Higher–order conditions beyond first–order optimality in control problems

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Why are higher-order conditions needed?

Finite-dimensional optimization

$J(x,y) = x^2 - y^2. \qquad \nabla J(0,0) = 0.$

First-order stationarity holds.

$$D^2 J(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}.$$

The Hessian has a negative direction:

$$J(0,\varepsilon) = -\varepsilon^2 < J(0,0).$$

The origin is a saddle.

Optimal control

- An extremal may satisfy the Pontryagin Maximum Principle and still fail to minimize the cost.
- Higher-order variations can reveal improving directions invisible at first order.
- Lie brackets encode these effects in control-affine systems.

Takeaway

First-order conditions detect stationarity. Higher-order conditions can exclude nonminimizing extremals.

Goh and Legendre-Clebsch Conditions for Nonsmooth Control Systems (F.A.,F. Rampazzo)

$$\begin{cases} \min\limits_{u\in\mathcal{U}} \Psi(T, x(u;T)) \qquad (\Psi \text{ is the final cost function}) \\ \\ x(u;\cdot) \text{ solution of } \begin{cases} \dfrac{dx}{dt} = f(x) + \sum\limits_{i=1}^{m} g_i(x)u^i, \quad (\text{the Dynamics}) \\ x(0) = \hat{x}, \\ \\ (T, x(T)) \in \mathfrak{T}, \quad (\mathfrak{T} \subset \mathbb{R}^{1+n} \text{ is the Target}), \end{cases} \end{cases} \tag{P}$$

The controls $u : [0, T] \to U$ take values in a subset $U \subset \mathbb{R}^m$.

Smooth picture

If the data $f, g_1, \dots g_m$, $\mathfrak{T}$ are C^1, a necessary condition for minima is the Maximum Principle

$H(x, p_0, p, u) = p_0 + p \cdot (f(x) + \sum_i g_i(x)u^i)$.

Theorem (Pontryagin Maximum Principle)

Let $(\overline{T}, \overline{u}, \overline{x})$ be a local minimizer for the problem (P). Then there exist multipliers $(p_0, p, \lambda) \in \mathbb{R}^* \times AC\left([0, \overline{T}]; (\mathbb{R}^n)^*\right) \times \mathbb{R}^*$, $\lambda \geq 0$ such that:

i) $(p_0, p, \lambda) \neq 0$;

ii) $\frac{dp}{dt}(t) = -\frac{\partial H}{\partial x}(\overline{x}(t), p_0, p, \overline{u}(t))$ *(adjoint equation)*

iii) $(p_0, p)(\overline{T}) \in -\lambda D\Psi\left(\overline{T}, \overline{x}(\overline{T})\right) - \mathcal{N}_{(\overline{T}, \overline{x}(\overline{T}))}\mathfrak{T}$

iv) $\max_{u \in U} H(\overline{x}(t), p_0, p(t), u) = H(\overline{x}(t), p_0, p(t), \overline{u}(t))$ *a.e.*

Goh and Legendre–Clebsch conditions (smooth data)

For a singular minimizing control with values in int(U), the PMP adjoint satisfies, for almost every t:

$$p(t) \cdot [g_i, g_j](\bar{x}(t)) = 0, \qquad i, j = 1, \dots, m \quad \text{(Goh)},$$
$$p(t) \cdot [f, g_j](\bar{x}(t)) = 0, \qquad j = 1, \dots, m \quad \text{(LC2)}.$$

For one control, $g = g_1$, with the regularity needed for the iterated bracket:

$$p(t) \cdot [g, [f, g]](\bar{x}(t)) \leq 0 \qquad \text{(LC3)}.$$

Lie bracket convention:

$$[X, Y] = DY \cdot X - DX \cdot Y.$$

Smooth generalizations: H. Frankowska–D. Tonon (2013), M. S. Aronna–F. Bonnans–B. S. Goh (2017).

For the CoV problem of minimizing

$$\int_a^b L(t, x, \dot{x}) dt$$

the **Legendre-Clebsch condition** reads

$$\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}(t, x(t), \dot{x}(t)) \geq 0, \quad \forall t \in [a, b]$$

i.e. the quadratic form is positive semidefinite (...sort of *convexity*).

Reachable and Profitable sets

For some $\delta > 0$, the extended *δ–reachable set* $\mathcal{R}_\delta \subset \mathbb{R}^{n+1}$, is defined in the following way:

$$\mathcal{R}_\delta =: \left\{ \begin{pmatrix} x(T) \\ \Psi(x(T)) \end{pmatrix}, \text{ s.t. } (u,x) \text{ is a s.t. } \|x - \overline{x}\|_{C^0} + \|u - \overline{u}\|_1 < \delta \right\}.$$

instead the *profitable set* is defined as below:

$$\mathcal{P} := \left(\mathfrak{T} \times \,]-\infty, \Psi(\overline{x}(T))\,[\right) \bigcup \left\{ \begin{pmatrix} \overline{x}(T) \\ \Psi(\overline{x}(T)) \end{pmatrix} \right\} \subset \mathbb{R}^{n+1}.$$

Approximating cones and separation

Fix a reference control $\bar{u}(\cdot)$ and consider small variations parametrized by $\varepsilon \in \mathbb{R}^N$.

$$\varepsilon \longmapsto x_\varepsilon(T)$$

The image of this map describes the local reachable set.

If the data are smooth, the endpoint map admits a linear approximation:

$$x_\varepsilon(T) = x_0(T) + L\,\varepsilon + o(\varepsilon).$$

Linearization and approximating cones

The image of the linear map L generates a cone

$$C := L(\mathbb{R}^N),$$

which approximates the reachable set at first order.

This cone is called a *Boltyanski approximating cone*.

Optimality can be reformulated as a separation problem between the reachable cone and the profitable set.

Boltyanski's approssimating cone

Definition. $\mathrm{E} \subseteq \mathbb{R}^n$ any subset, and $y \in \mathrm{E}$.
A convex cone C is called a
Boltyanski's approximating cone to E **at** y
if there exists a single-valued map $F : \mathbb{R}^N \to E \subseteq \mathbb{R}^n$, and a linear operator $L : \mathbb{R}^N \to \mathbb{R}^n$ such that $F(\varepsilon) = F(0) + L\varepsilon + o(\varepsilon)$ and $L(\mathbb{R}^N) = C$

Set-Separation Theorem for Boltyanski's approssimating cones

Definition. E_1 and E_2 are locally separated at y if $E_1 \cap E_2 \cap B(y, r) = \{y\}$ for some $r > 0$

Theorem. If E_1 and E_2 are locally separated at y
and C_1 [resp. C_2]
is a Boltyanski's approssimating cone at y
of E_1 [resp.E_2]
then there exists a linear form λ verifying

$$\lambda \cdot c_1 \geq 0 \qquad \lambda \cdot c_2 \leq 0 \qquad \forall (c_1, c_2) \in C_1 \times C_2$$

The non-smooth picture

If the data $f, g_1, \dots g_M$ are just Lipschitz continuous

UNAVOIDABLE ISSUES:

1) How do we **replace the Adjoint Equation**?

   ...can we utilize the one of the (first order) Nonsmooth Maximum principle?

   $$\frac{dp}{dt}(t) \in -\partial_x^C H(\overline{x}(t), p_0, p, \overline{u}(t))$$

2) What about **Lie brackets**

   $[X, Y]$ of **nonsmooth** fields X, Y?

   Might we use the *set-valued Lie brackets* $[X, Y]_{set}$ *(F.R.- H.Sussmann)?*

3) **Does a common theoretical frame include the non-smoothness of both 1) and 2)?**

Lie brackets of non-smooth vector field

Definition (F.R.-H.Sussmann, 2001)

Let X, Y be locally Lipschitz vector fields. We define the **set-valued Lie bracket of** X, Y as

$$[X, Y]_{set}(x) := co\left\{ v = \lim_{x_k \to x} DY(x_k)X(x_k) - DX(x_k)Y(x_k), \right\}$$

where limits are along sequences $(x_k)_{k\in\mathbb{N}} \subset Diff(X) \cap Diff(Y)$, $Diff(X)$ [resp. $Diff(Y)$] being the *full measure* subsets of points where X [resp. Y] are differentiable

This set-valued Lie bracket resulted successful for the generalization of
Commutativity Theorem(F.R,-H.Sussmann, 2007),
Frobenius Theorem(F.R., 2007),
Rashewski-Chow Theorem(E.Feleqi,R.Gupta,F.R., 2022).

Reachable and Profitable sets

For some $\delta > 0$, the extended *δ–reachable set* $\mathcal{R}_\delta \subset \mathbb{R}^{n+1}$, is defined in the following way:

$$\mathcal{R}_\delta =: \left\{ \begin{pmatrix} x(T) \\ \Psi(x(T)) \end{pmatrix}, \text{ s.t. } (u,x) \text{ is a s.t. } \|x - \overline{x}\|_{C^0} + \|u - \overline{u}\|_1 < \delta \right\}.$$

instead the *profitable set* is defined as below:

$$\mathcal{P} := \left(\mathfrak{T} \times \,]-\infty, \Psi(\overline{x}(T))\,[\right) \bigcup \left\{ \begin{pmatrix} \overline{x}(T) \\ \Psi(\overline{x}(T)) \end{pmatrix} \right\} \subset \mathbb{R}^{n+1}.$$

Quasi Differential Quotient for set-valued maps

Consider: a set-valued map $F : \mathbb{R}^n \to \mathbb{R}^m$, any point $\bar{x} \in \mathbb{R}^n$, and any subset (of directions)$\Gamma \subseteq \mathbb{R}^n$.

Definition

A compact subset $\Lambda \subset Lin(\mathbb{R}^n, \mathbb{R}^m)$ is a **Quasi Differential Quotient (*QDQ*) of F at $\bar{x}$ in the direction of Γ** if, for a $\delta^* > 0$ and a modulus $\rho : [0, +\infty[\to [0, +\infty[$, one has:

For every $\delta \in (0, \delta^*)$ there is a continuous map $(L_\delta, h_\delta) : \mathbb{R}^n \to Lin(\mathbb{R}^n, \mathbb{R}^m) \times \mathbb{R}^m$

such that

$$\left.\begin{array}{l} d(L_\delta(x), \Lambda) \leq \rho(\delta), \quad |h_\delta(x)| \leq \delta\rho(\delta) \\ \\ F(x) \ni F(\bar{x}) + L_\delta(x) \cdot (x - \bar{x}) + h_\delta(x) \end{array}\right\} \forall x \in (\bar{x} + B_\delta) \cap \Gamma$$

From linear maps to set–valued approximations

Consider the endpoint map $E : \varepsilon \mapsto x_\varepsilon(T)$.

In the nonsmooth case we approximate E at $\varepsilon = 0$ through δ–dependent linearizations:

$$E(\varepsilon) \ \in \ E(0) \ + \ L_\delta(\varepsilon) \ + h_\delta(\varepsilon), \qquad |h_\delta(\varepsilon)| \leq \delta \, \rho(\delta),$$

for all ε with $|\varepsilon| \leq \delta$.

A QDQ is the compact set Λ of linear maps such that $d(L_\delta(\varepsilon), \Lambda) \leq h(\delta)$: it replaces a single derivative by a controlled *family* of approximate linear models, and generates a family of approximating cones.

Very good news:

- The set of end-point $v(T)$ of the solutions v to the non-smooth Variational Inclusion

  $$\dot{v} \in \partial_x^C \left(f(\bar{x}(t)) + \sum_{i=1}^{m} g_i(\bar{x}(t))\bar{u}^i(t) \right) \cdot v$$

  $v(s) =$ some *variation*

  (which is dual to the Adjoint Inclusion of non-smooth analysis)

  is a QDQ of the flow map

  $$(t, \xi) \mapsto \text{solution to} \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\bar{u}^i \quad x(t) = \xi$$

- Every set-valued Lie bracket B **is a QDQ** of the local flow corresponding to a suitable control (depending on B)

Why an open mapping property is needed

Before separating reachable and profitable sets, one must ensure that the reachable set is not degenerate.

A non–punctured open mapping property guarantees that the family of QDQ linearizations generates a nontrivial cone of reachable variations.

This is the key geometric ingredient that makes set–separation meaningful in the nonsmooth setting.

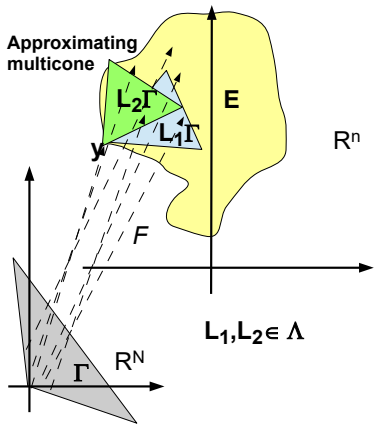
SO ...using the chain of QDQs
and set-separation arguments one is able to a Maximum Principle which includes the non-smooth Goh and Legendre-Clebsch conditions ...

A Set-Separation Theorem holds true for QDQ approximating multi-cones.

Begin with a notion of**"approximating multi-cone":**

Definition. $E \subseteq \mathbb{R}^n$ any subset, and $y \in E$.
A family $\mathcal{C}$ of convex cones is called a
QDQ-**approximating multi-cone to** E **at** y
if there exists a convex cone $\Gamma \subset \mathbb{R}^N$ of directions , a set-valued map $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$, and a Quasi Diff. Quotient Λ of F at $(0, y)$ in the direction of Γ, such that $F(\Gamma) \subseteq E$ and

, $\qquad \mathcal{C} = \{L \cdot \Gamma,\ L \in \Lambda\}.$

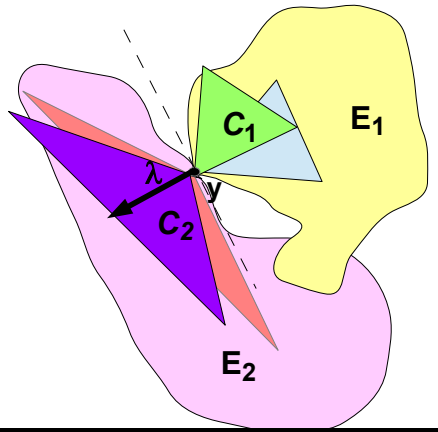


A Set-Separation Theorem holds true for approximating QDQ multi-cones!

Definition. E_1 and E_2 are locally separated at y if $E_1 \cap E_2 \cap B(y, r) = \{y\}$ for some $r > 0$

Theorem. If E_1 and E_2 are locally separated at y and $\mathcal{C}_1$ [resp. $\mathcal{C}_2$] is a QDQ approximating multi-cone at y of E_1 [resp.E_2] then there exist cones $C_1 \subset \mathcal{C}_1$, $C_2 \subset \mathcal{C}_2$ and of a linear form λ verifying

$$\lambda \cdot c_1 \geq 0 \qquad \lambda \cdot c_2 \leq 0 \qquad \forall (c_1, c_2) \in C_1 \times C_2$$



Main Theorem

Let us assume that $f, g_1 \in C^{1,1}(\mathbb{R}^n, \mathbb{R})$, $g_2, \dots, g_m \in C^{0,1}(\mathbb{R}^n, \mathbb{R})$ and $\psi \in C^{0,1}(\mathbb{R}, \mathbb{R})$. Let $(\overline{T}, \overline{u}, \overline{x})$ be a singular, weak local minimizer for problem (P), and let $\mathbb{T}$ be a QDQ-approximating multicone to the target set $\mathfrak{T}$ at $x(\overline{T})$. Then there exist multipliers $(p_0, p, \lambda) \in \mathbb{R}^* \times AC\Big([0, \overline{S}]; (\mathbb{R}^n)^*\Big) \times \mathbb{R}^*$, with $\lambda \geq 0$, such that the following facts are verified:

i) (non-triviality) $(p_0, p, \lambda) \neq 0$;

ii) (adjoint differential inclusion)

$$\frac{dp}{dt} \in -\partial_x^C H(\overline{x}(t), p_0, p, \overline{u}(t)) \qquad \forall t \in [0, \overline{T}] \text{ a.e.};$$

iii) (non-transversality)

$$(p_0, p)(\overline{T}) \in -\lambda \partial^C \Psi\Big(\overline{T}, \overline{x}(\overline{T})\Big) - \overline{\bigcup_{\mathcal{T} \in \mathbb{T}} \mathcal{T}^\perp};$$

iv) (Hamiltonian's maximization)

$$\max_{u \in U} H(\overline{x}(t), p_0, p(t), u) = H(\overline{x}(t), p_0, p(t), \overline{u}(t)) \qquad \forall t \in [0, \overline{T}] \text{ a.e.}$$

Moreover, the following higher order conditions are verified:

v) (NONSMOOTH GOH CONDITION)

$$0 \in p(t) \cdot [g_j, g_i]_{set}(\overline{x}(t)) \qquad i,j \in \{1,\dots,m\} \qquad \forall t \in [0,\overline{T}] \text{ a.e.;} \tag{1}$$

vi) (NONSMOOTH LEGENDRE-CLEBSCH CONDITION OF STEP 2)

$$0 \in p(t) \cdot [f, g_i]_{set}(\overline{x}(t)) \qquad i \in \{1,\dots,m\} \qquad \forall t \in [0,\overline{T}] \text{ a.e.;} \tag{2}$$

vii) (NONSMOOTH LEGENDRE-CLEBSCH CONDITION OF STEP 3 WITH $m = 1$)
for almost all $t \in [0,\overline{T}]$ such that f and $g := g_1$ are of class $C^{1,1}$ around $\overline{x}(t)$, one has

$$0 \geq \min\left\{p(t) \cdot [g,[f,g]]_{set}(\overline{x}(t))\right\}. \tag{3}$$

Bracket–like (Chen–type) variations: impulsive time–freezing

The construction relies on an *impulsive formulation* of the control problem, obtained by extending the dynamics and reparametrizing time.

Key point: time–freezing

- the physical time is momentarily *frozen*;
- control actions are applied in an auxiliary (impulsive) time scale;
- first-order effects cancel, leaving a higher-order displacement.

Bracket mechanism:

- short control arcs along e_i, e_j, $-e_i$, $-e_j$;
- each arc has impulsive duration of order $\sqrt{\varepsilon}$;
- first–order effects cancel by symmetry.

Effective displacement:

$$x_\varepsilon(T) - \bar{x}(T) = \varepsilon\,[g_i, g_j](\bar{x}(T)) + o(\varepsilon).$$

Time–freezing separates control oscillations from drift evolution.

From SICON to other structural degeneracies

Higher–order necessary conditions provide additional information on trajectories satisfying the Pontryagin Maximum Principle.

- There exist *normal extremals* satisfying first–order conditions but violating higher–order ones, hence they cannot be local minimizers;
- beyond normal extremals, *abnormal trajectories* may occur, for which first–order analysis is intrinsically insufficient.

As a consequence, extracting meaningful optimality information often requires a higher–order variational analysis.

The SICON result makes this analysis possible in a nonsmooth setting, where vector fields and endpoint maps lack regularity.

Beyond the SICON framework

Recently, I have identified three further classes of problems where higher–order analysis cannot be carried out using *standard* classical tools.

- In WP2, the SICON mechanism is combined with a state constraint, requiring additional nonsmooth tools;
- The other two problems involve smooth vector fields, but exhibit structural degeneracies not previously studied in the literature.

WP2: control–affine problem with a state constraint

$$\min_{u(\cdot)\in\mathcal{U}} \ \Psi\big(T, x(T)\big)$$

subject to

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))\, u^i(t), \qquad x(0) = \hat{x},$$

$$u(t) \in U \subset \mathbb{R}^m \ \text{ a.e.}, \qquad h(x(t)) \leq 0 \ \ \forall t \in [0, T],$$

and a terminal constraint

$$(T, x(T)) \in \mathfrak{T}.$$

Vector fields are Lipschitz; h may be nonsmooth.

WP2: same nonsmoothness, new admissibility issue

- The nonsmooth features are the same as in SICON:
  - adjoint differential inclusion (Clarke type);
  - set–valued Lie brackets in higher order terms.
- Therefore we rely on the same tools:

$$\text{QDQ} \quad + \quad [\cdot,\cdot]_{\text{set}}.$$

New difficulty: state–constraint admissibility.

SICON-type variations may violate $h(x(t)) \leq 0$ on the perturbed trajectory, hence they are not admissible and cannot be used directly.

WP2: why chattering needle variations for the PMP

In the joint preprint with M. Palladino, the nonsmooth Pontryagin Maximum Principle is obtained using QDQ and nonsmooth set–separation.

A crucial point concerns the choice of variations:

- Standard needle variations, localized in time, may violate the state constraint
$$h(x(t)) \leq 0,$$
and therefore fail to produce admissible trajectories.
- In contrast, *packages of chattering needle variations* allow one to distribute the perturbation over small time intervals, preserving admissibility.

Chattering variations generate averaged first–order effects while remaining tangential to the constraint, which makes them suitable for a geometric derivation of the PMP. This makes chattering variations the natural tool for a geometric derivation of the PMP

WP2 (work in progress): hybrid Chen–chattering variations

The SICON higher–order conditions are obtained through Chen–type variations, designed to eliminate first–order terms and isolate specific Lie brackets in the expansion.

Under state constraints, these variations are no longer admissible: they may violate $h(x(t)) \leq 0$ along the perturbed trajectory.

My current approach is to *combine*:

- Chen–type higher–order constructions (responsible for generating bracket terms),
- packages of chattering needle variations (responsible for preserving admissibility).

This leads to *hybrid variations* that are both higher–order sensitive and compatible with state constraints, and are expected to produce the desired higher–order conditions.

Expected higher–order necessary conditions

Let $(\bar{u}, \bar{x})$ be a trajectory satisfying the nonsmooth PMP, and let $p^\sharp(\cdot)$ denote the effective adjoint (including the contribution of the state constraint).

The expected higher–order conditions read, for a.e. t:

(Goh)

$$0 \in p^\sharp(t) \cdot [g_i, g_j]_{\text{set}}(\bar{x}(t)), \qquad i, j = 1, \dots, m;$$

(LC2)

$$0 \in p^\sharp(t) \cdot [f, g_i]_{\text{set}}(\bar{x}(t)), \qquad i = 1, \dots, m;$$

(LC3)

$$0 \geq \min \left\{ p^\sharp(t) \cdot [g, [f, g]]_{\text{set}}(\bar{x}(t)) \right\}.$$

$p^\sharp$ denotes the effective adjoint, including the contribution of the state constraint.

These conditions coincide with the SICON ones on interior arcs and extend them across boundary interactions.

Cusps, corners and regularity in smooth geometries

In smooth Riemannian and sub–Riemannian geometries, the smoothness of the metric imposes strong regularity on minimizing curves.

What is known:

- corners and angles are never minimizing, neither in the interior nor at the boundary (Hakavuori–Le Donne, Invent. Math. 2016);
- cuspidal trajectories are excluded in smooth sub–Riemannian geometries, both in the interior and at the boundary (see Agrachev–Barilari–Boscain, 2020);
- in high dimensions ($n \geq 5$), minimizing geodesics may fail to be C^2 *only at the boundary*, while the existence of interior nonsmooth minimizers remains a long–standing open problem (Chitour–Jean–Monti–Rifford–Sacchelli–Sigalotti–Socionovo, 2025).

Key point: these exclusions are a consequence of the *smoothness of the metric*, not of higher–order optimality conditions.

WP1: Lorentzian geometry and cuspidal extremals

In smooth geometric settings (e.g. sub–Riemannian geometry), trajectories with cusps or loss of regularity cannot be minimizing or maximizing.

In contrast, in Lorentzian geometries of Schwarzschild type:

- the metric is *nonsmooth* and degenerates near the horizon;
- cuspidal extremals satisfying the Pontryagin principle may exist;
- classical smooth geometric exclusions break down.

WP1: relativistic action in a Lorentzian setting

Let $(\mathcal{M}, g)$ be a Lorentzian manifold of Schwarzschild type. We consider the relativistic action

$$\mathcal{A}(\gamma) := \int_0^T \sqrt{-g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\, dt$$

and the variational problem

$$\max_{\gamma} \ \mathcal{A}(\gamma)$$

among absolutely continuous curves $\gamma : [0, T] \to \mathcal{M}$ such that

$$\gamma(0) = q_0,$$

and satisfying the *future–directed timelike constraint*

$$g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t)) < 0, \qquad \dot\gamma(t) \text{ future–directed a.e.}$$

The Lorentzian metric g is nonsmooth and degenerates at the horizon.

Control formulation and cuspidal extremal

Future–directed timelike curves can be written in control form as

$$\dot{x}(t) = u_0(t)\, e_0(x(t)) + u_1(t)\, e_1(x(t)),$$

where (e_0, e_1) is a local orthonormal frame adapted to the Lorentzian metric.

The admissible controls satisfy a cone constraint ensuring the future–directed timelike condition.

Within this formulation, I construct an extremal trajectory $\bar{x}(\cdot)$ satisfying the Pontryagin Maximum Principle and exhibiting a *cuspidal behavior*:

$$\dot{\bar{x}}(t) \text{ loses regularity at a point } t_c \in (0, T).$$

WP1: an explicit cuspidal extremal

I exhibit an extremal trajectory $\bar{x}(\cdot)$ with a cusp at $t = t_c$.

(Explicit form) In local coordinates (or in the reduced 2D model), the extremal is given by

$$\bar{x}(t) = \begin{cases} \bar{x}^-(t), & t \in [0, t_c], \\ \bar{x}^+(t), & t \in [t_c, T], \end{cases} \qquad \text{with } \bar{x}^-(t_c) = \bar{x}^+(t_c),$$

and

$$\dot{\bar{x}}^-(t_c) \neq \dot{\bar{x}}^+(t_c) \quad \text{(cuspidal / corner-type behavior)}.$$

The curve satisfies the future–directed timelike constraint and the Pontryagin Maximum Principle. Both first and second order necessary conditions are degenerate along $\bar{x}(\cdot)$.

Quasi-extremals-approximation

Using higher–order analysis, we construct a field of quasi–extremals satisfying higher–order conditions and converging to a cuspidal extremal.

By compactness, this allows us to prove that the cusp is a *maximizer* of the relativistic action, in agreement with the physical behavior near black holes.

Physical intuition (heuristic)

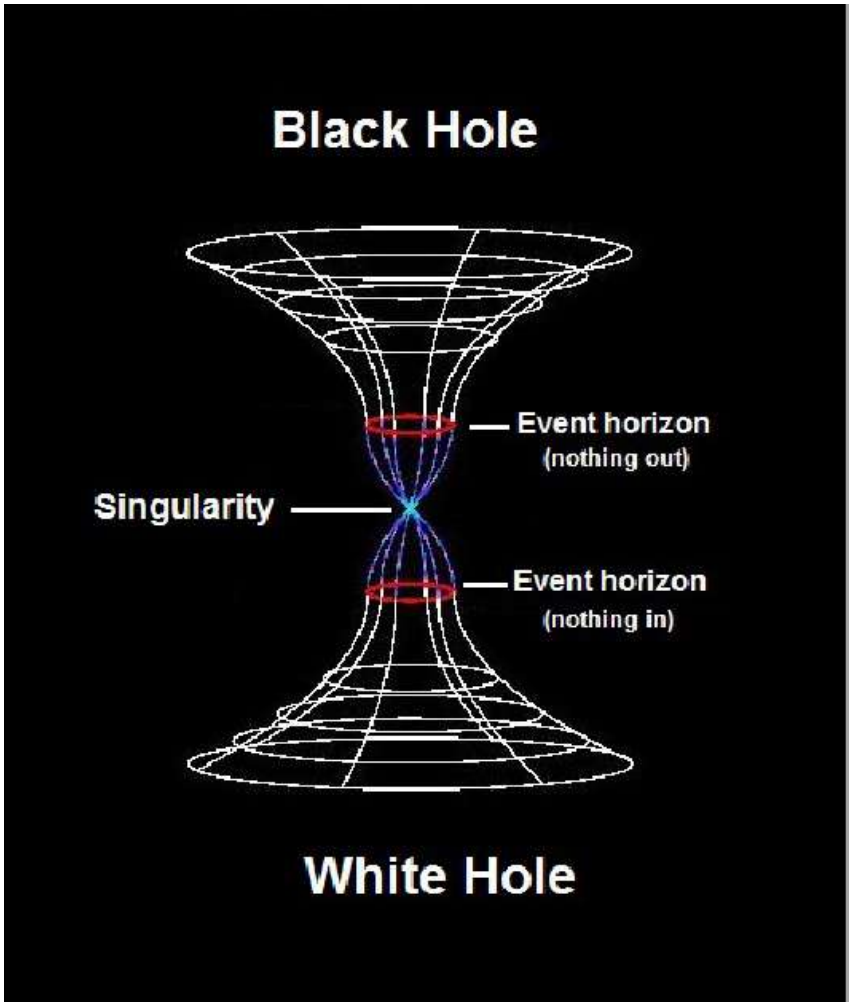
In stellar dynamics near a supermassive black hole, the *average* density of stars is known to develop a cuspidal profile (Bahcall–Wolf).

$$\rho(r) \sim r^{-7/4} \qquad \text{(statistical density profile)}$$

This cusp describes a *collective, statistical phenomenon*, not individual optimal trajectories.

This suggests that, at a purely geometric and variational level, cuspidal behavior may already appear for *single extremals*, as a consequence of the degeneracy of the Lorentzian metric near the horizon.

Geometric interpretation



Heuristic picture: degeneracy of the metric at the event horizon

WP3: Mean Field optimal control (ODE setting)

Let $\mu_0 \in \mathcal{P}_2(\mathbb{R}^n)$ and consider the problem

$$\min_{u(\cdot)\in L^\infty([0,T];U)} J(u) := \int_0^T L\big(x(t),\mu(t),u(t)\big)\,dt + \Phi\big(x(T),\mu(T)\big),$$

subject to

$$\dot{x}(t) = f\big(x(t),\mu(t)\big) + \sum_{i=1}^{m} g_i\big(x(t),\mu(t)\big)\,u^i(t), \qquad x(0) = x_0,$$

and to the consistency condition

$$\mu(t) = (X_t^u)_{\#}\mu_0,$$

where X_t^u denotes the flow associated with the controlled dynamics.

WP3: Measure constraint and variational structure

The measure $\mu(t)$ is not an independent variable: it is induced by the controlled flow.

Equivalently, μ satisfies the continuity equation

$$\partial_t \mu + \nabla_x \cdot \big(\mu\, b(x, \mu, u)\big) = 0, \qquad b(x, \mu, u) := f(x, \mu) + \sum_{i=1}^{m} g_i(x, \mu) u^i.$$

As a consequence, any control variation induces simultaneous variations of the trajectory and of the measure.

WP3 (work in progress): Chen variations in a mean–field setting

I consider mean–field ODE control problems where the dynamics depend on $\mu(t)$ through Lions derivatives.

The approach is based on:

- Chen–type higher–order variations (Hall basis coefficients), adapted to the mean–field dependence (and involving Lions derivatives);
- explicit bookkeeping of $\partial_\mu f$ and $\partial_\mu g_i$ in the expansion of the perturbed flow.

A key structural feature is that *symmetric* contributions are eliminated by construction of the Chen variations, so that only bracket–type terms survive at higher order.

Expected outcome: classical form of higher–order conditions

The expected necessary conditions include:

- nontriviality and (mean–field) Pontryagin maximum principle;
- non–transversality at the terminal constraint;
- higher–order conditions of Goh / Legendre–Clebsch type, with the *same formal structure as in the classical case* (i.e. expressed through Lie brackets),

but obtained through more delicate computations due to Lions derivatives.

The resulting necessary conditions have the same formal structure as classical Goh and Legendre–Clebsch conditions. **The novelty lies in the variational construction and in the presence of measure–dependent terms.**

Next step: extend the analysis to more general MFG models where variations must explicitly move the measure component to produce the correct higher–order terms.

Three mechanisms beyond the classical higher–order framework

These three regimes represent fundamental mechanisms by which the *classical higher–order theory no longer applies directly*:

$$\underbrace{\text{metric degeneracy}}_{\text{WP1}} \quad \underbrace{\text{nonsmooth vector fields + state constraints}}_{\text{WP2}}$$

$$\underbrace{\text{measure–dependent dynamics}}_{\text{WP3}}.$$

Together, these three work packages extend classical higher–order theory along independent directions.

They clarify how second and third order geometric information can be recovered—or must be modified—when smoothness or locality assumptions are relaxed, providing a unified variational perspective on problems in:

- Lorentzian geometry,
- nonsmooth dynamics,
- mean field control.

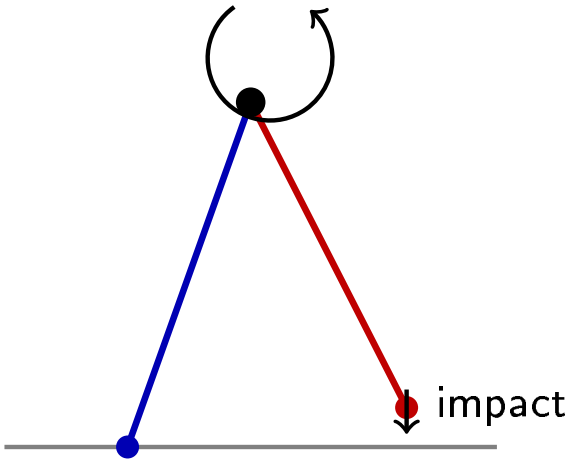
Higher-order optimality conditions in robotics

Application: Energy-efficient walking of a compass-like biped robot.

Ongoing collaboration with PRISMA LAB of Universita' degli studi di Napoli Federico II

Hybrid optimal control:

- Continuous dynamics between foot impacts;
- Velocity reset at impact;
- Periodic gaits and energy minimization.



Compass-like biped (schematic)

Current objective: Extend Goh and Legendre–Clebsch-type necessary conditions to hybrid periodic trajectories.

Aim

Rule out nonoptimal walking gaits beyond first-order tests.

Papers

- Angrisani, F., Palladino, M., *A* geometric approach to the non-smooth Pontryagin Maximum Principle for a problem with space state constraint, preprint.
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- Angrisani F, Rampazzo F., *Goh* Conditions for Minima of Nonsmooth Problems with Unbounded Controls, ESAIM: COCV 29, 17, (2023).
- Angrisani F., Rampazzo F., *Higher order necessary conditions for optimal controls not ranging in the interior* Communications in Optimization Theory, 40, (2026).

- Palladino, M. and Rampazzo, F., *A* geometrically based criterion to avoid infimum-gaps in Optimal Control J. Differential Equations 269, no. 11, 10107–10142, (2020).
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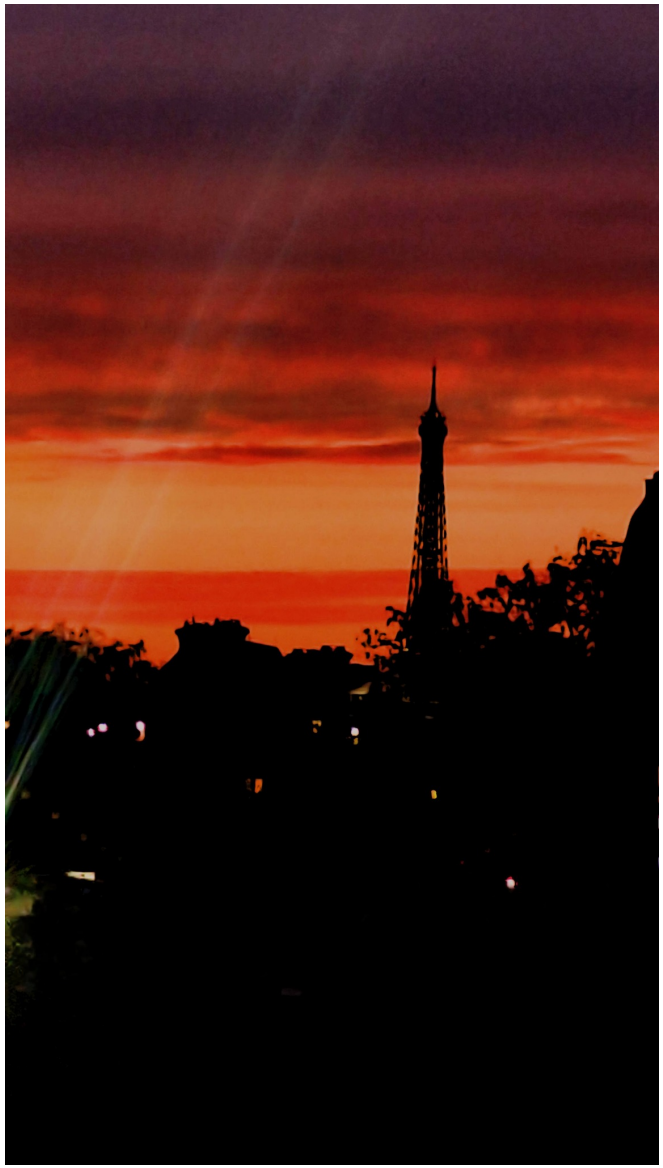
Thank you for your attention

Questions?



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