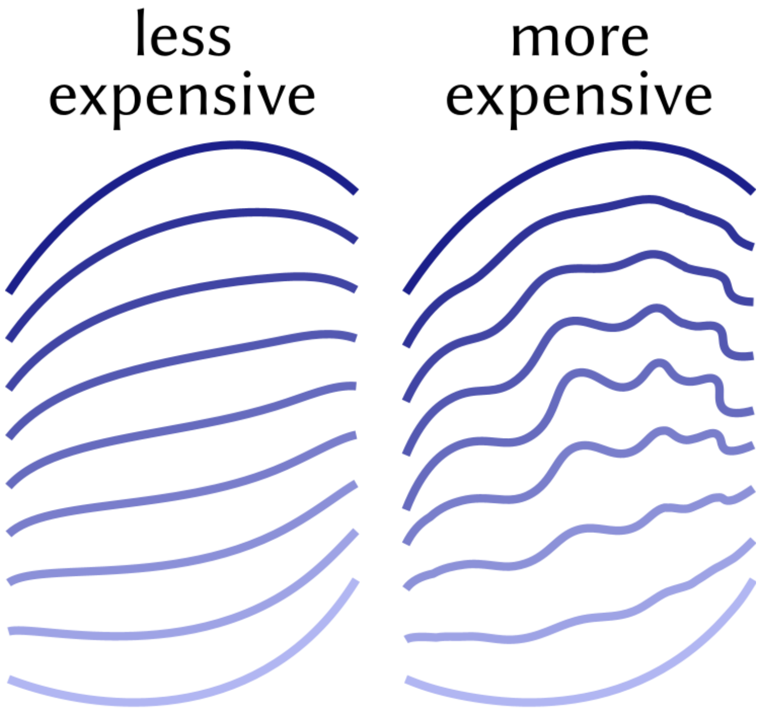
$$\mathcal{M} := \mathrm{Imm}(M) \;/\; \mathrm{SE}(3)$$

all immersions of the surface into space
(for meshes: vertex positions)

we ignore rigid transformations

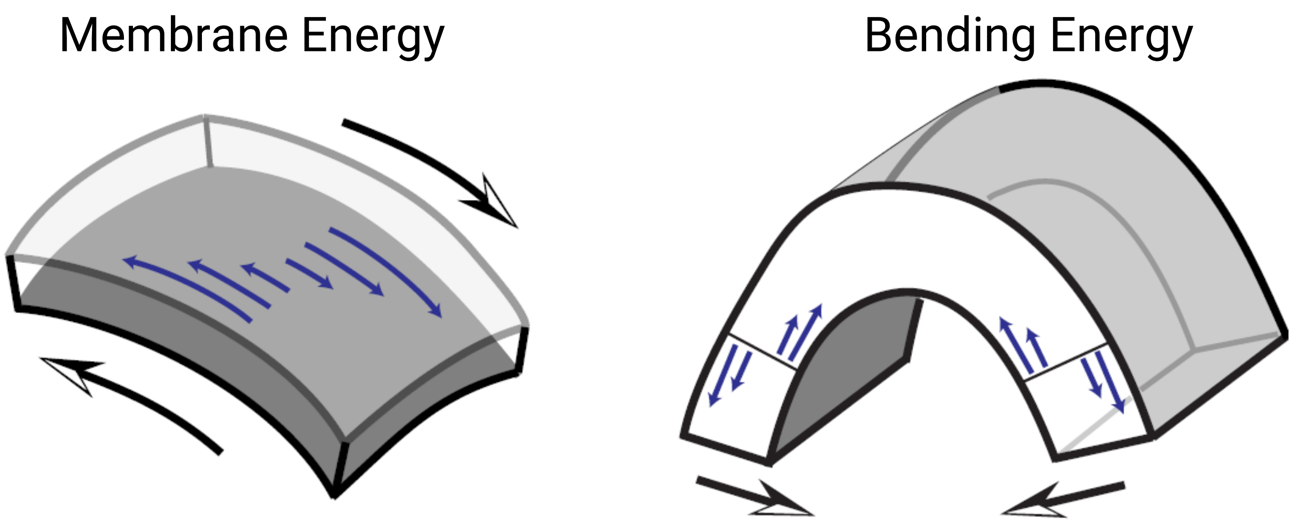
Elasticity-based Metric

Goal



Elastic Shell Energy

$$\mathcal{W}(\mathbf{x}, \tilde{\mathbf{x}}) := \mathcal{W}_{\text{membrane}}(\mathbf{x}, \tilde{\mathbf{x}}) + \mathcal{W}_{\text{bending}}(\mathbf{x}, \tilde{\mathbf{x}})$$

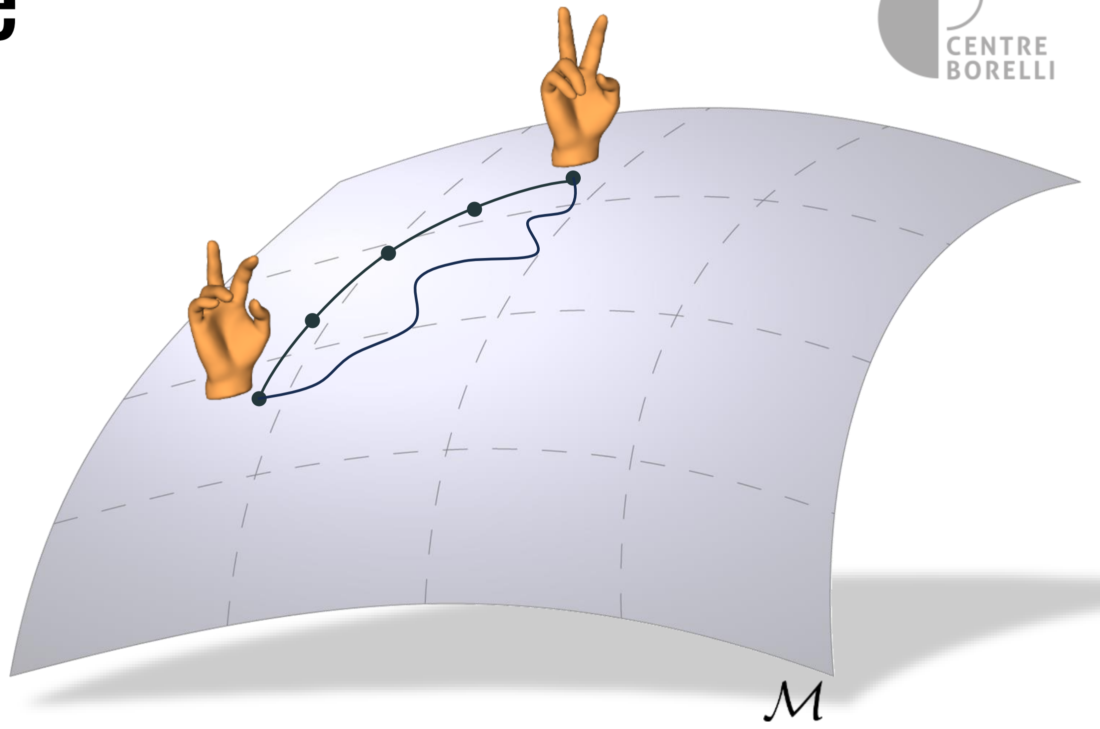


Rayleigh's Paradigm

cost of (small) deformation = heat lost due to internal friction

$$g_{\mathbf{x}}(u, v) := \frac{1}{2}\, \mathrm{d}^2_{\mathbf{y}} \mathcal{W}(\mathbf{x}, \mathbf{y})\big|_{\mathbf{y}=\mathbf{x}} (u, v)$$

Path Energy – Continuous & Discrete



Path Energy

$$\mathcal{E}(\mathbf{x}(t)) \coloneqq \int_0^1 g_{\mathbf{x}(t)}(\dot{\mathbf{x}}(t), \dot{\mathbf{x}}(t)) \, \mathrm{d}t$$

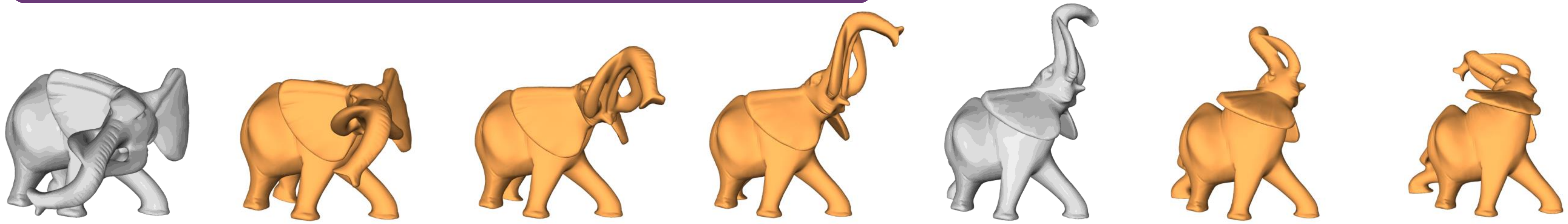
Geodesics

Minimizers of the path energy are geodesics

Time-discrete Geodesics [Heeren et al., 2012; Rumpf and Wirth, 2015]

A discrete geodesic $(x_0, \ldots, x_K)$ is a minimizer of

$$E(\mathbf{x}_0, \ldots, \mathbf{x}_K) = K \sum_{k=1}^{K} \mathcal{W}(\mathbf{x}_{k-1}, \mathbf{x}_k)$$



Question:

How can we augment our shape space to obtain self-avoiding surfaces?

Intersection-Free Shape Space

Ingredient: Repulsive Energy

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Definition: Repulsive Energy

A surface energy is called repulsive if it explodes (i.e. goes to infinity)

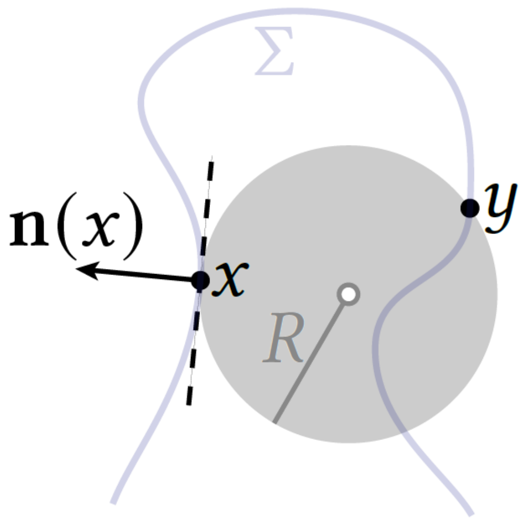
if the surface approaches self-intersection.

Tangent-Point Energy [Buck and Orloff, 1995]

$$\Phi(\mathbf{x}) := \int_\Sigma \int_\Sigma \frac{1}{R(\mathbf{n}(x), x, y)^\alpha} \, \mathrm{d}x \, \mathrm{d}y$$

with

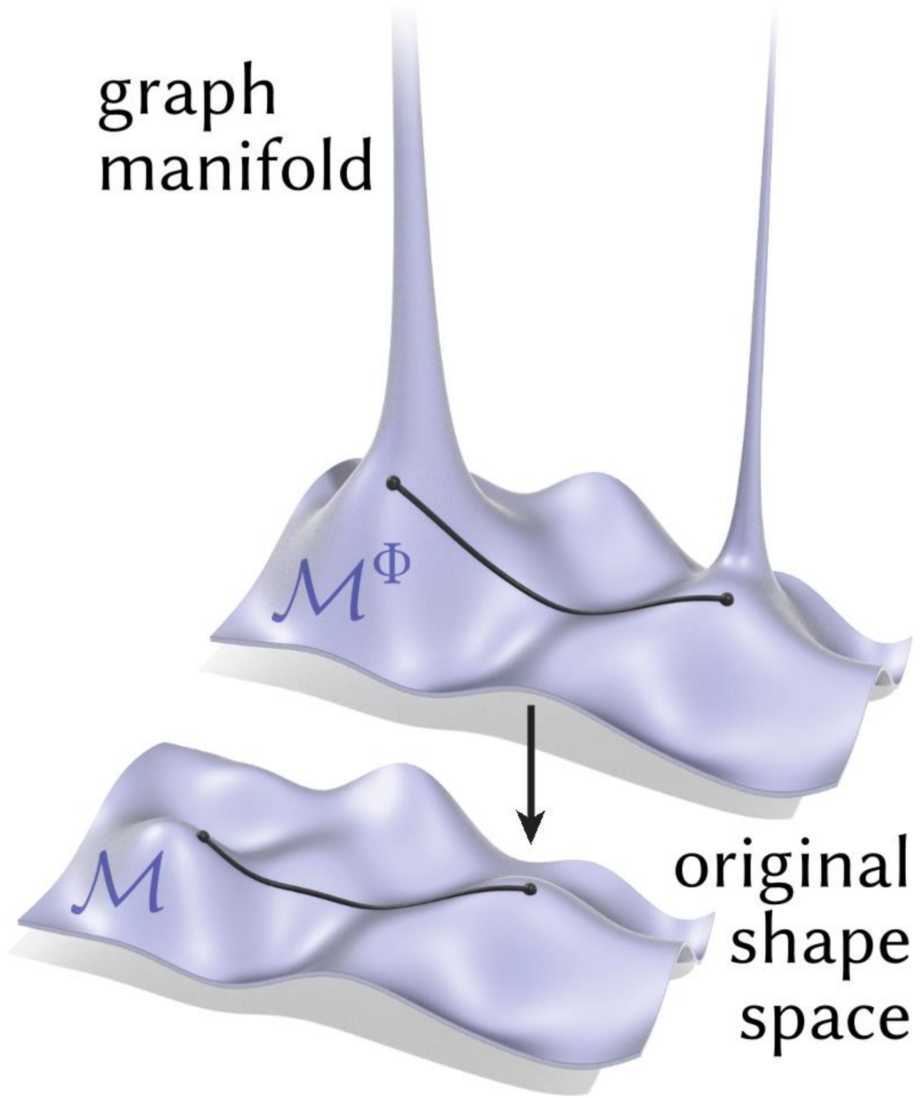
R = radius of smallest sphere tangent to *x* and passing through *y*



Repulsive for α > 4

Chris Yu, Caleb Brakensiek, Henrik Schumacher, Keenan Crane. 2021. "Repulsive Surfaces". *ACM Transactions on Graphics 40*

Graph Manifold



$$\mathcal{M}^\Phi := \{(\mathbf{x}, \Phi(\mathbf{x})) \mid \mathbf{x} \in \mathcal{M}\}$$

new metric | original metric | local change in potential

$$g^\Phi_\mathbf{x}(u, v) := g_\mathbf{x}(u, v) + \mathrm{d}_\mathbf{x}\Phi(u)\,\mathrm{d}_\mathbf{x}\Phi(v)$$

Crucial Observation

$$\mathrm{dist}_{g^\Phi}(\mathbf{x}, \mathbf{y}) = \infty \text{ if } \Phi(\mathbf{y}) = \infty$$

Repulsive Path Energy – Continuous & Discrete

Continuous Path Energy

$$\mathcal{E}(\mathbf{x}(t)) := \int_0^1 g_{\mathbf{x}(t)}(\dot{\mathbf{x}}(t), \dot{\mathbf{x}}(t))\,\mathrm{d}t + \int_0^1 d_{\mathbf{x}(t)}\Phi(\dot{\mathbf{x}}(t))^2\,\mathrm{d}t$$

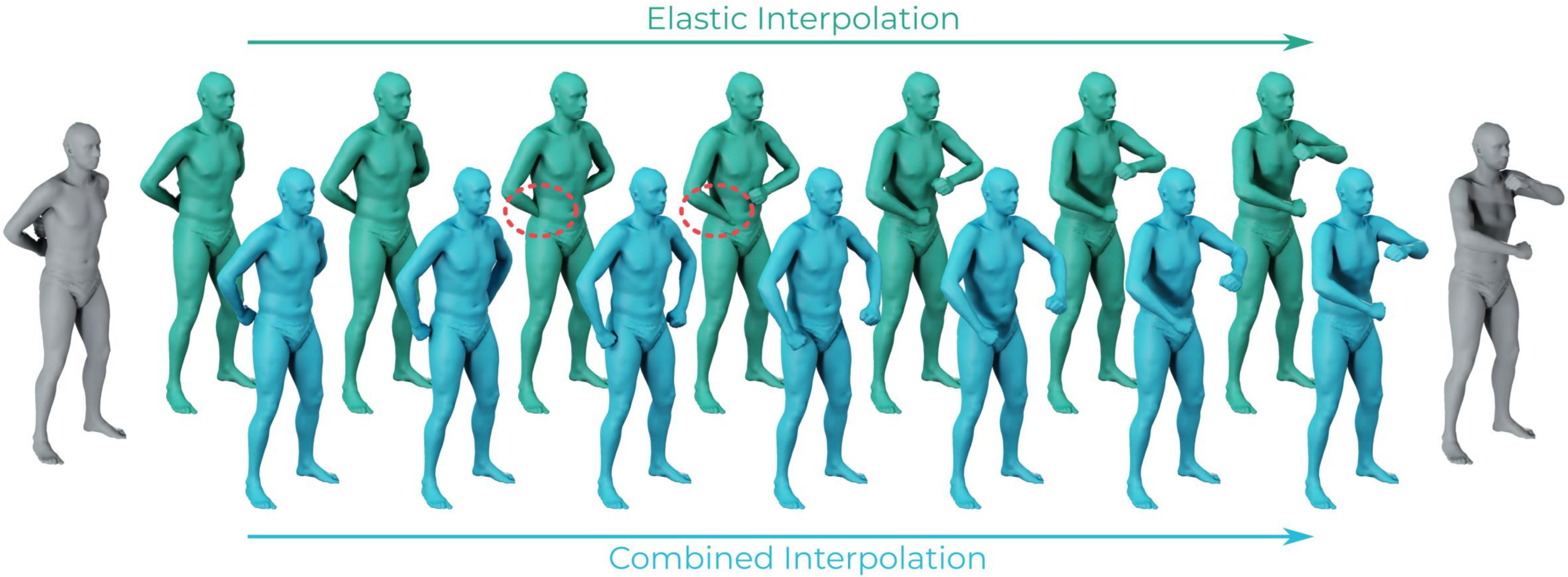
Discrete Path Energy

$$E(\mathbf{x}_0, \ldots, \mathbf{x}_K) = K\sum_{k=1}^{K}\mathcal{W}(\mathbf{x}_{k-1}, \mathbf{x}_k) + (\Phi(\mathbf{x}^{k-1}) - \Phi(\mathbf{x}^k))^2$$

Repulsive Path Energy – Continuous & Discrete

Discrete Path Energy

$$E(\mathbf{x}_0, \ldots, \mathbf{x}_K) = K \sum_{k=1}^{K} \mathcal{W}(\mathbf{x}_{k-1}, \mathbf{x}_k) + (\Phi(\mathbf{x}^{k-1}) - \Phi(\mathbf{x}^k))^2$$

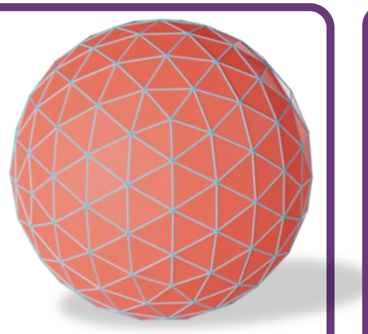


Discretization & Numerics

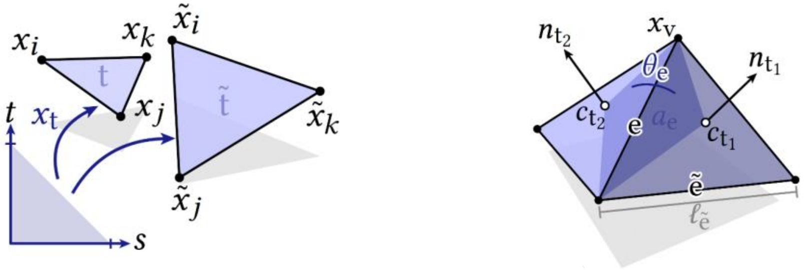
Discrete Surfaces

Triangle Mesh $\mathrm{M} = (\mathrm{V}, \mathrm{E}, \mathrm{T})$

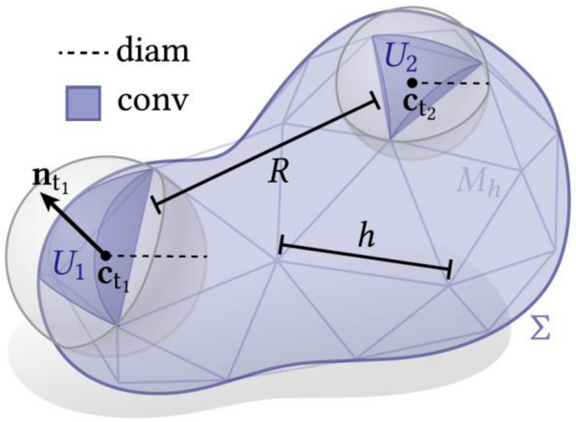
Piece-wise Affine Immersion $\mathbf{x}\colon \mathrm{M} \to \mathbb{R}^3$



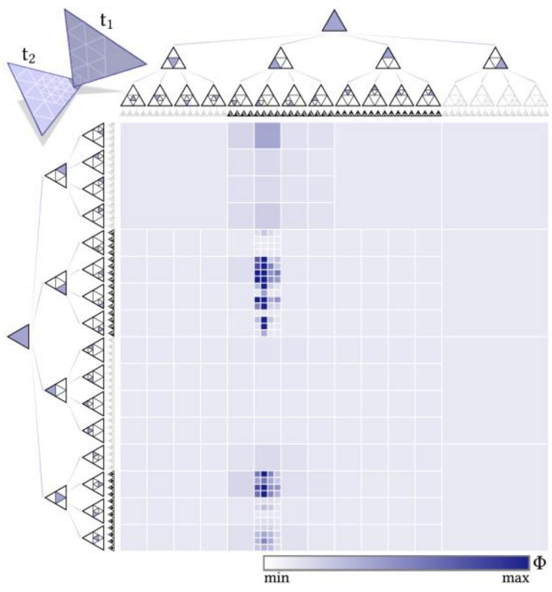
Discrete Shells Energy [Heeren et al., 2014]



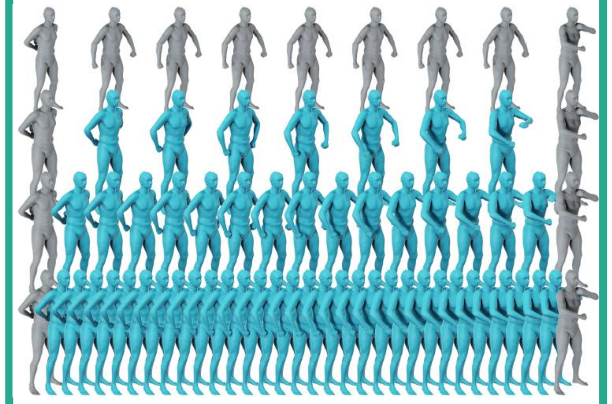
Multipole Approximation



Adaptive Refinement



Temporal Refinement



Spatial Prolongation

Trust-Region Based Optimization

Examples

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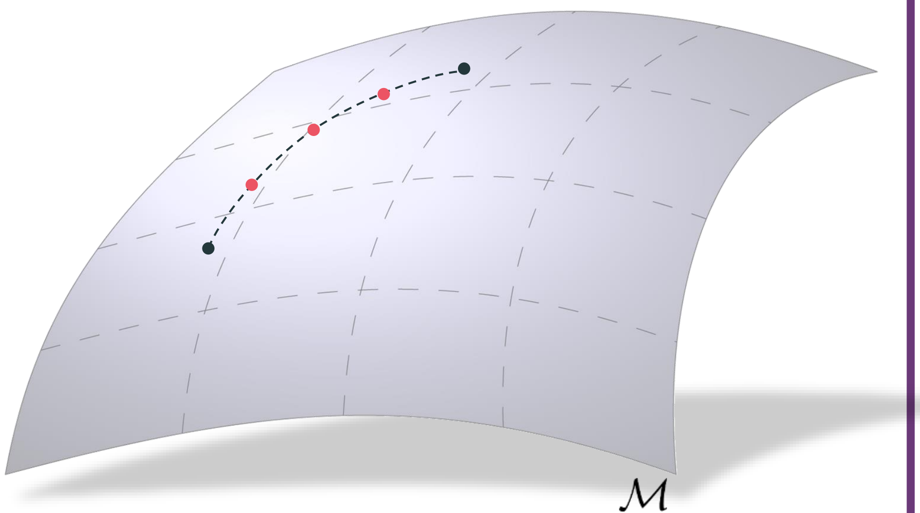
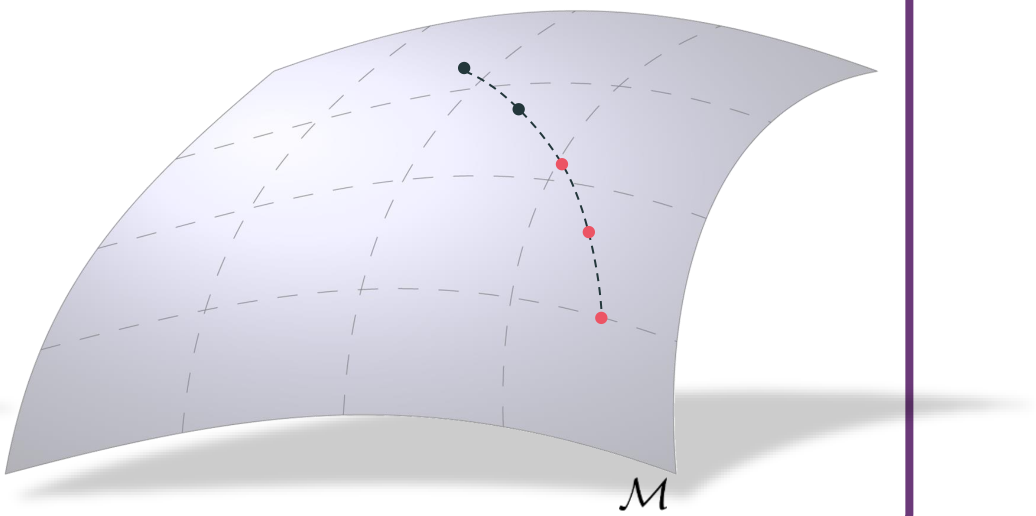
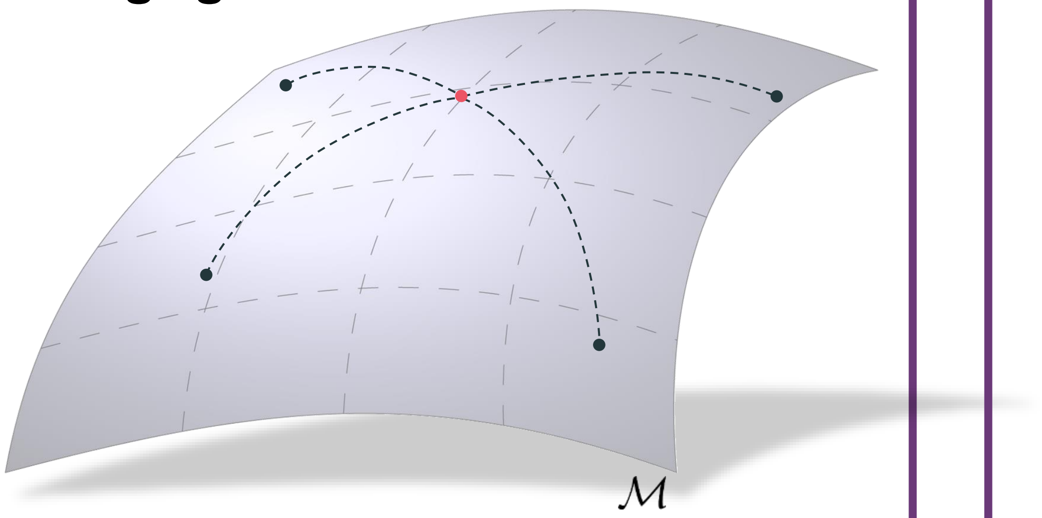
Interpolation



Extrapolation



Averaging

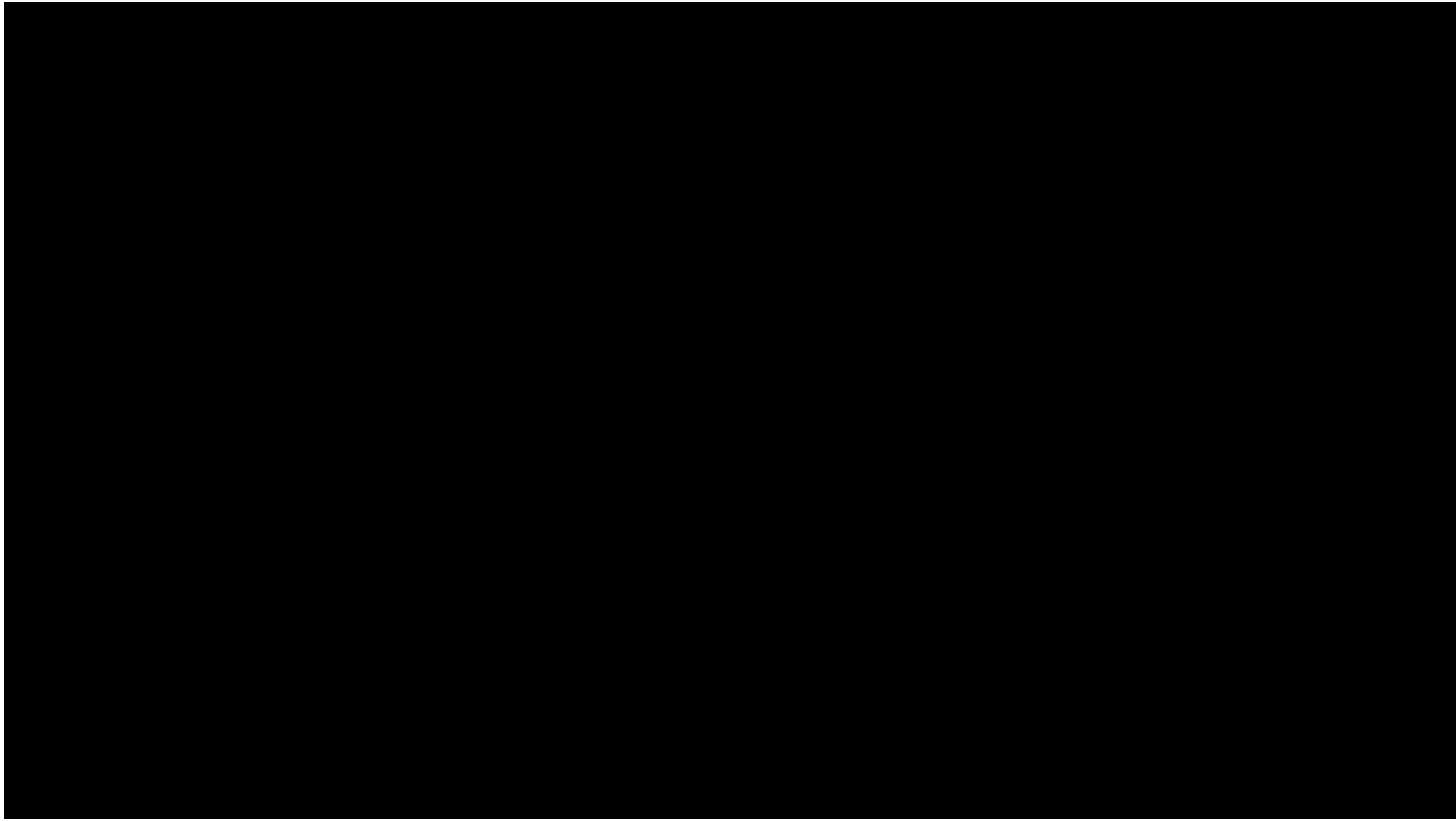


Bonus

Excellent

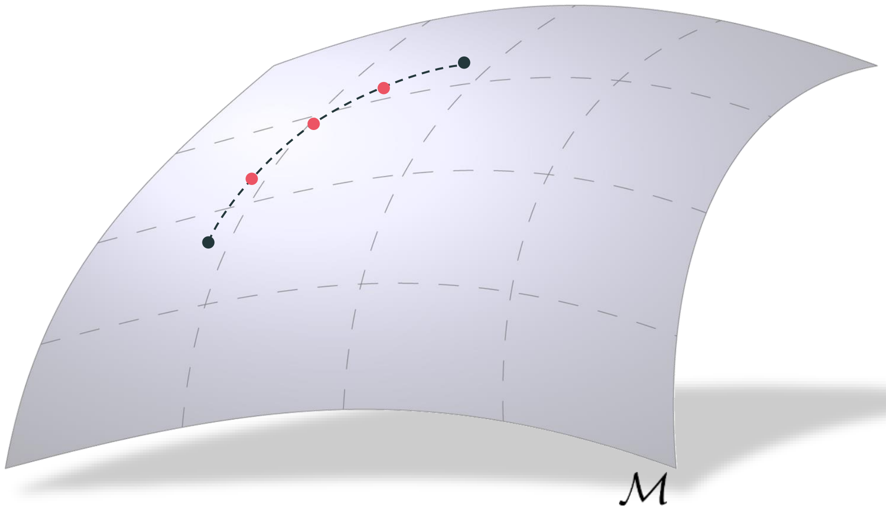
Avoiding Obstacles

Obstacle

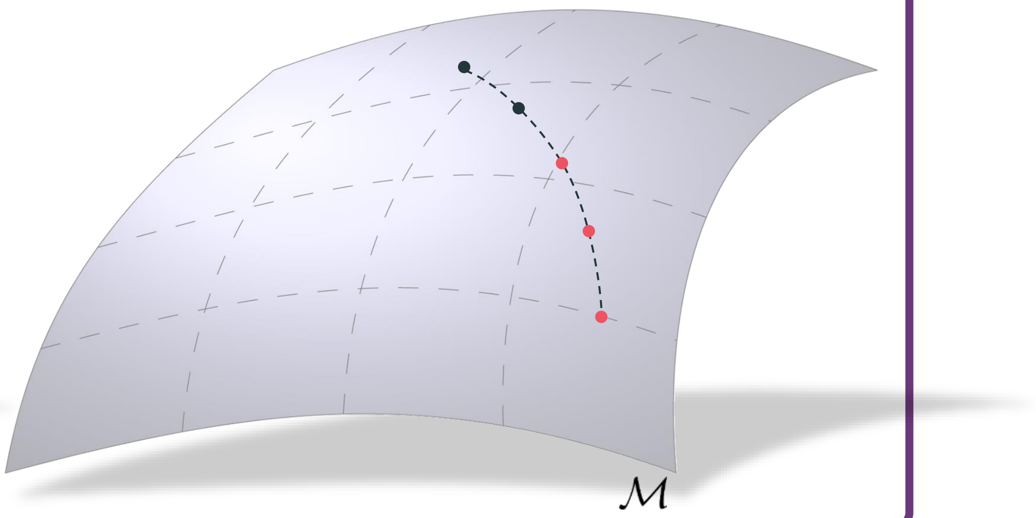


Simplifying Tedious Work

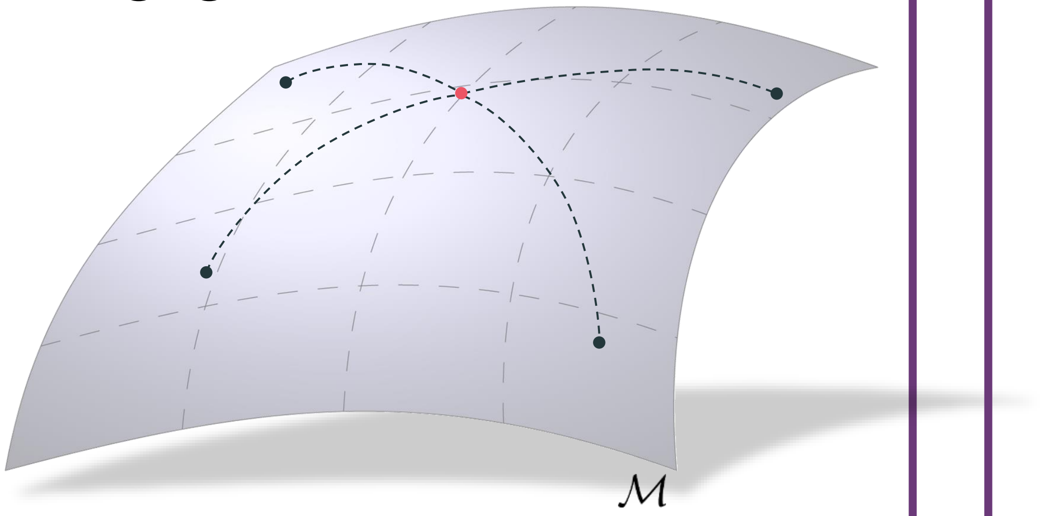
Interpolation



Extrapolation

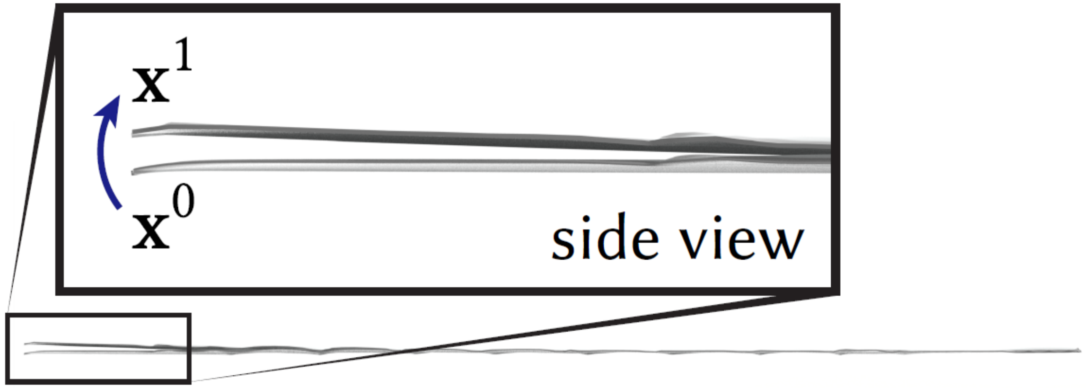


Averaging

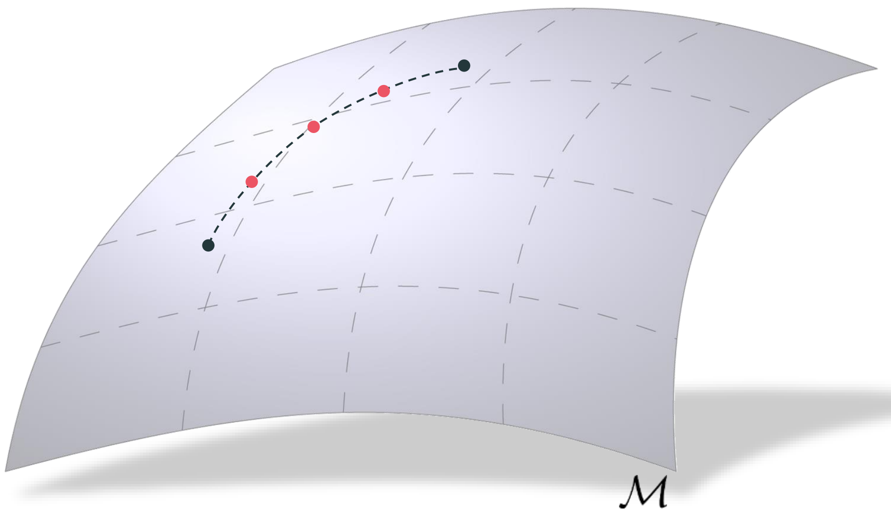


Bonus

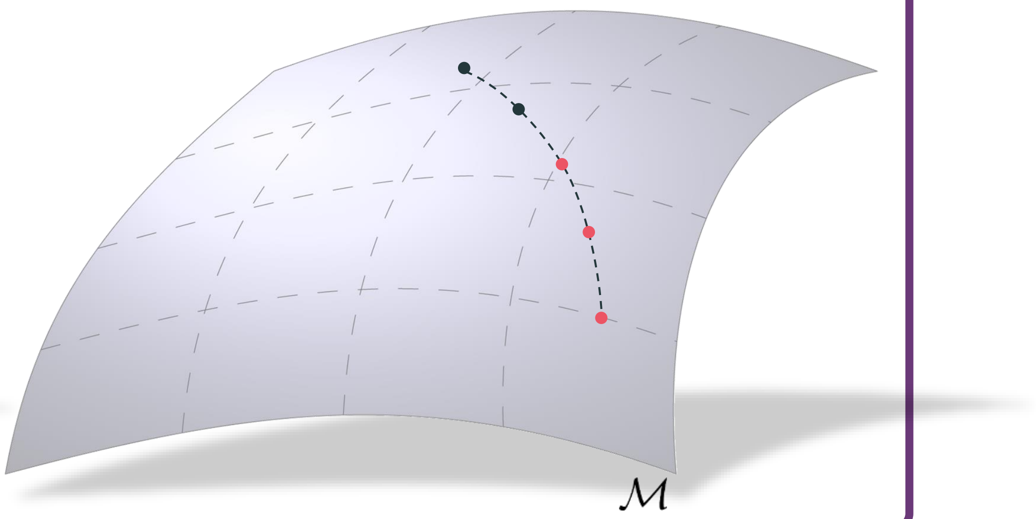
Curling a Leaf



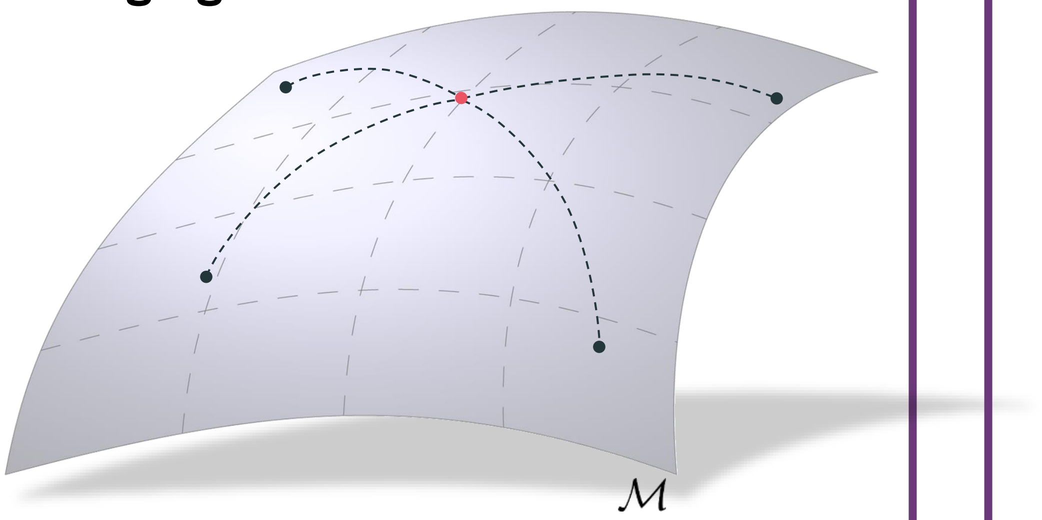
Interpolation



Extrapolation

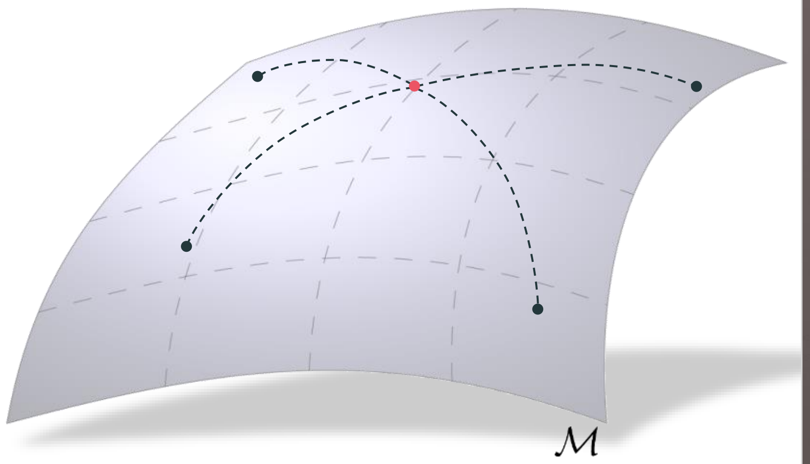


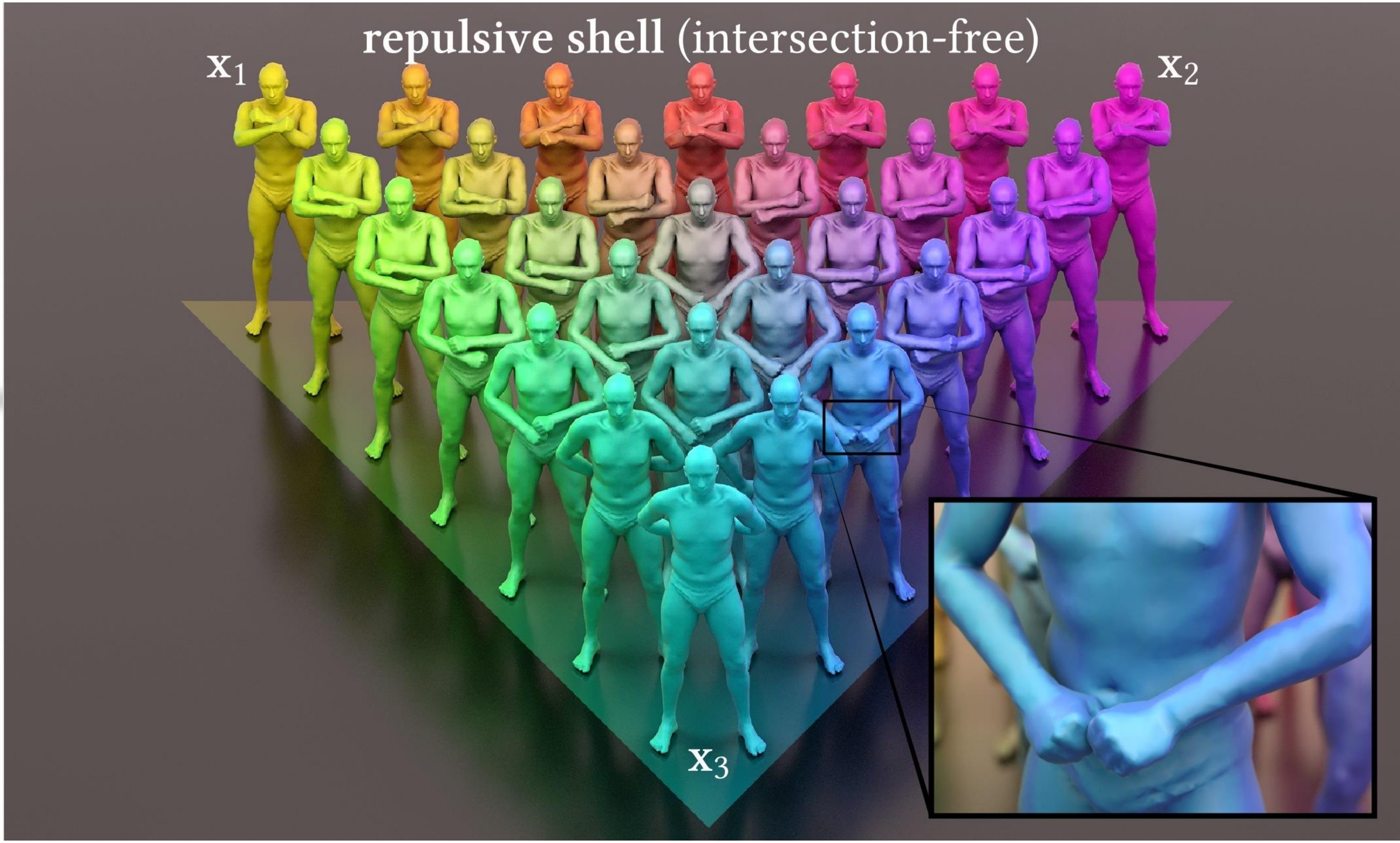
Averaging



Bonus

Averaging


$\mathcal{M}$


repulsive shell (intersection-free)
$\mathbf{x}_1$
$\mathbf{x}_2$
$\mathbf{x}_3$

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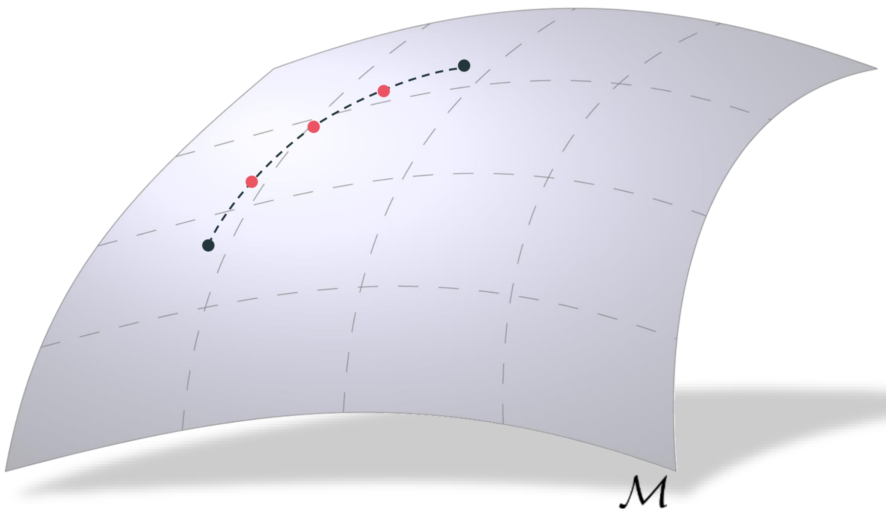
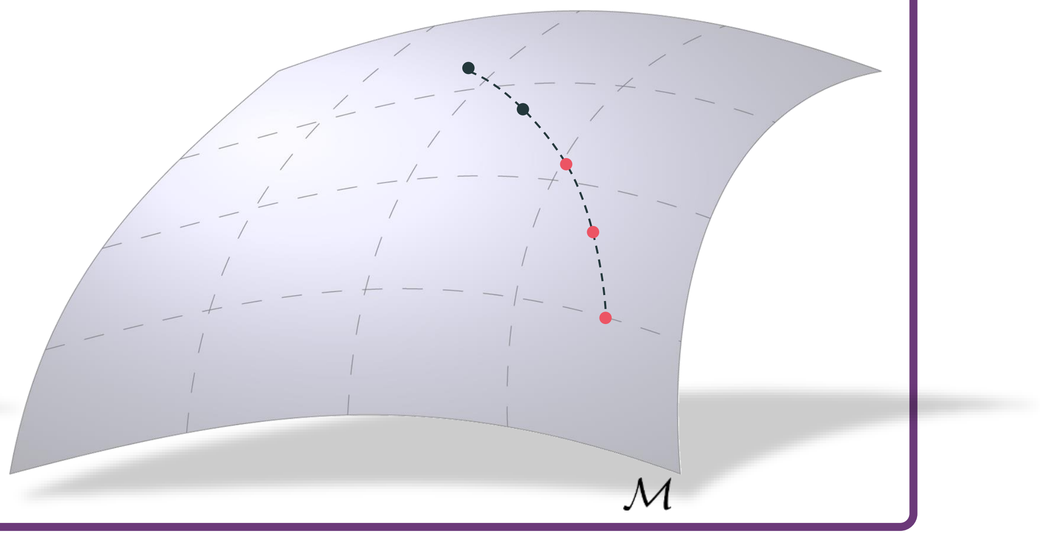
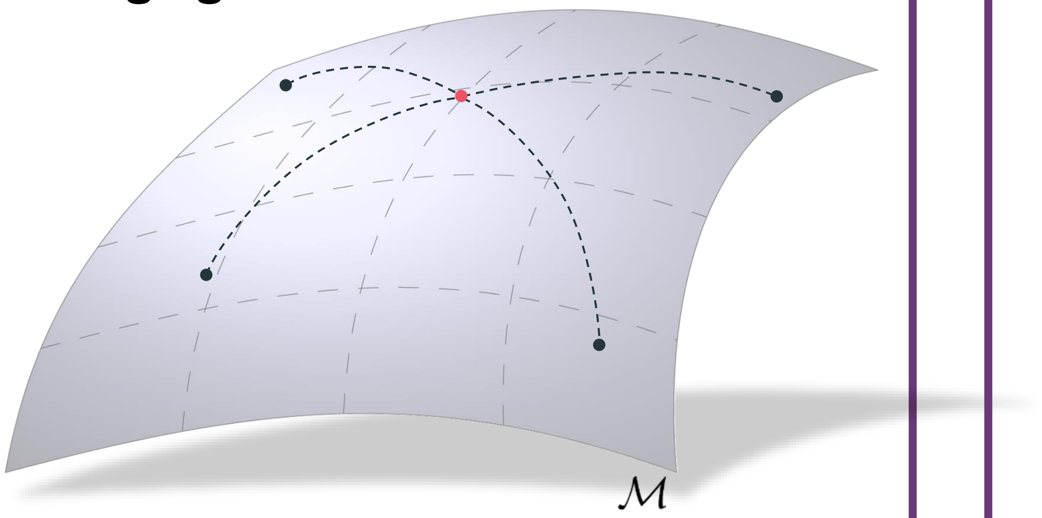
Interpolation



Extrapolation



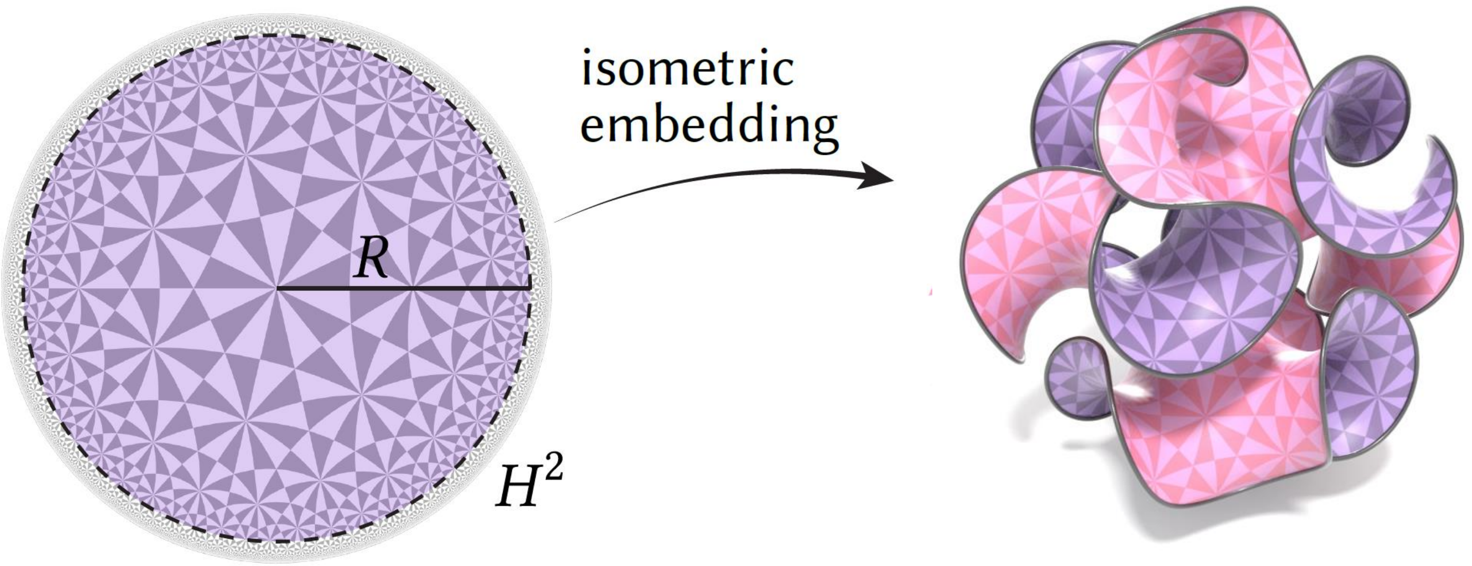
Averaging



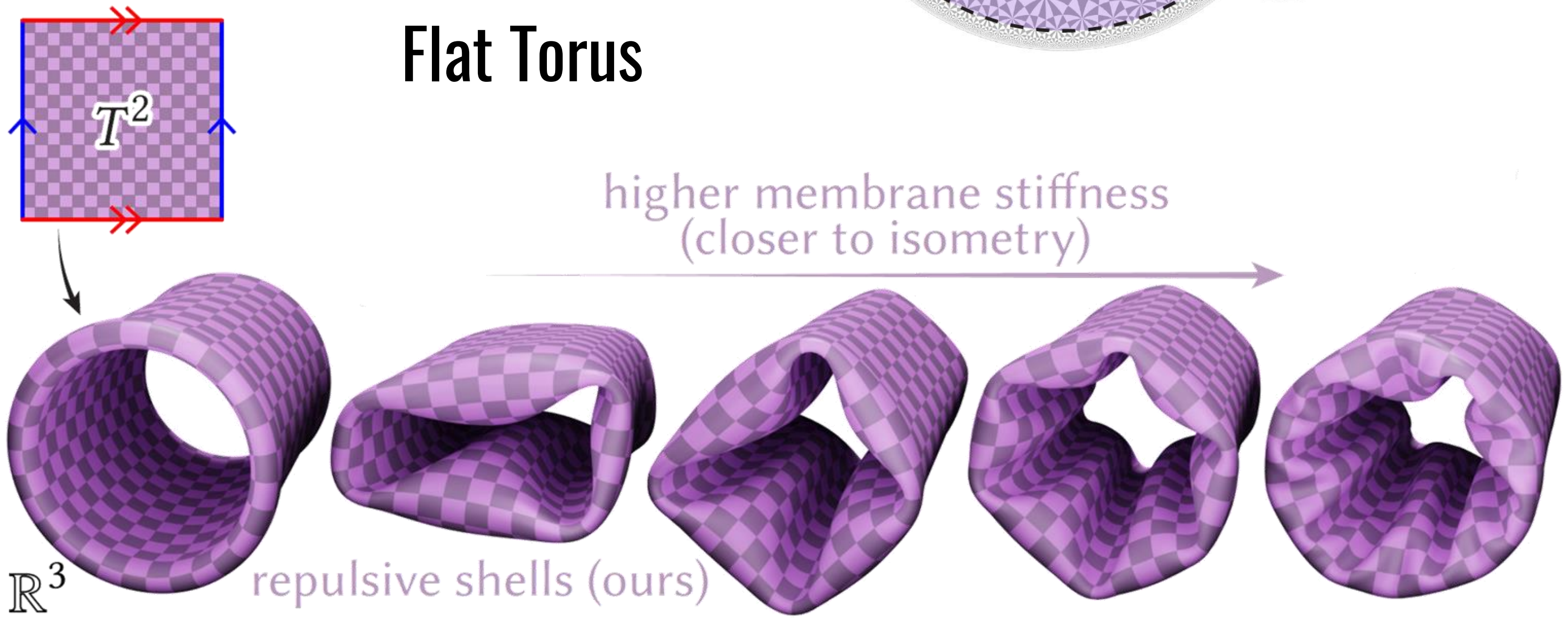
Bonus

Isometric Embeddings

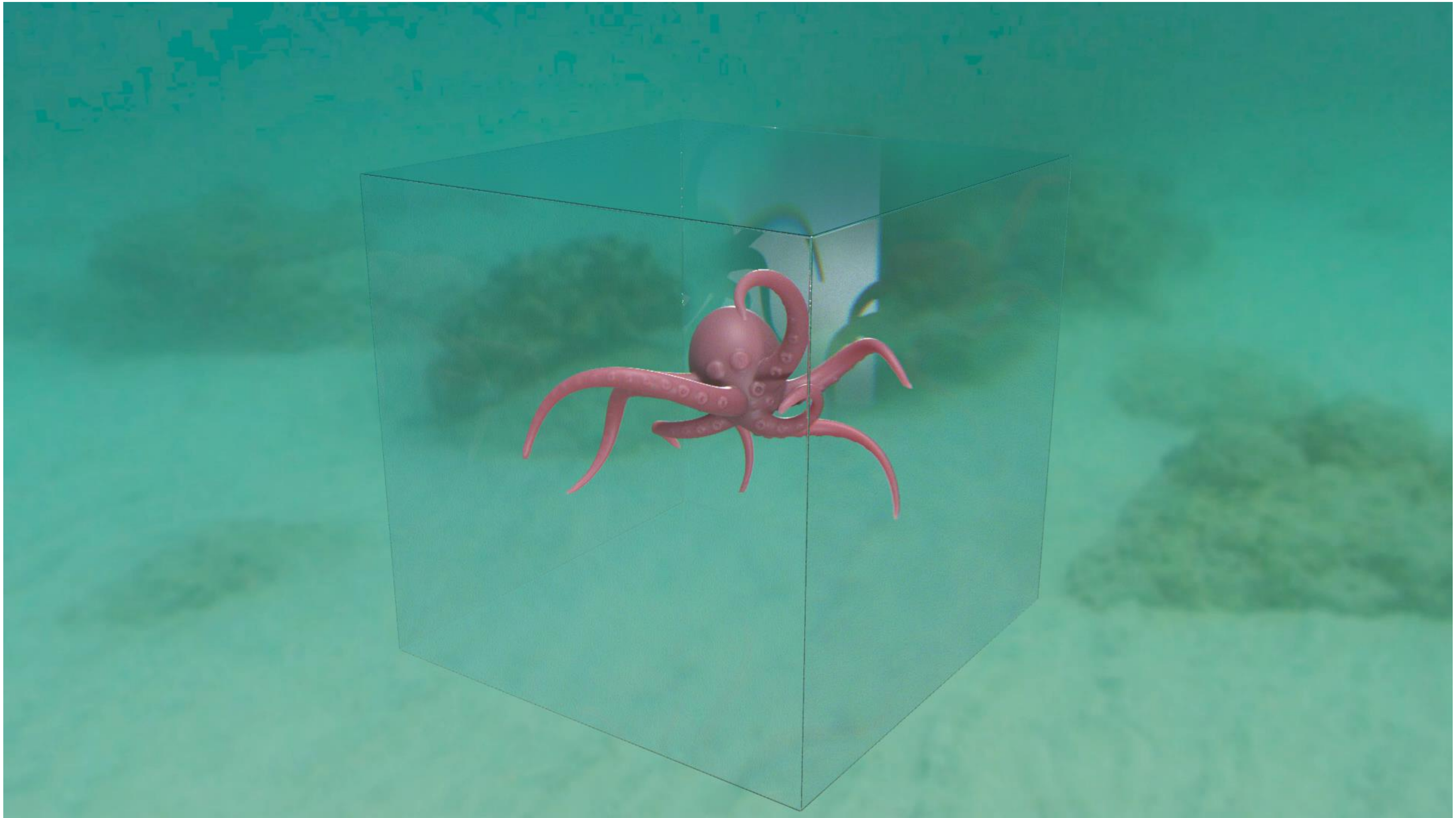
Hyperbolic Disk



Flat Torus

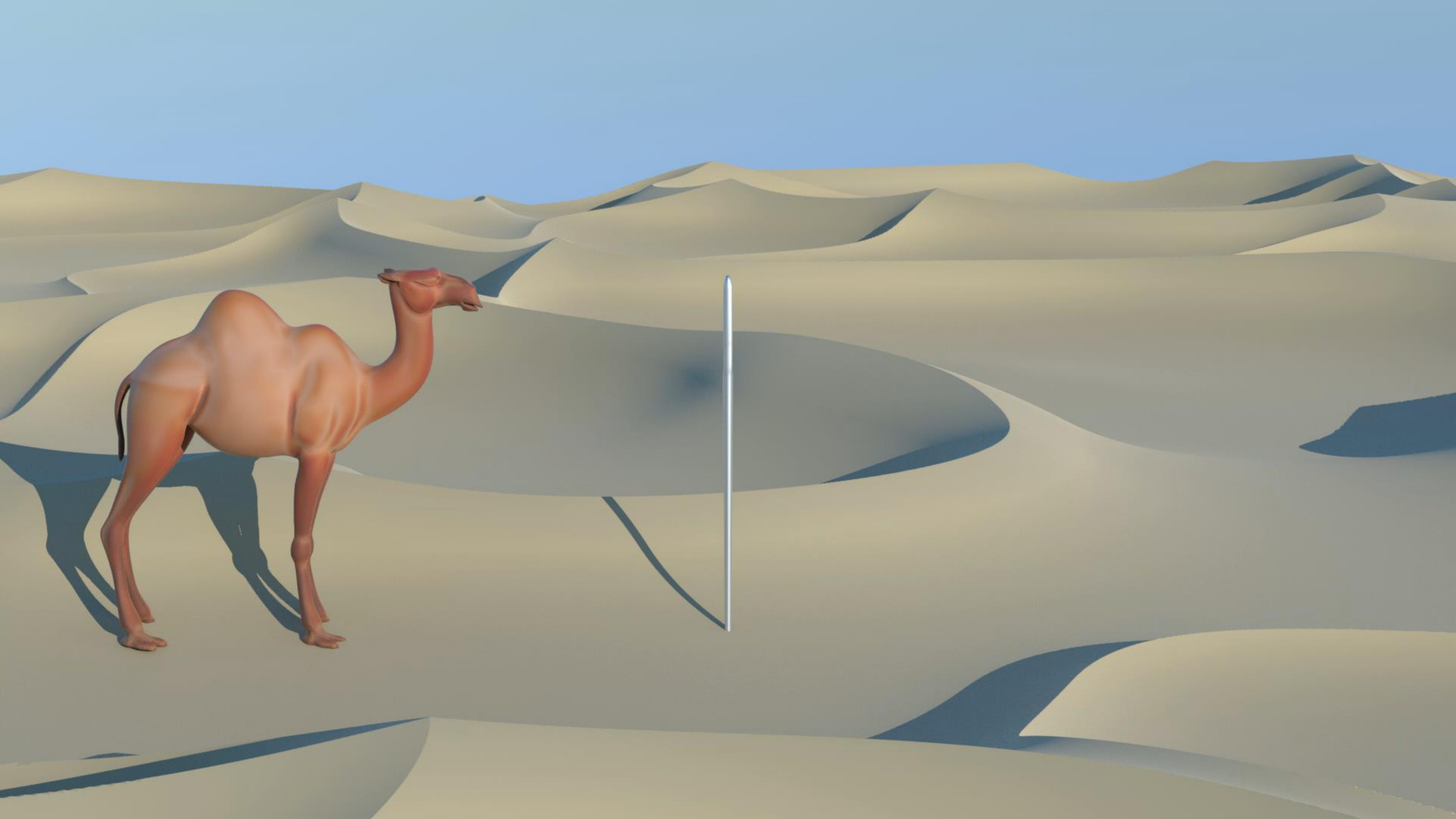


Octopus in a Box



“ It is easier for a camel to go through the eye of a needle,
than for a rich man to enter into the kingdom of God.”

Matthew 19:24



Thank you!

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"Repulsive Shells". *ACM Transaction on Graphics* (2024).

josua@josuasassen.com https://josuasassen.com