Configuration spaces, cluster algebras, and representation theory

Keywords. Configuration spaces, cluster algebras, representation theory, positive geometries, categorification.

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1 Institute and research environment

The position is to be held at the Laboratoire de Mathématiques de Versailles (LMV), in the Algebra and Geometry team. Scientific interactions within the team are fostered by a weekly seminar and specialized working groups, notably one on "Clusters, quivers and geometry" relevant to this project.

2 Context

Configuration spaces

One of the starting points of this proposals are the so called *u-equations*, introduced by physicists Koba and Nielsen in 1969 [10, 11, 12] in their study of tree-level string amplitudes. They can be defined in a simple combinatorial fashion: consider a disc with n points on its boundary, labelled counter-clockwise from 1 to n. Call $[i, j]$ the line segment joining non-adjacent points i and j ($i < j$). For two such segments, let $c\big([i, j], [k, \ell]\big)$ be the number of times they intersect in the interior of the disc. Then the u-equations are

$$u_{i,j} + \prod_{[k,\ell]} u_{k,\ell}^{c\left([i,j],[k,\ell]\right)} = 1.$$

Outside of the physics of scattering amplitudes, these u-equations have seen numerous interpretations.

- The algebraic variety that they define is Brown's "partial" compactification $\widetilde{\mathcal{M}}_{0,n}$ of the space $\mathcal{M}_{0,n}$ of configurations of n distinct points on a projective line [6].
- These varieties can be equipped with a differential form turning them into *positive geometries* in the sense of [1].
- They can be realized and further understood using Fomin and Zelevinsky's cluster algebras [7]. Specifically, they arise from cluster algebras of Dynkin type A.

This last observation led Arkani–Hamed, He and Lam to generalize the definition of $\widetilde{\mathcal{M}}_{0,n}$ from Dynkin type A to any cluster algebra of Dynkin type A, B, C, D, E, F, G [3]. The link with cluster algebras also opens the way for the application of methods from representation theory, using the *categorification of cluster algebras.*

Representation theory and categorification

Methods from representation theory have been used very successfully in the past two decades to study cluster algebras and related problems (see the survey [9]). In a work in progress [2], we are appliying this machinery to u-equations and configuration spaces. The general strategy is as follows. Let Λ be a finite-dimensional $\mathbb{C}$-algebra.

- For any indecomposable Λ-module M, let u_M be a variable.
- For pairs of indecomposable modules M, N, define a compatibility degree $c(M, N)$ as the dimension of certain spaces of extensions.
- Define u-equations $u_M + \prod_N u_N^{c(M,N)} = 1$.

This strategy vastly generalizes many cluster configuration spaces from [3], and allows for the construction of solutions of the u-equations in terms of certain generating series "counting" submodules of M called *F-polynomials*. It allows for the understanding of the locus of solutions of the u-equations in terms of toric varieties, and for the realization of the set of non-negative real solutions as polytopes such as the *associhedron*.

The main obstacles at this time are that the above methods only work if Λ admits only finitely many indecomposable modules, and that "non-simply laced" Dynkin types B, C, F, G are out of reach.

3 Objectives of the project

The project's aim is to extend representation-theoretic methods above to study cases beyond those currently understood. Here are particular aims and possible consequences for each.

- Extend the methods above to algebras Λ that admit infinitely many indecomposable objects. This will require replacing "module" by "geometric family of modules" in a manner reminiscent of motivic Hall algebras (see for instance [5]). Expected applications are the definition of cluster configuration spaces in the sense of [3] of infinite cluster type. This is the main objective of the project. It will likely require the use of tools from geometric representation theory, Hall algebras, and Auslander–Reiten theory.
- Extend the methods above to categorify u-equations in non-simply-laced Dynkin types. This is likely to be achieved, for example, using the work of [8] on the categorification of cluster algebras of non-simply-laced types.
- Study possible applications to related objects from the theory of scattering amplitudes, such as the *amplituhedron* [4] or *positive geometries.*

4 Candidate profile

Preference will be given to candidates proficient with a subset of the following: representation theory of associative algebras; cluster algebras; positive geometries; categorification; derived categories; polytopal geometry (associahedra and generalizations).

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