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Lattices of the oscillator group of signature (2,2)

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Abstract

A lattice of a solvable Lie group G is a discrete subgroup L such that G/L is compact. If lattices of nilpotent Lie groups are well studied, the ones of solvable but not nilpotent Lie groups are less known. The four dimentional general oscillator group $\mathrm{Osc}_{1,0}$ is an example of solvable but not nilpotent Lie group. It is the semidirect product of the three dimentional Heisenberg group H with the real line $\mathbb{R}$, where $t \in \mathbb{R}$ acts by $diag(e^{it}, e^{-it})$.

In this document we parametrize the set of lattices of $\mathrm{Osc}_{1,0}$ and classify them modulo $\mathrm{Aut}(\mathrm{Osc}_{1,0})$. Such lattices exist (see [1]). By virtue of a general Theorem by Saîto on solvable simply connected Lie groups with real roots (see [2], Theorem 5), these classes are the same as the abstract isomorphism classes of the underlying discrete subgroups. This classification is deeply related to the conjugacy classes by $\mathrm{GL}(2,\mathbb{Z})$ of the subset of $\mathrm{SL}(2,\mathbb{Z})$ consisting in elements of trace greater than 2. We also develop another classification of lattices of $\mathrm{Osc}_{1,0}$ - the classification up to commensurability - where the equivalence classes are greater. Although this second classification is simpler than the one up to isomorphism, it appears that both classifications are linked to quadratic number fields theory.

Résumé

Un réseau d'un groupe de Lie résoluble G est un sous-groupe discret L tel que G/L soit compact. Si les réseaux dans les groupes de Lie nilpotents sont bien étudiés, on ne connaît que peu de choses dans le cas des groupes de Lie résolubles non nilpotents. Le groupe $\mathrm{Osc}_{1,0}$ des Oscillateurs général de dimension 4 est un exemple de groupe de Lie résoluble non nilpotent. Il s'agit du produit semi-direct du groupe de Heisenberg H de dimension 3 par la droite réelle $\mathbb{R}$, où $t \in \mathbb{R}$ agit par $diag(e^{it}, e^{-it})$.

Dans ce document nous déterminons une paramétrisation des réseaux de $\mathrm{Osc}_{1,0}$ et les classifions modulo $\mathrm{Aut}(\mathrm{Osc}_{1,0})$. De tels réseaux existent (voir [1]). En vertu d'un Théorème général sur les groupes de Lie résolubles simplement connexes à racines réelles, dû à Saîto (voir [2], Théorème 5), ces classes sont confondues avec les classes d'isomorphisme abstrait des sous-groupes discrets sous-jascents. Cette classification est profondément liée aux classes de conjugaison par $\mathrm{GL}(2,\mathbb{Z})$ du sous-ensemble de $\mathrm{SL}(2,\mathbb{Z})$ des matrices de trace strictement plus grande que 2. Nous développons également une autre classification des réseaux de $\mathrm{Osc}_{1,0}$ - la classification à commensurabilité près - pour laquelle les classes sont plus grandes. Bien que cette seconde classification soit bien plus simple que la première, il s'avère que toutes deux sont liées à la théorie des corps de nombres quadratiques.

Context

This document is the report of the work produced in my three and a half months internship, done remotely with the Institute of Mathematics and Computer Science of the University of Greifswald, as part of the M1 Hadamard of the ENS Paris-Saclay. The project was supervised by Pr. Ines Kath, managing director of the Institute and head of the chair of Analysis.

My role in this project was to parametrize and classify the lattices of the general Oscillator group $\mathrm{Osc}_{1,0}$.

Acknowledgements

I warmly thank Ines Kath for meeting with me almost every day so that we could move forward on the project despite the condition of internship at distance. The talks we had were really enriching mathematically and gave me a taste of the research world. I am also gratefull to Mathias Fischer, member of the chair of Analysis of the University of Greifswald, for the advice he shared in meetings, which helped me to understand the issues we had to face in this project, and in a research work in general.

Contents

1. Introduction

The two four dimensional general oscillator groups $\mathrm{Osc}_{0,1}$ and $\mathrm{Osc}_{1,0}$ are both Lie groups defined as semidirect products of the three dimensional Heisenberg group H with the real line $\mathbb{R}$. In $\mathrm{Osc}_{0,1}$, which is the usual oscillator group, $t \in \mathbb{R}$ acts on $H/Z(H)$ by the rotation e^{it}, whereas in the case of $\mathrm{Osc}_{1,0}$, $t \in \mathbb{R}$ acts by $diag(e^{it}, e^{-it})$.

A lattice of a solvable Lie group G is a discrete subgroup L such that G/L is compact. If lattices of nilpotent Lie groups are well studied, the ones of solvable but not nilpotent Lie groups are less known. The two general Oscillator groups are worth studying because they are both examples of solvable but not nilpotent Lie groups. Moreover, they are the only indecomposable four dimensional Lie groups admitting a bi-invariant pseudo-Riemannian metric: $\mathrm{Osc}_{0,1}$ admits a bi-invariant Lorentzian metric and $\mathrm{Osc}_{1,0}$ admits a metric of signature $(2,2)$.

As part of this project, we aim at parametrizing and classifying modulo $\mathrm{Aut}(\mathrm{Osc}_{1,0})$ the set $\mathbb{L}$ of lattices of $\mathrm{Osc}_{1,0}$.

Although the definitions of $\mathrm{Osc}_{1,0}$ and $\mathrm{Osc}_{0,1}$ look alike, they confer different properties to their groups: for example, $\mathrm{Osc}_{1,0}$ has real roots, while $\mathrm{Osc}_{0,1}$ has not. Moreover they lead to different problems in the classification of lattices. The group $\mathrm{Osc}_{0,1}$ has already been thoroughly studied. Its lattices have been classified in [3] up to automorphism of $\mathrm{Osc}_{0,1}$ and in [4] up to inner automorphism. Their classification is linked to conjugacy classes by $\mathrm{GL}(2,\mathbb{Z})$ of the set of matrices of finite order of $\mathrm{SL}(2,\mathbb{Z})$ - which is also the set of all matrices of $\mathrm{SL}(2,\mathbb{Z})$ with trace < 2, together with I_2. These classes have already been studied and we know their 5 explicit representatives.

In the case of $\mathrm{Osc}_{1,0}$, the classification of the set $\mathbb{L}$ of lattices of $\mathrm{Osc}_{1,0}$ - which is not empty according to [1] - is deeply related to the study of the action ρ by conjugation of $\mathrm{GL}(2,\mathbb{Z})$ on the set of infinite order $\mathrm{SL}(2,\mathbb{Z})$ matrices - which is also the subset of matrices with trace greater than 2, denoted $\mathbb{S}$. In particular, we are led to seak for the commutant $C(B)$ in $\mathrm{GL}(2,\mathbb{Z})$ of a matrix $B \in \mathbb{S}$, which is also its stabilisator for ρ. It is possible if B is given explicitly, but if one wants an expression of $C(B)$ in terms of B, the problem is complicated. It is linked, but not equivalent, to the problem of number theory consisting in finding the group of units of the quadratic number field $\mathbb{Q}(\lambda)$ - where $\lambda \in \mathbb{R}\backslash\mathbb{Q}$ is an eigenvalue of $B \in \mathbb{S}$. Similarly, if it is possible to determine if two explicit matrices are conjugate in $\mathrm{GL}(2,\mathbb{Z})$ - i.e are in the same orbit for ρ -, one cannot find explicit representatives of each conjugacy class (i.e each orbit). The difficulty to describe this action lies in the fact that $\mathbb{S}$ consists in matrices diagonalizable with irrational real spectrum. This action has been studied in the literature. It is especially linked to geometry of continued fractions (see [5], Chapter 7), but also to the number theory problem of classification of quadratic binary forms with positive non square discriminant (see [6], part 4.4).

In virtue of a general Theorem by Saîto on simply connected solvable Lie groups with real roots (see [2]), classes modulo $\mathrm{Aut}(\mathrm{Osc}_{1,0})$ are the same as the abstract isomorphism classes of the underlying discrete subgroups. This was not the case in the study of the lattices of $\mathrm{Osc}_{0,1}$ led in [3] and [4], and here it allows us to take both points of view, the ones of isomorphism classes and the one of equivalence classes. We will see that they give complementary results.

Notations. $\mathrm{GL}(2,\mathbb{Z})$ is the set of matrices of $M_2(\mathbb{Z})$ with deteminant ± 1. $\mathrm{SL}(2,\mathbb{Z})$ is the subgroup of $\mathrm{GL}(2,\mathbb{Z})$ of matrices with determinant 1, and $\mathrm{SL}(2,\mathbb{Z})^-$ the set of matrices of $\mathrm{GL}(2,\mathbb{Z})$ with determinant -1. We will sometimes use the notation $[[a,b][c,d]]$ for the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For $\lambda \in \mathbb{C}$, we will denote $\mathbb{Q}[\lambda] = \{P(\lambda) \mid P \in \mathbb{Q}[X]\}$ the sub-$\mathbb{Q}$-algebra of $\mathbb{C}$ generated by λ. Recall that if $\lambda \in \bar{\mathbb{Q}}$, $\mathbb{Q}[\lambda]$ is a number field denoted $\mathbb{Q}(\lambda)$.

In this report, **all the Propositions, Lemmas, Corollaries and Theorems that are marked with a star (*) have their Proof in the Annexe A**. All these proofs were put in place as part of the internship.

2. The Oscillator group and its lattices

Definition 2.1. *The Heisenberg group H is defined as the group $\mathbb{R}\times\mathbb{C} \simeq \mathbb{R}\times\mathbb{R}^2$ with multiplication given by:*

$$(z,\xi).(z',\xi') = (z + z' + \frac{1}{2}\omega(\xi,\xi'), \xi + \xi')$$

with ω the $\mathbb{R}$-bilinear form defined by $\omega(\xi,\xi') = {}^t\xi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xi' = Im(\bar{\xi}\xi')$.
A lattice of H is a discrete subgroup Γ of H, such that H/Γ is compact.

Denote $e_1 = (1,0)$ and $e_2 = (0,1) \in \mathbb{R}^2$. For $r \in \mathbb{N}_{>0}$, let Γ_r be the discrete subgroup of H generated by $\{(\frac{1}{r},0),(0,e_1),(0,e_2)\}$. We have:

$$\Gamma_r = \left\{(z,\eta) \in H \mid \eta \in \mathbb{Z}^2, z \in \frac{1}{2}\eta_1\eta_2 + \frac{\mathbb{Z}}{r}\right\} \text{ for } r \text{ odd;}$$

$$\Gamma_r = \left\{(z,\eta) \in H \mid \eta \in \mathbb{Z}^2, z \in \frac{\mathbb{Z}}{r}\right\} \text{ for } r \text{ even.}$$

From [7], p.294, we deduce the following description of the automorphism group $\mathcal{A}$ of the Heisenberg group:

Proposition 2.1. * *A map $\alpha : H \to H$ is an automorphism of the Heisenberg group H if, and only if*

$$\alpha = \begin{pmatrix} \det(S) & {}^t\delta \\ 0 & S \end{pmatrix}$$

for some $S \in \mathrm{GL}(2,\mathbb{Z})$.

In particular, we will denote $\bar{F}_S$ the automorphism of H defined by $diag(\det(S), S)$ for $S \in \mathrm{GL}(2,\mathbb{R})$ and $\bar{F}_\eta$ the conjugation by η defined on H (which is equal to some $\begin{pmatrix} 1 & {}^t\delta \\ 0 & I_2 \end{pmatrix}$ in matricial language). For all $P \in \mathrm{GL}(2,\mathbb{R})$, we set:

$$\Gamma_P^r = \bar{F}_{P^{-1}}(\Gamma_r) = \left\{(\det(P^{-1})z, P^{-1}\xi) \mid (z,\xi) \in \Gamma_r\right\}. \quad (1)$$

Corollary 2.1. * *For all $P \in \mathrm{GL}(2,\mathbb{R})$, the set Γ_P^r is a lattice of H.*

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Definition 2.2. *The generalized oscillator group $\mathrm{Osc}_{1,0}$ is defined as the semi-direct product of H with $\mathbb{R}$, where $\mathbb{R}$ acts on H by*

$$t.(z,\xi) = (z, e^{tA}\xi) \quad \forall t \in \mathbb{R}, \forall (z,\xi) \in H.$$

Thus the multiplication in $\mathrm{Osc}_{1,0}$ is given by

$$(z,\xi,t).(z',\xi',t') = \left(z + z' + \frac{1}{2}\omega(\xi, e^{tA}\xi'), \xi + e^{tA}\xi', t + t'\right).$$

A lattice of $\mathrm{Osc}_{1,0}$ *is a discrete subgroup* L *such that* $\mathrm{Osc}_{1,0}/L$ *is compact.*

It is known that H is a nilpotent Lie group and that $\mathrm{Osc}_{1,0}$ is a solvable but not nilpotent Lie group with $H \times \{0\}$ as commutator subgroup. In the following we will identify H (resp. the subgoups of H, especially the ones of the form Γ^r_P) with the subgroup $H \times \{0\}$ of $\mathrm{Osc}_{1,0}$ (resp. the subgroup $\Gamma^r_P \times \{0\}$ of $\mathrm{Osc}_{1,0}$) via the canonical injection.

We will denote $\mathbb{L}$ the set of all lattices of $\mathrm{Osc}_{1,0}$. Let's recall the following general result (see [8], Cor. 3.5) on lattices in Lie groups: If G is a solvable Lie group and N its maximal nilpotent connected normal Lie subgroup, and if Γ is a lattice in G, then $\Gamma \cap N$ is a lattice in N.

Applying this result to $\mathrm{Osc}_{1,0}$ and H, we get:

Proposition 2.2. * *For all* $L \in \mathbb{L}$, $L \cap H$ *is a lattice of* H.

For $P \in \mathrm{GL}(2,\mathbb{R})$ and $s \in \mathbb{R}_{>0}$, set

$$B_{P,s} = Pe^{sA}P^{-1} \in \mathrm{GL}(2,\mathbb{R}).$$

Proposition 2.3. * *For all* $P \in \mathrm{GL}(2,\mathbb{R})$ *and* $s \in \mathbb{R}_{>0}$, *the matrix* $B_{P,s}$ *defined above is in* $\mathrm{SL}(2,\mathbb{R})$. *Moreover, we have* $Tr(B) > 2$ *and* $s = \ln\lambda$, *with* $\lambda = \frac{1}{2}(Tr(B) + \sqrt{Tr(B)^2 - 4})$.

This leads us to introduce the set

$$\mathcal{G} = \{B \in \mathrm{SL}(2,\mathbb{R}), Tr(B) > 2\}.$$

According to proposition 2.3, for $s \in \mathbb{R}_{>0}$ and $P \in \mathrm{GL}(2,\mathbb{R})$, the matrix $B_{P,s} = Pe^{sA}P^{-1}$ is in $\mathcal{G}$. Reciprocally, the Proof of Proposition 2.3 gives that for all $B \in \mathcal{G}$, there are unique:

- $s_B > 0$ such that $\ln(s_B)$ is an eigenvalue of B (defined as $s_B = e^{\lambda_B}$ where λ_B is the unique eigenvalue of B greater than 1),

- $C_B = \{P \in \mathrm{GL}(2,\mathbb{R}) \mid B = Pe^{s_B A}P^{-1}\} \neq \varnothing$.

We denote $(C_B, s_B) \in \mathcal{P}(\mathrm{GL}(2,\mathbb{R})) \times \mathbb{R}_{>0}$ the ordered pair associated to a matrix $B \in \mathcal{G}$.

For $r \in \mathbb{N}_{>0}$, $B \in \mathcal{G}$, $P \in C_B$, $x = (x_1, x_2) \in \mathbb{R}^2$, let $L^r_{B,P,x}$ be the subgroup of $\mathrm{Osc}_{1,0}$ defined by

$$L^r_{B,P,x} = \left\langle \Gamma^r_P, (0, P^{-1}x, s_B) \right\rangle, \tag{2}$$

where Γ^r_P is defined in (1) and is a lattice of H according to Corollary 2.1. Proposition 2.2 gives that if $L^r_{B,P,x}$ is a lattice of $\mathrm{Osc}_{1,0}$, then $L^r_{B,P,x} \cap H$ is a lattice of H. We want to know at which condition on B, P, x and r this intersection is exactly the lattice Γ^r_P of H.

Definition 2.3. *Let* $r \in \mathbb{N}_{>0}$. *We define* $\mathcal{B}_r$ *as the set of all ordered pairs* $(B,x) \in \mathrm{GL}(2,\mathbb{Z}) \times \mathbb{R}^2$, *with* $B = [[a,b][c,d]]$, *satisfying the following conditions:*
If r *is even,* $x \in \frac{\mathbb{Z}}{r} \times \frac{\mathbb{Z}}{r}$;
if r *is odd:*
if ab *and* cd *are even,* $x \in \frac{\mathbb{Z}}{r} \times \frac{\mathbb{Z}}{r}$;
if ab *is even and* cd *is odd,* $x \in \frac{\mathbb{Z}}{r} \times \left(\frac{1}{2} + \frac{\mathbb{Z}}{r}\right)$;
if ab *is odd and* cd *is even,* $x \in \left(\frac{1}{2} + \frac{\mathbb{Z}}{r}\right) \times \frac{\mathbb{Z}}{r}$.

Proposition 2.4. * *Let $B \in \mathcal{G}$, $P \in C_B$ and $x \in \mathbb{R}^2$. For $L = L^r_{B,P,x}$ to be a lattice of* $\mathrm{Osc}_{1,0}$ *with $L \cap H = \Gamma^r_P$, it is necessary and sufficient that $(B, x) \in \mathcal{B}_r$.*

In the Annexe B, we give two examples of lattices in order to illustrate Proposition 2.4. It follows from this Proposition that the property of being a lattice of $\mathrm{Osc}_{1,0}$ with intersection to H equal to Γ^r_P for $L^r_{B,P,x}$ only depends on $B \in \mathcal{G}$, $r \in \mathbb{N}_{>0}$ and $x \in \mathbb{R}^2$.

Remark 2.1. *In the proof of Proposition 2.4, we establish that a couple (B, x) is in $\mathcal{B}_r$ if, and only if, $B \in \mathrm{GL}(2,\mathbb{Z})$ and:*

$$cx_1 - ax_2 \in \frac{1}{2}ac + \frac{\mathbb{Z}}{r} \quad \text{and} \quad dx_1 - bx_2 \in \frac{1}{2}bd + \frac{\mathbb{Z}}{r}. \tag{3}$$

(the equivalence is only proven in the case where $r = 1$, but with a similar reasoning, we get the general case).

We will denote the set $\mathrm{GL}(2,\mathbb{Z}) \cap \mathcal{G}$ by $\mathbb{S}$. For each $T > 2$, we also define

$$\mathbb{S}_T = \{B \in \mathbb{S} \mid tr(B) = T\}. \tag{4}$$

Remark 2.2. *Notice that $\mathbb{S}$ is exactly the set of all matrices of* $\mathrm{GL}(2,\mathbb{Z})$ *that are conjugate to an element of* $\mathrm{SO}(1,1) \setminus \{\pm I_2\}$*, which is also the set of all matrices of* $\mathrm{SL}(2,\mathbb{Z})$ *with a real positive spectrum different from $\{1\}$.*

For a given matrix $B \in \mathbb{S}$ and a given $r \in \mathbb{N}_{>0}$, we introduce

$$\mathbb{L}^r_B = \left\{L^r_{B,P,x} \mid (B, x) \in \mathcal{B}_r, P \in C_B\right\} \subset \mathbb{L} \tag{5}$$

the class of all lattices of the form of $L^r_{B,P,x}$, with $P \in C_B$ and x satisfying $(B, x) \in \mathcal{B}_r$. We also denote

$$\mathbb{L}_{\mathbb{S}_T} = \bigcup_{r \in \mathbb{N}^*} \bigcup_{B \in \mathbb{S}_T} \mathbb{L}^r_B \tag{6}$$

for a fixed trace $T > 2$, and

$$\mathbb{L}_{\mathbb{S}} = \bigcup_{T > 2} \mathbb{L}_{\mathbb{S}_T}. \tag{7}$$

3. The automorphism group of $\mathrm{Osc}_{1,0}$

In this section we aim at describing the automorphism group of the oscillator group. This is a first necessary step to approach the paramtrization and the description of the equivalence classes of $\mathbb{L}$, which we do in Section 4.

Definition 3.1. *For $\eta \in \mathbb{R}^2$, let $F_\eta : \mathrm{Osc}_{1,0} \to \mathrm{Osc}_{1,0}$ be the conjugation by $(0, \eta, 0)$. For $u \in \mathbb{R}$, define $F_u : \mathrm{Osc}_{1,0} \to \mathrm{Osc}_{1,0}$ by*

$$F_u(z, \xi, t) = (z + ut, \xi, t).$$

For $S \in GL(2,\mathbb{R})$ such that $SA = \mu AS$ with $\mu \in \{-1, 1\}$, let $F_S : \mathrm{Osc}_{1,0} \to \mathrm{Osc}_{1,0}$ be defined by

$$F_S(z, \xi, t) = (\det S z, S\xi, \mu t).$$

Remark 3.1. *If $S \in \mathrm{GL}(2,\mathbb{R})$ satisfies $SA = \mu AS$, then $\det(A)\mu^2 = \det(A)$ implies $\mu \in \{-1, 1\}$. Thus the condition $\mu \in \{-1,1\}$ in the previous definition is not necessary.*
Moreover, if we set $S = [[s_1, s_2][s_3, s_4]]$, then $SA = \mu AS$ implies $s_2 = \mu s_3$, $s_1 = \mu s_4$,

$$S = \begin{pmatrix} s_1 & s_2 \\ \mu s_2 & \mu s_1 \end{pmatrix}$$

and $\mu = \frac{\det S}{s_1^2 - s_2^2}$ (well defined because $S \in \mathrm{GL}(2,\mathbb{R})$ and thus $s_1^2 - s_2^2 \neq 0$).

Theorem 3.1. * *The automorphisms of $\mathrm{Osc}_{1,0}$ are all the functions of the form*

$$F_u \circ F_\eta \circ F_S$$

with $u \in \mathbb{R}, \eta \in \mathbb{R}^2$ and $S \in \mathrm{GL}(2,\mathbb{R})$ such that $SA = \mu AS$ with $\mu \in \{-1, 1\}$.

Each map F_η, F_S, F_u is an automorphism of $\mathrm{Osc}_{1,0}$ and induces an automorphism of H. Moreover, the restriction to H of F_η is exactly $\bar{F}_\eta$, the restriction of F_u to H is the identity, and the restriction to H of F_S is exactly $\bar{F}_S$.

4. Equivalence classes of lattices

4.1. Parametrization of $\mathbb{L}$

Definition 4.1. *Two lattices L and L' of $\mathrm{Osc}_{1,0}$ are said to be* ***isomorphic as lattices*** *if there exists an automorphism F of $\mathrm{Osc}_{1,0}$ such that $L = F(L')$. In this case they will also be said* ***equivalent*** *or in the same equivalence class.*

The relation "being equivalent" is an equivalence relation on $\mathbb{L}$, that we denote $\sim$. We denote $\mathcal{L} = \mathbb{L}/\sim$ the set of all equivalence classes of $\mathbb{L}$. Similarly, $\mathcal{L}_{\mathbb{S}}$ (resp. $\mathcal{L}_B^r$, $\mathcal{L}_{\mathbb{S}_T}$) is the set of all equivalence classes of $\mathbb{L}_{\mathbb{S}}$ (resp. $\mathbb{L}_B^r$, $\mathbb{L}_{\mathbb{S}_T}$). It is clear that each equivalence class of an element L of $\mathbb{L}_{\mathbb{S}}$ (resp. $\mathbb{L}_{\mathbb{S}_T}$, $\mathbb{L}_B^r$) is the restriction of the equivalence class of L in $\mathbb{L}$ to $\mathbb{L}_{\mathbb{S}}$ (resp. $\mathbb{L}_{\mathbb{S}_T}$, $\mathbb{L}_B^r$).

For $r \in \mathbb{N}_{>0}$, $B \in \mathcal{G}$, $P \in C_B, x \in \mathbb{R}^2$, $\delta \in \mathbb{R}^2$, $z \in \mathbb{R}$, let's define the set

$$L(r, B, P, x, \delta, z) = F_{P^{-1}\delta}\left(\left\langle \Gamma_P^r, \left(\det(P^{-1})z, P^{-1}x, s_B\right)\right\rangle\right) \tag{8}$$

(recall that s_B is defined as the logarithm of the greatest eigenvalue of B). We also introduce the set

$$\mathcal{P}_1 = \left\{(r, B, P, x, \delta, z) \in \mathbb{N}_{>0} \times \mathbb{S} \times \mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \mid P \in C_B, (B, x) \in \mathcal{B}_r\right\}.$$

Proposition 4.1. * *A subgroup L of $\mathrm{Osc}_{1,0}$ is a lattice of $\mathrm{Osc}_{1,0}$ such that $L \cap H = \bar{F}_\eta(\Gamma_P^r)$ if, and only if, it can be written in the form $L(r, B, P, x, \delta, z)$ with $(r, B, P, x, \delta, z) \in \mathcal{P}_1$. Moreover, in this case $L(r, B, P, x, \delta, z)$ is equivalent to $L_{B,P,x}^r$.*

Reciprocally, we wonder if each lattice of $\mathrm{Osc}_{1,0}$ can be written as a $L(r, B, P, x, \delta, z)$ for $(r, B, P, x, \delta, z) \in \mathcal{P}_1$. Let $\mathcal{A}_0$ denote the identity component of $\mathcal{A}$ (the automorphism group of H) and SL(H) be the subroup of all $F \in \mathcal{A}_0$ whose restriction to the center of H (which is $\{(z, 0, 0) \mid z \in \mathbb{R}\}$) is the identity mapping. Then Theorem 1.10 pp.303 of [7] directly gives:

Proposition 4.2. *For Γ being a lattice of H, there exist an $F \in \mathrm{SL(H)}$, a unique element $h \in \mathbb{R}_{>0}$ and a unique $r \in \mathbb{N}_{>0}$ such that $\Gamma = F \circ \bar{F}_{hI_2}(\Gamma_r)$.*

We know from Proposition 2.2 that for each lattice $L \in \mathbb{L}$, $L \cap H$ is a lattice of H. Then Proposition 4.2 gives that there exist $r \in \mathbb{N}_{>0}$, $h \in \mathbb{R}_{>0}$ and $F \in \mathrm{SL(H)}$ such that $L \cap H = F \circ \bar{F}_{hI_2}(\Gamma_r)$. According to Proposition 2.1, the map F is of the form $\bar{F}_\eta \circ \bar{F}_{\tilde{P}^{-1}}$ for some $\tilde{P} \in \mathrm{GL}(2,\mathbb{Z})$ and $\eta \in \mathbb{R}^2$. Thus each lattice L of $\mathrm{Osc}_{1,0}$ has an intersection with H of the form :

$$L \cap H = \bar{F}_\eta(\Gamma_P^r) = \left\{(0,\delta,0)(z,\eta,0)(0,\eta,0)^{-1} \mid (z,\delta,0) \in \Gamma_P^r\right\},$$

for some $P = \frac{1}{h}\tilde{P} \in \mathrm{GL}(2,\mathbb{R})$, $h \in \mathbb{R}_{>0}$, some $r \in \mathbb{N}_{>0}$ and some $\eta \in \mathbb{R}^2$.

Thus each lattice of $\mathrm{Osc}_{1,0}$ is of the form $\langle \bar{F}_\eta(\Gamma_P^r), (a,b,s)\rangle$, with $P \in \mathrm{GL}(2,\mathbb{R})$, $r \in \mathbb{N}_{>0}$, $\eta \in \mathbb{R}^2$, $(a,b,s) \in \mathrm{Osc}_{1,0}$, $s > 0$ (s is not null because otherwise $\mathrm{Osc}_{1,0}/L$ would not be compact). Moreover, if we set $\delta = P\eta$ so that $F_\eta = F_{P^{-1}\delta}$, we have

$$L = F_{P^{-1}\delta}\left(\langle \Gamma_P^r, F_{-P^{-1}\delta}(a,b,s)\rangle\right) = F_{P^{-1}\delta}\left(\left\langle \Gamma_P^r, (\det(P^{-1})z, P^{-1}x, s)\right\rangle\right) = L(r,B,P,x,\delta,z), \quad (9)$$

with $(\det(P^{-1})z, P^{-1}x, s) = F_{-P^{-1}\delta}(a,b,s)$. Then Proposition 4.1 gives that $(r,B,P,x,\delta,z) \in \mathcal{P}_1$. Then the following Proposition is clear:

Proposition 4.3. *(parametrization of $\mathbb{L}$) The set $\mathcal{P}_1$ is a parametrization of $\mathbb{L}$ via*

$$p_1 : \begin{cases} \mathcal{P}_1 \longrightarrow \mathbb{L} \\ (r,B,P,x,\delta,z) \mapsto L(r,B,P,x,\delta,z). \end{cases} \quad (10)$$

We can directly deduce a parametrization of $\mathbb{L}_\mathbb{S}$:

$$\mathcal{P}_1^\mathbb{S} = \left\{(r,B,P,x) \in \mathbb{N}_{>0} \times \mathbb{S} \times \mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}^2 \mid P \in C_B, (B,x) \in \mathcal{B}_r\right\} \quad \text{via} \quad (11)$$

$$p_1^\mathbb{S} : \begin{cases} \mathcal{P}_1^\mathbb{S} \longrightarrow \mathbb{L} \\ (r,B,P,x) \mapsto L(r,B,P,x,0,0). \end{cases} \quad (12)$$

Proposition D.1 (developped in Annexe D) gives more precisions on a parametrization that is equivalent to $\mathcal{P}_1$.
One can also notice form Proposition 4.1 that each element of $\mathbb{L}$ is equivalent to an element of $\mathbb{L}_\mathbb{S}$. This means that each element of $\mathcal{L}$ has a representative in $\mathbb{L}_\mathbb{S}$, and

$$\mathcal{L} \simeq \mathcal{L}_\mathbb{S} \quad (13)$$

via the map $\Phi_\mathbb{S}$ defined as follows : denote $\Pi_\mathbb{L}$ the projection of $\mathbb{L}$ on $\mathcal{L}$ and $\Pi_{\mathbb{L}_\mathbb{S}}$ the projection of $\mathbb{L}_\mathbb{S}$ on $\mathcal{L}_\mathbb{S}$. Let $\Pi_\mathbb{L}(L) \in \mathcal{L}$. According to Proposition 4.1, there exists $L^r_{B,P,x} \in \mathbb{L}_\mathbb{S}$ such that $L \sim L^r_{B,P,x}$. Then we define $\Phi_\mathbb{S}(\Pi_\mathbb{L}(L)) = \Pi_{\mathbb{L}_\mathbb{S}}(L^r_{B,P,x})$. The map $\Phi_\mathbb{S}$ is well defined and is a bijection, with inverse $\Phi_\mathbb{S}^{-1} : \Pi_{\mathbb{L}_\mathbb{S}}(L_{B,P,x}) \mapsto \Pi_\mathbb{L}(L_{B,P,x})$.

According to the bijection (13), if one is interested in describing $\mathcal{L}$, one can concentrate on $\mathcal{L}_\mathbb{S}$, which we do in subsection 4.2.

4.2. Equivalence classes of $\mathbb{L}_\mathbb{S}$

Recall that the set $\mathbb{L}_\mathbb{S}$ admits $\mathcal{P}_1^\mathbb{S}$ as set of parametrization. In following Propositions 4.4 and 4.5, we give criteria to determine if two lattices of $\mathbb{L}_\mathbb{S}$ are equivalent. As said in the introduction, their proofs are in Annexe A. In both of them we use the following Lemma 4.1.

Lemma 4.1. * *Let* $\alpha = F_\delta \circ F_S \in \mathrm{Aut}(\mathrm{Osc}_{1,0})$ *with* $\delta \in \mathbb{R}^2$ *and* $S \in \mathrm{GL}(2,\mathbb{R})$*. The automorphism* $\alpha_{\|H}$ *of* H *maps* Γ_r *onto itself if, and only if,* $(S,\delta) \in \mathcal{B}_r$*.*

Proposition 4.4. * *Let* $(r,B,P,x),(r',B',P',x') \in \mathcal{P}_1^{\mathbb{S}}$*. Suppose that* $L^r_{B,P,x} \sim L^{r'}_{B',P',x'}$*. Then*

- $r = r'$,
- *Either* B *and* B' *or* B^{-1} *and* B' *are conjugate in* $\mathrm{GL}(2,\mathbb{Z})$ *(which implies* $s_B = s_{B'}$*).*

Moreover, if they are isomorphic via a map F*, we can always write this map* $F = F_u \circ F_{P^{-1}\eta} \circ F_S$*, and then we have a condition on* η : $(PSP^{-1},\eta) \in \mathcal{B}_r$*.*

Proposition 4.5. *Let* $(r,B,P,x) \in \mathcal{P}_1^{\mathbb{S}}$*,* $\tilde{B} \in \mathrm{GL}(2,\mathbb{Z})$ *and* $\tilde{P} \in C_{\tilde{B}}$*. If either* B *and* $\tilde{B}$ *or* B^{-1} *and* $\tilde{B}$ *are conjugate with respect to a matrix in* $\mathrm{GL}(2,\mathbb{Z})$*, then there is an automorphism* F *of* $\mathrm{Osc}_{1,0}$ *and an* $\tilde{x} \in \mathbb{R}^2$ *such that* $(\tilde{B},\tilde{x}) \in \mathcal{B}_r$ *and* $F(L^r_{B,P,x}) = L^r_{\tilde{B},\tilde{P},\tilde{x}}$*.*

Remark 4.1. *Notice that in the proof of Proposition 4.5 (given in Annexe A), we determine an explicit couple* $(F,x') \in \mathrm{Aut}(\mathrm{Osc}_{1,0}) \times \mathbb{R}^2$ *satisfying the conditions of the Proposition. In particular,*

$$x' = \tilde{P}[SP^{-1}x - e^{\mu s A}P^{-1}\eta + P^{-1}\eta]$$

works, where $\eta \in \mathbb{R}^2$ *satisfies* $(E,\eta) \in \mathcal{B}_r$ *for some* $E \in \mathrm{GL}(2,\mathbb{Z})$ *such that* $\tilde{B} = EB^{\pm 1}E^{-1}$ *(it is possible to find* E *when* B *and* $\tilde{B}$ *are given explicitly).*

The results of Propositions 4.4 and 4.5 lead us to introduce an equivalence relation $\sim$ on $\mathbb{S}$, defined by

$$B \sim \tilde{B} \iff \exists M \in \mathrm{GL}(2,\mathbb{Z}), \quad \exists \mu \in \{-1,1\}, \quad B^\mu = M\tilde{B}M^{-1}, \tag{14}$$

and $\mathcal{S}$ the quotient set $\mathcal{S} = \mathbb{S}/\sim$.

For each $T > 2$, $\mathbb{S}_T$ is stable by conjugation by elements of $\mathrm{GL}(2,\mathbb{Z})$ and by inversion. Thus $\sim$ induces an equivalence relation on each $\mathbb{S}_T$, and we denote $\mathcal{S}_T = \mathbb{S}_T/\sim$. Then

$$\mathcal{S} = \bigsqcup_{T>2} \mathcal{S}_T, \tag{15}$$

If $B, B' \in \mathbb{S}$ are not equivalent, Proposition 4.4 gives that the two lattices $L^r_{B,P,x}$ and $L^{r'}_{B',P',x'}$ are not isomorphic (for any other parameters): it implies that the sets $(\mathcal{L}_{\mathbb{S}_T})_{T>2}$ are disjoints. Moreover, the equivalence classes of $\mathbb{L}_{\mathbb{S}_T}$ ($T > 2$) are also equivalence classes in $\mathbb{L}_{\mathbb{S}}$, and not only restrictions of equivalence classes of $\mathbb{L}_{\mathbb{S}}$: then each $\mathcal{L}_{\mathbb{S}_T}$, for $T > 2$, is a subset of $\mathcal{L}_{\mathbb{S}}$. Since it is clear that their union is equal to $\mathcal{L}_{\mathbb{S}}$, they form a partition of $\mathcal{L}_{\mathbb{S}}$ and we have:

$$\mathcal{L}_{\mathbb{S}} = \bigsqcup_{T>2} \mathcal{L}_{\mathbb{S}_T}. \tag{16}$$

This implies that the description of $\mathcal{L}$ simplifies to the classification of each $\mathcal{L}_{\mathbb{S}_T}$, $T > 2$. Let us introduce, for $T > 2$, $\mathcal{C} \in \mathcal{S}_T$, $r \in \mathbb{N}_{>0}$, the set

$$\mathbb{L}^r_{\mathcal{C}} = \bigcup_{B\in\mathcal{C}} \mathbb{L}^r_B$$

and $\mathcal{L}^r_{\mathcal{C}}$ the set of all equivalence classes of $\mathbb{L}^r_{\mathcal{C}}$. From Proposition 4.5, we get that each $\mathcal{L}^r_{\mathcal{C}}$ is a subset of $\mathcal{L}_{\mathbb{S}_T}$. Moreover, if $r \neq r'$ or $\mathcal{C} \neq \mathcal{C}'$, Proposition 4.4 gives that $\mathcal{L}^r_{\mathcal{C}}$ and $\mathcal{L}^{r'}_{\mathcal{C}'}$ are disjoint. Since the union of the $(\mathcal{L}^r_{\mathcal{C}})_{\mathcal{C}\in\mathcal{S}_T, r\in\mathbb{N}_{>0}}$ is clearly equal to $\mathcal{L}_{\mathbb{S}_T}$, we get:

$$\mathcal{L}_{\mathbb{S}_T} = \bigsqcup_{r\in\mathbb{N}^*} \bigsqcup_{\mathcal{C}\in\mathcal{S}_T} \mathcal{L}^r_{\mathcal{C}}. \tag{17}$$

This means that $\mathcal{L}_{\mathbb{S}} = \bigsqcup_{r\in\mathbb{N}^*} \bigsqcup_{T>2} \bigsqcup_{\mathcal{C}\in\mathcal{S}_T} \mathcal{L}^r_{\mathcal{C}}$ according to (16), and $\mathcal{L} \overset{\phi_{\mathbb{S}}}{\simeq} \bigsqcup_{r\in\mathbb{N}^*} \bigsqcup_{T>2} \bigsqcup_{\mathcal{C}\in\mathcal{S}_T} \mathcal{L}^r_{\mathcal{C}}$ according to (13).

Then it would be natural to study the set $\mathcal{L}^r_{\mathcal{C}}$ for each $\mathcal{C} \in \mathcal{S}_T$, $r \in \mathbb{N}_{>0}$. In Section 5 we study the set $\mathcal{L}^r_B$ for fixed $B \in \mathcal{C}$ and $r \in \mathbb{N}_{>0}$, that will appear in Section 6 to be in one-to-one correspondence with $\mathcal{L}^r_{\mathcal{C}}$.

Remark 4.2. *Equation (16) translates the fact that a subset of $\mathbb{L}_{\mathbb{S}}$ is an equivalence class in $\mathbb{L}_{\mathbb{S}}$ if, and only if it is an equivalence class in some $\mathbb{L}_{\mathbb{S}_T}$ for some $T > 2$, and that in this case T is unique.*

5. Isomorphism classes and study of $\mathcal{L}^r_B$

5.1. Isomorphism classes and equivalence classes

Definition 5.1. *Two lattices L and L' of the oscillator group are said isomorphic if they are isomorphic as abstract groups. In this case they will also be said of the same isomorphism class.*

If two lattices are equivalent, they are isomorphic. The converse implication is not true in general: in the case of the group $\mathrm{Osc}_{0,1}$ studied in [3] and [4], the isomorphism classes and the equivalence classes of lattices are different. The case of $\mathrm{Osc}_{1,0}$ is different. Let's recall a Theorem due to Saîto (see Theorem 5 od [2]):

Let G, G' be two simply connected solvable Lie groups with real roots and D be a uniform discrete subgroup of G. Then each homomorphism ϕ from D to a discrete subgroup D' of G' can be uniquely extended to a homomorphism ψ from G to G'. If in particular D' is uniform in G', ψ is surjective. If ϕ is injective, ψ is too.

We know $\mathrm{Osc}_{1,0}$ is a solvable Lie group with real roots, i.e the eigenvalues of all the transformations Ad_g, $g \in \mathrm{Osc}_{1,0}$, in the adjoint action of G are real. We do not give a proof of this fact in this document, but it is a direct consequence of the fact that the matrix $A = [[0,1][1,0]]$ we use to define $\mathrm{Osc}_{1,0}$ as a semidirect product has real eigenvalues. We refer to a lesson on Lie groups for more details on the definition of the adjoint representation Ad, for example the one of Jean-François Dat, "Cours introductif de M2, Groupes et Algèbres de Lie" or the one of Benoît Stroh, "Groupes Compacts et Groupes de Lie".

Moreover, it is known that $\mathrm{Osc}_{1,0}$ is simply connected. Then, recalling that a lattice in a solvable Lie group is an uniform discrete subgroup, Theorem 5 of [2] directly gives:

Theorem 5.1. *Two lattices L and L' of $\mathrm{Osc}_{1,0}$ are isomorphic if, and only if, they are equivalent.*

5.2. The Discrete Oscillator Groups

Definition 5.2. *Let $r \in \mathbb{N}_{>0}$. The discrete Heisenberg group with parameter r is defined by*

$$H_r = \left\langle \alpha, \beta, \gamma \mid \alpha\beta\alpha^{-1}\beta^{-1} = \gamma^r, \alpha\gamma = \gamma\alpha, \beta\gamma = \gamma\beta \right\rangle.$$

A discrete oscillator group is a semidirect product $H_r \rtimes_S \mathbb{Z}.\delta$ of H_r with $\mathbb{Z}.\delta$ with respect to a homomorhism S from $\mathbb{Z}.\delta$ to the automorphism group of H_r, such that the map $\overline{S(\delta)}$ induced by $S(\delta)$ on $H_r/Z(H_r)$ is a matrix of $\mathbb{S}$ in $\mathrm{GL}(2,\mathbb{R})$.

It is a typical result that H_r and $H_{r'}$ are isomorphic if, and only if $r = r'$. For $r \in \mathbb{N}_{>0}$, H_r is isomorphic as abstract group to $\varGamma_r$ and to any $\varGamma^r_P$, $P \in \mathrm{GL}(2,\mathbb{R})$.
A direct consequence of Proposition 2.2 is that the intersection $L \cap H$ of a lattice L of $\mathrm{Osc}_{1,0}$ with H is a discrete Heisenberg group H_r for some $r \in \mathbb{N}_{>0}$. Thus L considered as an abstract group is isomorphic to a discrete oscillator group, and one can wonder which one it is.

Definition 5.3. *Let $r \in \mathbb{N}_{>0}$, $B = [[a,b][c,d]] \in \mathbb{S}$ and $(k,l) \in Z^2$. We define the homomorphism*

$$S_B^{k,l} = \begin{pmatrix} B & & 0 \\ k & l & 1 \end{pmatrix}$$

of H_r, which means that $S_B^{k,l}(\delta)(\alpha) = \alpha^a\beta^c\gamma^k$, $S_B^{k,l}(\delta)(\beta) = \alpha^b\beta^d\gamma^l$ and $S_B^{k,l}(\delta)(\gamma) = \gamma$.

Notice that for each $B \in \mathbb{S}$ and $r \in \mathbb{N}_{>0}$, $\bar{S_B^{k,l}} = B$ is an element of $\mathbb{S}$ and $H^r \rtimes_{S_B^{k,l}} \mathbb{Z}.\delta$ is a discrete oscillator group. In the following we will denote

$$\varGamma(r,B,k,l) := H_r \rtimes_{S_B^{k,l}} \mathbb{Z}.\delta.$$

Proposition 5.1. * *Let $r \in \mathbb{N}_{>0}$, $B = [[a,b][c,d]] \in \mathbb{S}$ and $x = (x_1,x_2)$ such that $(B,x) \in \mathcal{B}_r$, and $P \in C_B$. Then $L^r_{B,P,x}$ is isomorphic to $\varGamma(r,B,k,l)$, for $k = r(cx_1 - ax_2 - \frac{ac}{2}) \in \mathbb{Z}$ and $l = r(dx_1 - bx_2 - \frac{bd}{2}) \in \mathbb{Z}$.*

Since each lattice of $\mathrm{Osc}_{1,0}$ is equivalent to some $L^r_{B,P,x}$, it means that each lattice of $\mathrm{Osc}_{1,0}$ is isomorphic as abstract group to some discrete oscillator group $\varGamma(r,B,k,l)$ for $B \in \mathbb{S}$ and $(k,l) \in \mathbb{Z}^2$.

Remark 5.1. *Every discrete oscillator group $\varGamma(r,B,k,l)$, with $B \in \mathbb{S}$ is isomorphic to a lattice of* $\mathrm{Osc}_{1,0}$*. Indeed, since B is invertible, it is always possible to find a solution to the system:*

$$\begin{pmatrix} k + \frac{rac}{2} \\ l + \frac{rbd}{2} \end{pmatrix} = r \begin{pmatrix} c & a \\ d & b \end{pmatrix} \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}.$$

But then

$$r(cx_1 - ax_2) = \frac{rac}{2} + k, \quad r(dx_1 - bx_2) = \frac{rbd}{2} + l,$$

which clearly imply relation (3) of Remark 2.1. Thus $(B,x) \in \mathcal{B}_r$. Moreover, by assumption $B \in \mathbb{S} \subset \mathcal{G}$. Then (B,x) satisfies all the conditions of Proposition 2.4, so that $L^r_{B,P,x}$ is a lattice of $\mathrm{Osc}_{1,0}$.

For $B \in \mathbb{S}$ and $r \in \mathbb{N}_{>0}$, let $\mathbb{O}(B,r)$ denote the set of all discrete oscillator groups of the form $\varGamma(r,B,k,l)$ with $k,l \in \mathbb{Z}$, and $\mathcal{O}(B,r)$ denote the set of isomorphim classes of $\mathbb{O}(B,r)$. Let $\varPi^r_B$ and ${\varPi'_B}^r$ denote respectively the canonical projections of $\mathbb{L}^r_B$ on $\mathcal{L}^r_B$ and of $\mathbb{O}(B,r)$ on $\mathcal{O}(B,r)$.

Proposition 5.2. * *For fixed $B = [[a,b][c,d]] \in \mathbb{S}$ and $r \in \mathbb{N}_{>0}$, $\mathcal{L}^r_B$ is in one-to-one correspondence with $\mathcal{O}(B,r)$ via:*

$$\varPsi^{(1)}_{B,r} : \begin{cases} \mathcal{L}^r_B \simeq \mathcal{O}(B,r); \\ \varPi^r_B(L^r_{B,P,x}) \mapsto {\varPi'_B}^r \left[\varGamma\big(r,B,r\left[cx_1 - ax_2 - \frac{ac}{2}\right], r\left[dx_1 - bx_2 - \frac{bd}{2}\right]\big)\right]; \end{cases} \tag{18}$$

This bijection only depends on B and r.

5.3. The reversing symmetry group of a matrix

Definition 5.4. *For $B \in \mathbb{S}$ and $K \in \mathrm{GL}(2,\mathbb{Z})$, we say that K is a symmetry of B if $KBK^{-1} = B$, and K is a reversing symmetry of B if $KBK^{-1} = B^{-1}$. We say that $B \in \mathbb{S}$ is reversible - resp. symmetrizable - if B has a reversing symmetry - resp. a symmetry K with determinant -1.*

The group of all symmetries of B is exactly its commutant $C(B)$ in $\mathrm{GL}(2,\mathbb{Z})$. Let $\mathcal{R}(B)$ denote the set of all symmetries and reversing symmetries of B, which is clearly a subgroup of $\mathrm{GL}(2,\mathbb{Z})$ with $C(B)$ as normal subgroup. This group is described in [9].

5.4. Description of $\mathcal{L}_B^r$

In this subsection, we fix $B \in \mathbb{S}$ and $r \in \mathbb{N}_{>0}$, and since there is no ambiguity, we will denote $\mathit{\Gamma}(k,l) = \mathit{\Gamma}(r,B,k,l)$. We define an equivalence relation on $\mathbb{Z}^2$ by:

$$(k,l) \overset{B,r}{\sim} (k',l') \iff \mathit{\Gamma}(k,l) \simeq \mathit{\Gamma}(k',l') \tag{19}$$

Since there is no ambiguity, in this section we will denote $\overset{B,r}{\sim}$ by $\sim$. It is clear that the map

$$\Psi_{B,r}^{(2)} : \begin{cases} \mathcal{O}(B,r) \longrightarrow \mathbb{Z}^2/\sim \\ \mathit{\Pi}'_{B,r}[\mathit{\Gamma}(k,l)] \mapsto \pi_B^r(k,l) \end{cases} \tag{20}$$

is a bijection, where π_B^r is the canonical projection of $\mathbb{Z}^2$ on $\mathbb{Z}^2/\sim$. Then

$$\mathcal{L}_B^r \overset{\Psi_{B,r}^{(2)} \circ \Psi_{B,r}^{(1)}}{\simeq} \mathbb{Z}^2/\sim. \tag{21}$$

Proposition 5.3. * *If one of the four following conditions is satisfied, then $(k,l) \sim (k',l')$.*
(i). $k \equiv k'[r]$ and $l \equiv l'[r]$;
(ii). There exists $(t_1,t_2) \in \mathbb{Z}^2$ such that $(k,l) = (k',l') + (t_1,t_2).(B - I_2)$;
(iii). There exists $\bar{K} \in \mathcal{R}(B)$ such that $(k',l') = \det(\bar{K}).(k,l).\bar{K}$.

Theorem 5.2. * *The equivalence relation $\overset{B,r}{\sim}$ defined on $\mathbb{Z}^2$ is generated by relations of the form (i), (ii) and (iii) of Proposition 5.3.*

Theorem 5.2, which is the main result of Section 5, provides that the description of $\mathcal{L}_B^r$ can simplify to the study of 3 relations of $\mathbb{Z}^2$. Its proof directly gives:

Corollary 5.1. * *Let (k,l) and $(k',l') \in \mathbb{Z}^2$. The isomorphisms K mapping $\mathit{\Gamma}(k,l)$ onto $\mathit{\Gamma}(k',l')$ are all of the form*

$$K_{|H} = \begin{pmatrix} \bar{K} & & 0 \\ t_1 & t_2 & \det(\bar{K}) \end{pmatrix}$$

and $K(\delta) = \alpha^x \beta^y \gamma^z \delta^\mu$, $t_1,t_2,x,y,z \in \mathbb{Z}$ and $\mu \in \{-1,1\}$, $\bar{K} \in \mathrm{GL}(2,\mathbb{Z})$ and $\bar{K}B = B^\mu \bar{K}$. Reciprocally, a map of this form is an isomorphism.

5.5. The set $\mathbb{Z}^2/\sim$

In this subsection we translate Theorem 5.2 in terms of the action of $\mathcal{R}(B)$ on a quotient of $\mathbb{Z}^2$, which is easier to manipulate. Let's fix $B = [[a,b][c,d]] \in \mathbb{S}_T$ and $r \in \mathbb{N}_{>0}$. Let's dfine

$$L_1(B,r) = \mathbb{Z}^2.(B - I_2) + r\mathbb{Z}^2 = span_{\mathbb{Z}}\{(a-1,b),(c,d-1),(r,0),(0,r)\},$$

which is a lattice of $\mathbb{R}^2$. In this subsection we denote $L_1 = L_1(B,r)$ since B and r are fixed. Relation *(iii)* of Theorem 5.2 leads us to introduce an action of $\mathcal{R}(B)$ on $\mathbb{Z}^2$ at right, defined by

$$\forall K \in \mathcal{R}(B), \forall (k,l) \in \mathbb{Z}^2, \rho(K)(k,l) = \det(K).(k,l).K. \tag{22}$$

Denote $\Pi_{L_1} : \mathbb{Z}^2 \to \mathbb{Z}^2/L_1$ and $\Pi_{B,r} : \mathbb{Z}^2 \to \mathbb{Z}^2/\overset{B,r}{\sim}$ the canonical projections.

Proposition 5.4. * *L_1 is stable under this action, and this induces an action $\bar{\rho}$ of $\mathcal{R}(B)$ on $\mathbb{Z}^2/L_1$. Denote $\omega_{\mathcal{R}(B)}(k,l)$ the orbit of $(k,l) \in \mathbb{Z}^2/L_1$ for this action. Then there is a bijection $\Psi^{(3)}_{B,r} : (\mathbb{Z}^2/L_1)/\mathcal{R}(B) \simeq \mathbb{Z}^2/\overset{B,r}{\sim}$ defined by $\Psi^{(3)}_{B,r}\left(\omega_{\mathcal{R}(B)}\left(\Pi_{L_1}(k,l)\right)\right) = \Pi_{B,r}(k,l)$.*

Let's define:

$$\text{For } K = [[t_1,t_2][t_3,t_4]] \in \mathcal{R}(B), \quad (K \mod r) = \begin{pmatrix} (t_1 \mod r) & (t_2 \mod r) \\ (t_3 \mod r) & (t_4 \mod r) \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}_\mathrm{r});$$

$$\mathcal{R}(B)_r = \{(K \mod r) \mid K \in \mathcal{R}(B)\}.$$

Proposition 5.5. * *One can define an action at right $\rho_{B,r}$ of $\mathcal{R}(B)_r$ on $\mathbb{Z}^2/L_1$, by :*

$$\rho_{B,r}(K \mod r)(\Pi_{L_1}(k,l)) = \bar{\rho}(K)(\Pi_{L_1}(k,l)).$$

Moreover, $(\mathbb{Z}^2/L_1)/\mathcal{R}(B) = (\mathbb{Z}^2/L_1)/\mathcal{R}(B)_r$.

Then if one wants to determine $\mathbb{Z}^2/\overset{B,r}{\sim}$, one just has to first compute L_1, and then study the set $\left(\mathbb{Z}^2/L_1(B,r)\right)/\mathcal{R}(B)$ or $\left(\mathbb{Z}^2/L_1(B,r)\right)/\mathcal{R}(B)_r$.

6. Conclusion of Sections from 3 up to 5

In Sections from 3 up to 5, our goal was to parametrize $\mathbb{L}$ and to get criteria on the equivalence classes of $\mathbb{L}$ in order to establish a bijection between $\mathcal{L}$ and a set that we know better or that we can study.

The points of view of equivalence classes and isomorphism classes, although equivalent (Theorem 5.1), have given us complementary results: the study of equivalence classes allows us to find the explicit automorphisms between two lattices (Propositions 4.1, 4.4, 4.5 and D.5 - see Annexe D - for example), and it is easier to find necessary conditions for two lattices to be equivalent (see Proposition 4.4), while it is easier to find sufficient conditions on two lattices to be isomorphic (see Proposition 5.3). For example Theorem 5.2 and Proposition D.4 (developped in Annexe D using the tool of normalized lattices) give the same result, but Proposition D.4 gives additional information on the automorphism mapping $L^r_{B,P,x}$ on $L^{\tilde{r}}_{\tilde{B},\tilde{P},\tilde{x}}$ and the explicit form of the sequence of relations that generate the equivalence relation on $\mathbb{Z}^2$ defined in (19), while Theorem 5.2 allows us to take a more abstract point of view. Moreover, the point of view of equivalence classes gives us a parametrization of $\mathbb{L}$ and allows us to introduce the normalized lattices (developped in Annexe D), that permit to understand better this parametrization (see Proposition D.1) and can be useful if one wants to go further and classify lattices up to inner automorphism.

Moreover, a direct consequence of Proposition 5.1 is that the isomorphism class of a lattice $L^r_{B,P,x}$ does not depend on $P \in C_B$. Thus for $P, \tilde{P} \in C_B$, $L^r_{B,P,x}$ and $L^r_{B,\tilde{P},x}$ are isomorphic as abstract groups. Then we have, for $T > 2$, $\mathcal{C} \in \mathcal{S}_T$, $B \in \mathcal{C}$ and $r \in \mathbb{N}_{>0}$,

$$\mathcal{L}^r_{\mathcal{C}} \simeq \mathcal{L}^r_B \tag{23}$$

via the bijection $\Psi^{(4)}_{B,r}$ defined as follows : let Π^r_B denote the projection of $\mathbb{L}^r_B$ on $\mathcal{L}^r_B$ and $\Pi^r_{\mathcal{C}}$ the projection of $\mathbb{L}^r_{\mathcal{C}}$ on $\mathcal{L}^r_{\mathcal{C}}$ (which is actually the restriction to $\mathbb{L}^r_{\mathcal{C}}$ of the projection $\Pi_{\mathbb{L}_{\mathbb{S}}}$ of $\mathbb{L}_{\mathbb{S}}$ on $\mathcal{L}_{\mathbb{S}}$). Let's fix $P \in C_B$. Let $L^{r'}_{B',P',x'} \in \mathbb{L}^r_{\mathcal{C}}$, with $B' \in \mathbb{S}$ and $(B',x') \in \mathcal{B}_{r'}$. Then by definition, $B' \in \mathcal{C}$ and $r = r'$. Thus, according to Proposition 4.5, there exists $\tilde{x} \in \mathbb{R}^2$ such that $L^{r'}_{B',P',x'} \sim L^r_{B,P,\tilde{x}}$ (we can find such an $\tilde{x}$ with Remark 4.1, one we know $E \in \mathrm{GL}(2,\mathbb{Z})$ satisfying $B = EB'^{\pm 1}E^{-1}$). Let's define $\Psi^{(4)}_{B,r}\left(\Pi^r_{\mathcal{C}}(L^{r'}_{B',P',x'})\right) = \Pi^r_B(L^r_{B,P,\tilde{x}})$. The map $\Psi^{(4)}_{B,r}$ is well defined, because it does not depend of the vector $\tilde{x}$ such that $L^{r'}_{B',P',x'} \sim L^r_{B,P,\tilde{x}}$. It is a bijection, with inverse $\Psi^{(4)\,-1}_{B,r} : \Pi_B(L^r_{B,P',x}) \mapsto \Pi^r_{\mathcal{C}}(L^r_{B,P,x}) \in \mathcal{L}^r_{\mathcal{C}}$.

For $B \in \mathbb{S}$ annd $r \in \mathbb{N}_{>0}$, let $\Psi_{B,r} = \Psi^{(3)\,-1}_{B,r} \circ \Psi^{(2)}_{B,r} \circ \Psi^{(1)}_{B,r} \circ \Psi^{(4)}_{B,r}$ (where $\Psi^{(1)}_{B,r}$ is defined in (18), $\Psi^{(2)}_{B,r}$ is defined in (20) and $\Psi^{(3)}_{B,r}$ in Proposition 5.4).

Let us summarize the results we have found in Sections 4 and 5. From Section 4, recall that we have an explicit bijection:

$$\mathcal{L} \overset{\Phi_{\mathbb{S}}}{\simeq} \bigsqcup_{T>2} \mathcal{L}_{\mathbb{S}_T} \tag{24}$$

where the bijection is explicit and canonical, and (17) and Section 5 give, $\forall T > 2$:

$$\mathcal{L}_{\mathbb{S}_T} = \bigsqcup_{r\in\mathbb{N}_{>0}} \bigsqcup_{\mathcal{C}\in\mathcal{S}_T} \mathcal{L}^r_{\mathcal{C}} \quad \text{where} \quad \forall \mathcal{C} \in \mathcal{S}_T, \forall B \in \mathcal{C} \quad \mathcal{L}^r_{\mathcal{C}} \overset{\Psi_{B,r}}{\simeq} \left(\mathbb{Z}^2/L_1(B,r)\right)/\mathcal{R}(B)_r, \tag{25}$$

and all the bijections $\Psi_{B,r}$ are determined in terms of B and r. One can make two observations:

- The description of $\left(\mathbb{Z}^2/L_1(B,r)\right)/\mathcal{R}(B)_r$ raises the problem of the study the reversing symmetry group of $B \in \mathbb{S}_T$. It is linked to the set $\mathcal{S}_T$ in the sense that $\mathcal{R}(B)$ is the set of all matrices T that make B equivalent to itself for $\sim$ (i.e $TBT^{-1} = B^{\pm 1}$). The problem is that we do not have an expression of $\mathcal{R}(B)$ in terms of $B \in \mathbb{S}_T$, $T > 2$.
- The second disjoint union in (25) is indexed on $\mathcal{S}_T$. We would be tempted to reunite all the bijections of (25) in the formula

$$\mathcal{L}_{\mathbb{S}_T} \simeq \bigsqcup_{i\in\{1,\dots,n\}} \bigsqcup_{r\in\mathbb{N}_{>0}} E(B_i, r), \tag{26}$$

  where $(B_i)_{i\in\{1,\dots,n\}}$ ($n \in \mathbb{N}_{>0}$) is a system of representatives of $\mathcal{S}_T$ (we know that it is finite, see section 7) and $E(B,r)$ is the set of representatives of $(\mathbb{Z}^2/L_1(B,r))/\mathcal{R}(B)_r$. However we cannot give an expression of $\mathcal{R}(B)$ in terms of $B \in \mathbb{S}_T$, and thus of $E(B,r)$ in terms of B and r. Thus we need explicit representatives of $\mathcal{L}_{\mathbb{S}_T}$ if we want the bijection (26) to be well determined (and thus usable in practice). We will see in Section 9 that it is possible to get, for an explicit $T > 2$, a system of representatives of $\mathcal{S}_T$ (and thus simplify (25)), but we do not have an expression of the set of representatives of $\mathcal{S}_T$ in terms of $T > 2$.

As said in the introduction, these two problems raised by Sections 4 and 5 (the fact that we cannot get an expression of a system of representative of $\mathbb{S}_T$ in terms of $T > 2$ and of $\mathcal{R}(B)$ in terms of B) and that impede to get a bijection between $\mathcal{L}_{\mathbb{S}_T}$ and a set written in terms of T are linked to the fact that $\mathbb{S}_T$ consists in matrices diagonalizable with real spectrum. In Sections from 7 up to 9, we give a method to determine, for an explicit $T > 2$, an explicit bijection of the form of (26).

Remark 6.1. *Notice that writing*

$$\mathcal{L} \simeq \bigsqcup_{r\in\mathbb{N}_{>0}} \bigsqcup_{i\in\mathbb{N}} \left(\mathbb{Z}^2/L_1(B'_i,r)\right)/\mathcal{R}(B'_i)_r,$$

where $(B'_i)_{i\in\mathbb{N}}$ is a system of representatives of $\mathcal{S}$, would not be convenient because even if we will find a method to determine an explicit set of representatives of each $\mathcal{S}_T$ in next section, it will not give us a method to determine a set of representatives of $\mathcal{S}$ in a finite number of computations. Thus this bijection is abstract and not usable in practice (to determine if two lattices are equivalent for example).

7. Description of $\mathcal{S}_T$

In this section we develop the study of the set $\mathcal{S}_T = \mathbb{S}_T/\sim$ $(T > 2)$, which is central in the classification up to isomorphism of lattices of $\mathrm{Osc}_{1,0}$ (see Section 6).

By definition, it is related to the conjugacy classes of $\mathbb{S}_T$ by $\mathrm{SL}(2,\mathbb{Z})$. There are many ways to study these classes, for example the one of Karpenkov in [5], that we describe in Annexe E. In this Section, we will rather use the point of view of Aicardi in [6]. We also have to take into account the fact that $\mathcal{S}_T$ is not exactly $\mathbb{S}_T/\mathrm{SL}(2,\mathbb{Z})$, but consists in the sets $C \cup C^{-1}$, where $C \in \mathbb{S}_T/\mathrm{GL}(2,\mathbb{Z})$ and C^{-1} is the set of all inverse matrices of elements of C.

7.1. Conjugation by $\mathrm{SL}(2,\mathbb{Z})$ and quadratic binary forms

In this subsection we use the results of [6], more particularly the ones of section 4.4, in which Aicardi studies a classification of the integer quadratic binary forms with non square positive discriminant, up to the action of $\mathrm{SL}(2,\mathbb{Z})$.

The set $\mathcal{Q}$ of quadratic binary forms is in bijection with $\mathbb{Z}^3$ via

$$\begin{cases} \mathbb{Z}^3 \simeq \mathcal{Q} \\ (k,m,n) \mapsto [q : (x,y) \mapsto kx^2 + mxy + ny^2]. \end{cases}$$

Aicardi introduces the change of coordinates $(k,m,n) \mapsto [K,D,S] = [k, m+n, m-n]$ (where the new coordinates are denoted in brackets to avoid confusions). Then $(k,m,n) \in \mathbb{Z}^3$ if, and only if, $[K,D,S] \in \mathbb{Z}^3$ and $D \equiv S \bmod 2$. From this change of coordinates, $\mathbb{Z}^3$ (and thus $\mathcal{Q}$) is in bijection with the set $H = \{[K,D,S] \in \mathbb{Z}^3, D \equiv S \bmod 2\}$. It is known that $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathcal{Q}$ via

$$\forall A \in \mathrm{SL}(2,\mathbb{Z}), \forall q \in \mathcal{Q} \quad (A.q)(x,y) = q((x,y).{}^tA).$$

Then $\mathrm{SL}(2,\mathbb{Z})$ acts on the set H via the bijection $\mathcal{Q} \simeq \mathbb{Z}^3 \simeq H$. Let's set

$$H^0 = \left\{[K,D,S] \in \mathbb{Z}^3 \mid |S| < |D|, D > 0\right\}$$

and for $T > 2$:

$$\Delta_T = T^2 - 4, \quad H_T = \left\{[K,D,S] \in \mathbb{Z}^3 \mid K^2 + D^2 - S^2 = \Delta_T\right\} \quad \text{and} \quad H_T^0 = H^0 \cap H_T.$$

The action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{Z}^3$ induces an action on H_T, whose Aicardi gives results on the orbits.

One can actually translate all these results in terms of the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{S}_T$. Let's define

$$\phi_T : \begin{cases} H_T \to \mathbb{S}_T \\ [K,D,S] \mapsto \begin{pmatrix} \frac{K}{2} + \frac{\sqrt{\Delta_T + 4}}{2} & \frac{D-S}{2} \\ \frac{D+S}{2} & \frac{\sqrt{\Delta_T+4}}{2} - \frac{K}{2} \end{pmatrix}; \end{cases}$$

It is easy to check that ϕ_T is well defined. It is a bijection with inverse map defined by the following: for $B \in \mathbb{S}_T$, one can write the matrix B under the form

$$B = \begin{pmatrix} k + \frac{T}{2} & m - n \\ m + n & \frac{T}{2} - k \end{pmatrix} \quad \text{with} \quad k + \frac{T}{2} \in \mathbb{Z}, \quad m + n \in \mathbb{Z}. \tag{27}$$

Then $\phi_T^{-1}(B) = (2k, 2m, 2n)$. To see this, one just has to notice that

$$\det(B) = 1 \Leftrightarrow k^2 + m^2 - n^2 = \frac{T^2}{4} - 1 \Leftrightarrow (2k)^2 + (2m)^2 - (2n)^2 = T^2 - 4 \Leftrightarrow (2k, 2m, 2n) \in H_T^0.$$

ϕ_T induces a bijection between H_T^0 and $\phi_T(H_T^0) = \{B = [[a,b][c,d]] \in \mathbb{S}_T \mid a,b,c,d > 0\}$, that we will still denote H_T^0 in the following. Moreover,

$$\forall A \in \mathrm{SL}(2,\mathbb{Z}), \forall (k,m,n) \in H_T \quad \phi_T(A.(k,m,n)) = A\phi_T(k,m,n)A^{-1}.$$

Thus all the results of [6] on the action of $\mathrm{SL}(2,\mathbb{Z})$ on H_T are also true for the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{S}_T$. Here, $T > 2$ implies that $\Delta_T > 0$, and more particularly, $\Delta_T = T^2 - 4$ is not a square number. This case is studied by Aicardi in section 4.4 of [6]. In the following, when we use the notions and results of this section, we will systematically translate them in matricial terms by identifying the elements of $\mathbb{S}_T$ (resp. of H_T^0) with their preimage under ϕ_T. Let's define

$$N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

In the following of the section, we fix $T \in \mathbb{N}_{>2}$. Let us recall a definition introduced by Aicardi in [6]:

Definition 7.1. *A cycle of length $t > 1$ is a cyclic sequence of distinct matrices $[B_1, B_2, ..., B_t]$ such that $B_i = T_{i-1}B_{i-1}T_{i-1}^{-1}$ $(i = 2, ..., t)$ and $B_1 = T_t M_t T_t^{-1}$, where each of the matrices $T_1, T_2, ..., T_t$ is N or S.*

For each matrix $B \in \mathbb{S}_T$, the orbit $\omega(B)$ of B for conjugation by $\mathrm{SL}(2,\mathbb{Z})$ contains exactly one cycle in H_T^0 (see Theorem 4.10 of [6]). In order to determine it, let's translate the method given by Aicardi in matricial terms. For each matrix $B \in \mathbb{S}_T$, there exists $B' \in H_T^0 \cap \omega(B)$. Then we know from Lemma 4.7 of [6] that for each matrix $B' \in H_T^0$ the orbit of B' has exactly one cycle in H_T^0, and moreover B' is in this cycle. We build by induction (as in the Proof of Lemma 4.7 of [6]) the finite sequence of matrices $(B_1, ..., B_n)$ of the cycle and the finite sequence of operators $[T_1, ..., T_n]$ by setting $B_1 = B'$ and: if $NB_kN^{-1} \in H_T^0$, then $T_k = N$ and $B_{k+1} = NB_kN^{-1}$, and otherwise, according to Lemma 4.6 of [6], we have $SB_kS^{-1} \in H_T^0$. Thus we set $T_k = S$ and $B_{k+1} = SB_kS^{-1}$. Thus we get a sequence $(B_1, ...)$ of matrices being in H_T^0. But according to Lemma 4.5 of [6], H_T^0 contains only a finite number of points, thus this sequence is periodic and there exists a minimal t such that $B_{t+1} = B' = B_1$.

Hence the cycle $(B', ..., B_t)$, denoted by $\gamma_B(B', T_1, ..., T_t)$ - with $B_1 = B'$ -, which does not depend on the choice of the representative B' of the orbit of B lying in H_T^0. In the case where $B \in H_T^0$, since B is already a representative of its class lying in H_T^0, we will allow to denote its cycle by $\gamma_B(T_1, ..., T_t)$, where $B_1 = B$.

Remark 7.1. *In order to simplify the notations, we will often allow to identify an index, being an element of $\mathbb{Z}$, with its projection on $\mathbb{Z}/t\mathbb{Z}$. Then we can write $\gamma_B(T_1, ..., T_t) = \gamma_B(T_{i+1}, ..., T_{i+t})$ for all $B \in \mathbb{S}_T$ and $\gamma_B(T_1, ..., T_t) = \gamma_{B_{i+1}}(T_{i+1}, ..., T_{i+t})$ for all $B \in H_T^0$, $i \in \mathbb{Z}$.*

7.2. Conjugation by $\mathrm{SL}(2,\mathbb{Z})$ and reduced cycles

In this subsection, we introduce subcycles of these cycles, that we call **reduced cycles**. These cycles allow us to consider fewer matrices in each conjugacy class (see figure 1). Moreover, we will see that they are easy to determine for a given matrix $B \in H_T^0$ (see Proposition 7.2), and that this notion fits well with our goal to describe $\mathcal{S}_T$ (see Section 7.3).

For $B = [[a,b][c,d]] \in H_T^0$ and $n \in \mathbb{N}_{>0}$, a computation gives :

$$N^n B N^{-n} = \begin{pmatrix} a+nc & b+n(d-a)-n^2c \\ c & d-nc \end{pmatrix}, \quad S^n B S^{-n} = \begin{pmatrix} a-nb & b \\ c+n(a-d)-n^2b & d+nb \end{pmatrix}. \tag{28}$$

Proposition 7.1. * *For all $B \in \mathbb{S}_T$, there exist $B_i = [[a_i,b_i][c_i,d_i]]$ and $B_j = [[a_j,b_j][c_j,d_j]]$ in the cycle $\gamma_B(B_1,T_1,...,T_t)$ such that $a_i = \max(a_i,b_i,c_i,d_i)$ and $d_j = \max(a_j,b_j,c_j,d_j)$.*

A matrix $B \in H_T^0$ satisfying the conditions of Proposition 7.1 will be said to be *reduced*. Notice that the definition that we use here of a reduced matrix is not the same as the one usually given, for example the one of [5] (see Annexe 3).

Definition 7.2. *A reduced cycle of length $t > 1$ is a cyclic sequence of distinct matrices $[B_1,B_2,...,B_t]$ such that $B_i = T_{i-1}B_{i-1}T_{i-1}^{-1}$ $(i = 2,...,t)$ and $B_1 = T_tB_tT_t^{-1}$, where each of the matrices $B_i = [[a,b][c,d]]$ satisfies $a_i = \max(a_i,b_i,c_i,d_i)$ or $d_i = \max(a_i,b_i,c_i,d_i)$, and each of the matrices T_i is of the form N^{q_i} or S^{q_i}, such that :*

$$T_{i-1} \text{ is a power of } N \Longleftrightarrow T_i \text{ is a power of } S \Longleftrightarrow T_{i+1} \text{ is a power of } N$$

(whith $i+1$ identified with 1 if $i = n$ and and $i-1$ identified with n if $i = 1$, according to Remark 7.1). The matrices T_i will be called ***operators associated to the cycle****.*

Next Theorem 7.1 shows the existence and uniqueness of a reduced cycle of H_T^0 in each conjugacy class by $\mathrm{SL}(2,\mathbb{Z})$. It requires the following technical Lemma 7.1.

Lemma 7.1. * *Let $B \in H_T^0$. $B_i \in \gamma_B(T_1,...,T_t)$ satisfies $B_{i+1} = SB_iS^{-1}$ and $B_i = NB_{i-1}N^{-1}$ if, and only if, $a_i = \max(a_i,b_i,c_i,d_i)$. Similarly, $B_i \in \gamma_B(T_1,...,T_t)$ satisfies $B_{i+1} = NB_iN^{-1}$ and $B_i = SB_{i-1}S^{-1}$ if, and only if, $d_i = \max(a_i,b_i,c_i,d_i)$.*

Theorem 7.1. * *For all $B \in \mathbb{S}_T$, the orbit of B contains exactly one reduced cycle in H_T^0, denoted $\gamma_B^R(B_1',T_1',...,T_n')$. Moreover, this reduced cycle is a subcycle of $\gamma_B(B_1,T_1,...,T_t)$.*

For $B \in H_T^0$, we will denote this cycle by $\gamma_B^R(B_1,T_1,...,T_n) = [B_1,...,B_t]$ (notice that B is not necessarily in its reduced cycle), and sometimes only γ_B^R. If B is a reduced matrix, then it is in its reduced cycle. Thus in this case we will also denote it $\gamma_B(T_1,...,T_t)$. It follows from the uniqueness of the reduced cycle and the fact that it is a subcycle of $\gamma_B(B_1,T_1,...,T_t)$, that the reduced cycle associated to a matrix of H_T^0 is an invariant of the orbit $\omega(B)$ of B.

We see with the proof of the Proof of Theorem 7.1 that the construction of the reduced cycle associated to a matrix B actually consits in extracting all the reduced matrices of its cycle. In figure 1, we give the example of a cycle in H_8^0. One can see that the reduced cycle contained in it consists in fewer matrices, and this justifies the introduction of the notion of reduced cycles.

For a matrix $B \in H_T^0$ and $\gamma_B^R(B_1,T_1,...,T_t) = \gamma_{B_1}^R(T_1,...,T_t)$, the unique reduced cycle associated to B, we set $(q_1,...,q_t)_B$ the cyclic sequence defined by $T_i = S^{q_i}$ or N^{q_i}. This sequence will be called **sequence of powers** associated to B.

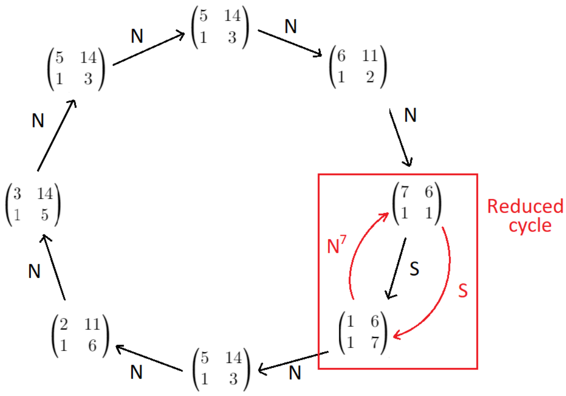

Figure 1: An example of a cycle in H_8^0 and of its reduced cycle. The rows represent the action of conjugation (by either S or N).

Although Theorem 7.1 gives an algorithm to determine the reduced cycle associated to a matrix $B \in H_T^0$, it is not very efficient because we need to compute the cycle γ_B to determine γ_B^R. Proposition 7.2 gives a more efficient algorithm (that we will develop in Section 7.4), where we directly compute the sequence of powers without having to compute γ_B:

Proposition 7.2. *Let $B \in \mathbb{S}_T$. Let $\gamma_B^R(B_1, T_1, ..., T_t)$ be the unique reduced cycle and $(q_1, ..., q_t)_B$ be the sequence of powers associated to B. For $i \in \{1, ..., t\}$ (where we identify $n+1$ and 1), let (k_i, r_i) be the quotient and the rest of the Euclidian division of a_i by b_i, and let (k_i', r_i') be the quotient and the rest of the euclidian division of d_i by c_i.*
If T_i is a power of N, then $(q_i, d_{i+1}) = (k_i', r_i')$ if $c_i \neq 1$, and $(q_i, d_{i+1}) = (d_i - 1, 1)$ if $c_i = 1$.
If T_i is a power of S, then $(q_i, a_{i+1}) = (k_i, r_i)$ if $b_i \neq 1$, and $(q_i, a_{i+1}) = (a_i - 1, 1)$ if $b_i = 1$.

7.3. Description of $\mathcal{S}_T$

Let's set

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Notice that

$$\forall B \in \mathrm{SL}(2, \mathbb{Z}) \quad WB^{-1}W^{-1} = ^t B. \tag{29}$$

Let $\gamma = [B_1, ..., B_t]$ be a reduced cycle contained in H_T^0, and $B \in \mathbb{S}_T, T_1, ..., T_t \in \{N^p, S^q \mid p, q \in \mathbb{Z}\}$ such that $\gamma = \gamma_B^R(B_1, T_1, ..., T_t)$. We define $\forall i \in \{1, ..., t\}$:

- $B_i' = UB_iU^{-1}$ and $T_i' = UT_iU^{-1}$.
- $B_i'' = WB_{t-i+1}^{-1}W_i^{-1} = ^t B_{t-i+1}$ and $T_i'' = W_iT_{t-i+1}^{-1}W_i^{-1} = ^t T_{t-i+1}$.

Then we get two cyclic sequences: $[B_1', ..., B_t']$, denoted $\bar{\gamma}$, and $[B_1'', B_t'', ..., B_2''] = [^tB_t, ..., ^tB_1]$, denoted $^t\gamma$.

Proposition 7.3. *The cyclic sequences $\bar{\gamma}$ and $^t\gamma$ are both reduced cycles contained in H_T^0. Moreover, the operators associated to these cycles are respectively the T_i', $i \in \{1, ..., t\}$ and the T_i'', $i \in \{1, ..., t\}$, and we have*

$$\bar{\gamma} = \gamma_{UBU^{-1}}^R(B_1', T_1', ..., T_t') \quad \textit{and} \quad ^t\gamma = \gamma_{B^{-1}}^R(B_1'', T_1'', ..., T_t'').$$

Then both conjugation by U and the inversion composed by conjugation by W map reduced cycles of H_T^0 on reduced cycles of H_T^0. For γ being a reduced cycle, we will say that $\bar{\gamma}$ is the conjugate reduced cycle of γ and that ${}^t\gamma$ is its transposed reduced cycle. One can notice the following straightforward properties:

$$\overline{{}^t\gamma} ={}^t \overline{\gamma}, \quad {}^t({}^t\gamma) = \gamma \quad \text{and} \quad \overline{\overline{\gamma}} = \gamma.$$

Lemma 7.2. * *Let $B, B' \in \mathbb{S}_T$. $B \sim B'$ if, and only if, one of the following equalities is true:*

- $\gamma_B^R = \gamma_{B'}^R$, *in which case B and B' are conjugate in* $\mathrm{SL}(2,\mathbb{Z})$;
- $\gamma_B^R ={}^t \gamma_{B'}^R$, *in which case there exists $M \in \mathrm{SL}(2,\mathbb{Z})$ such that $B' = MB^{-1}M^{-1}$;*
- $\gamma_B^R = \overline{\gamma_{B'}^R}$, *in which case there exists $M \in \mathrm{SL}(2,\mathbb{Z})^-$ such that $B' = MBM^{-1}$;*
- $\gamma_B^R ={}^t \overline{\gamma_{B'}^R} = \overline{{}^t\gamma_{B'}^R}$, *in which case $M \in \mathrm{SL}(2,\mathbb{Z})^-$ such that $B' = MB^{-1}M^{-1}$.*

Then we get the following Proposition:

Proposition 7.4. * *For each matrix $B \in \mathbb{S}_T$, the set of all reduced cycles contained in the class of B in $\mathcal{S}_T$ is exactly $\left\{\gamma_B^R, \overline{\gamma_B^R}, {}^t\gamma_B^R, \overline{{}^t\gamma_B^R}\right\}$. More precisely, the class of B in $\mathcal{S}_T$ contains exactly*

- *One reduced cycle (which is exactly γ_B^R) if, and only if, B is conjugate to B^{-1} and B by an element of* $\mathrm{GL}(2,\mathbb{Z})\setminus\mathrm{SL}(2,\mathbb{Z})$ *if, and only if, the class of B in $\mathcal{S}_T$ is equal to its conjugacy class in* $\mathrm{SL}(2,\mathbb{Z})$*;*
- *Four reduced cycles (which are exactly $\gamma_B^R, \overline{\gamma_B^R}, {}^t\gamma_B^R$ and $\overline{{}^t\gamma_B^R}$) if, and only if, B is not reversible and not symmetrisable;*
- *Two reduced cycles otherwise.*

Moreover, since $\mathbb{S}_T$ has a finite number of conjugacy classes by $\mathrm{SL}(2,\mathbb{Z})$ - this number is studied in [10] -, there is a finite number of reduced cycles in H_T^0.

Let $n_{rc}(T)$ be the number of conjugacy classes by $\mathrm{SL}(2,\mathbb{Z})$ of $\mathbb{S}_T$ (which is also the number of reduced cycles in H_T^0). Let C_1 denote the set of all distinct $\gamma_0, ..., \gamma_{n_1(T)-1}$ being the reduced cycles in H_T^0 that are equal to their conjugate and their transpose cycles, and $n_1(T)$ its cardinal (i.e the number of conjugacy classes that are equal to an equivalence class of $\mathcal{S}_T$). According to Proposition 7.4, the number of reduced cycles in H_T^0 that are different from their conjugate cycle and to their reduced cycle is a multiple of 4, and one can choose $\gamma'_0, ..., \gamma'_{n_2(T)-1}$ $(n_2(T) > 0)$ such that the set of reduced cycles in H_T^0 that are not equal to their conjugate cycle and to their reduced cycle is set C_2 of all distinct $\gamma'_0, {}^t\gamma'_0, \overline{\gamma'_0}, {}^t(\overline{\gamma'_0}), ..., \gamma'_{n_2(T)-1}, {}^t\gamma'_{n_2(T)-1}, \overline{\gamma'_{n_2(T)-1}}, {}^t(\overline{\gamma'_{n_2(T)-1}})$. Let $n_3(T) = \frac{n_{rc}(T)-n_1(T)-4n_2(T)}{2}$. Then $2n_3(T) = n_{rc}(T) - n_1(T) - 4n_2(T)$ is the number of reduced cycles $\gamma \in H_T^0$ such that $\{\gamma, {}^t\gamma, \overline{\gamma}, {}^t\overline{\gamma}\}$ is of cardinal 2, and thus $2n_3(T)$ is even and $n_3(T) \in \mathbb{N}$. Then one can choose $\gamma''_0, ..., \gamma''_{n_3(T)-1} \in H_T^0$ such that the set of reduced cycles of H_T^0 that are either equal to their transpose but not to their conjugate (case (1)) or equal to their conjugate but not to their transpose (case (2)) is the set C_3 of all distinct $\gamma''_i, \tilde{\gamma}_i''$, $i \in \{0, ..., n_3(T) - 1\}$, where $\tilde{\gamma}''_i = \overline{\gamma''_i}$ in case (1) and ${}^t\gamma''_i$ in case (2).

Thus the set of all the reduced cycles in H_T^0 (which represent the conjugacy classes of $\mathbb{S}_T$ in $\mathrm{SL}(2,\mathbb{Z})$) is $R_T = C_1 \cup C_2 \cup C_3$, and the set of all reduced cycles in H_T^0 up to conjugation and transposition (which represent the equivalence classes of $\mathcal{S}_T$) is

$$R_T^c = \left\{\gamma_0, ..., \gamma_{n_1(T)-1}, \gamma'_0, ..., \gamma'_{n_2(T)-1}, \gamma''_0, ..., \gamma''_{n_3(T)-1}\right\}.$$

One can notice that we get that the number of equivalence classes of $\mathbb{S}_T$ is

$$n_{eq}(T) = n_1(T) + n_2(T) + n_3(T).$$

Let's give the additional following result, which gives a criterion for a matrix of H_T^0 to be reversible/symmetrizable:

Corollary 7.1. * *Let γ be a reduced cycle in H_T^0, and let $B = [[a,b][c,d]] \in \gamma$. Then :*

- $\gamma = {}^t\gamma$ *if, and only if,* ${}^tB \in \gamma$.
- $\gamma = \overline{\gamma}$ *if, and only if,* $UBU^{-1} = [[d,c][b,a]] \in \gamma$.

Remark 7.2. *Corollary 7.1 implies that $B = [[a,b][c,d]] \in \gamma$ is reversible if, and only if, ${}^tB = [[a,c][b,d]] \in \gamma$ or $[[d,b][c,a]] \in \gamma$.*

7.4. A systematic method to describe $\mathcal{S}_T$

The previous study gives us a method to determine the equivalence classes of $\mathbb{S}_T$ for a given T. Let $T > 2$. We know that each reduced cycle of H_T^0 contains at least one reduced matrix $B_1 = [[a_1,b_1][c_1,d_1]]$ (for example we can take B_1 such that $a_1 = \max(a_1,b_1,c_1,d_1)$).

Let's construct $\gamma_{B_1}^R(T_1,...,T_t) = [B_1,...,B_t]$. Suppose that $B_1,...,B_n$ are built. From Lemma 7.1, one can determine if T_n is a power of N or S (wether if $\max(a_n,b_n,c_n,d_n)$ is a_n or b_n), and one can also determine this power q_n from Proposition 7.2. Thus we get $B_{n+1} = T_nB_nT_n$ with $T_n = N^{q_n}$ or S^{q_n}. This construction stops when we get $B_n = B_1$, and we get $\gamma_{B_1}^R$. Then we directly have ${}^t\gamma_{B_1}^R$ (by transposition) and $\overline{\gamma_{B_1}^R}$ (by conjugation). Then, in particular, Remark 7.2 gives if B_1 is reversible or not.

Now suppose that we have determined k reduced cycles in H_T^0 and their conjugate/transpose reduced cycles, for $k \geq 1$. Then one can seak if there exists $B = [[a,b][c,d]] \in H_T^0$ satisfying $a = \max(a,b,c,d)$ which does not lie in any of these reduced cycles, or their transpose/conjugate reduced cycles (equivalently, $B, {}^tB$ or UBU^{-1} does not lie in one of the reduced cycles we have found). If it does not, the construction stops. If it does, we construct the cycle of B as we did in the previous paragraph. This new cycle is disjoint from the k other reduced cycles we have already found and their conjugate/transposed cycles, because of the definition of B. The construction stops at some point because there exists only a finite number of matrices $B = [[a,b][c,d]] \in H_T^0$ satisfying $a = \max(a,b,c,d)$, more precisely there are at most

$$\sum_{a=1}^{\lfloor \frac{T}{2} \rfloor} N(a(T-a)-1)$$

such matrices, where $N(d)$ is the number of positive integer dividing d (because and $2a \geq a+d = T$ and $a = T-d \leq T-1$, and for a fixed a we need to have $bd = a(T-a)-1$).

With this method, we get exactly one reduced cycle for each equivalence class of matrices of trace T. Since the reduced cycles contain a finite number of matrices, one can choose a representative for each equivalence class in the reduced cycle associated.

Notice that we find a finite number of equivalence classes for $\sim$, which fits with the typical result saying that for a given trace $T > 2$, there is a finite number of conjugacy classes in $\mathbb{S}_T$.

Some examples of computation of $\mathcal{S}_T$ for small T are given in the Annexe C.1.

8. Reversing symmetry groups

In this section we give a more detailed description of the reversing symmetry group of a matrix $B \in \mathbb{S}_T$.

First notice that if $B = TB'^{\mu}T^{-1}$ with $T \in \mathrm{GL}(2,\mathbb{Z})$ and $\mu \in \{-1,1\}$, then $\mathcal{R}(B) = T\mathcal{R}(B')T^{-1}$, $C(B) = TC(B')T^{-1}$ and M is a reversing symmetry of B if, and only if, TMT^{-1} is a reversing symmetry of B'. This gives in particular that the property of being reversible (resp. symmetrisable) and the structure of the reversing symmetry groups are both invariant of equivalence classes in $\mathrm{GL}(2,\mathbb{Z})$.

8.1. The symmetry group of a matrix of $\mathbb{S}_T$

Let $T > 2$. The eigenvalues of each matrix of $\mathbb{S}_T$ are equal to λ_T, $\frac{1}{\lambda_T}$, with

$$\lambda_T = \frac{T + \sqrt{T^2 - 4}}{2} > 1.$$

Let $B \in \mathbb{S}_T$. A typical result gives that $E \in \mathrm{GL}(2,\mathbb{Z})$ satisfies $BE = EB$ if, and only if, $E = P(B)$ with $P \in \mathbb{Q}[X]$ and $\det(P(B)) = \pm 1$. Moreover, it is known that $\mathbb{Q}[B] \simeq \mathbb{Q}(\lambda_T)$ via $\phi_B : P(B) \mapsto P(\lambda_T)$ (because λ_T is an eigenvalue of B). If $P(B) = x + yB \in C(B)$, then $\phi_B(P) = P(\lambda_T) = \mu$ is an eigenvalue of P and is in $\mathbb{Q}(\lambda_T)$. Moreover, since $P(B)$ has its entries in $\mathbb{Z}$ and $\det(P) = \pm 1$, then $\chi_{P(B),\mathbb{Q}}$ is in $\mathbb{Z}[X]$ and its constant coefficient is equal to ± 1. But $\chi(\mu) = 0$, and thus

$$\mu \in \mathcal{O}^{\times}_{\mathbb{Q}(\lambda_T)}, \tag{30}$$

where $\mathcal{O}_{\mathbb{Q}(\lambda_T)}$ is the ring of integers of $\mathbb{Q}(\lambda_T)$ and $\mathcal{O}^{\times}_{\mathbb{Q}(\lambda_T)}$ is the set of invertible elements of the ring $\mathcal{O}_{\mathbb{Q}(\lambda_T)}$, also called the set of units of $\mathbb{Q}(\lambda_T)$.

Lemma 8.1. * *We have*

$$\mathcal{O}^{\times}_{\mathbb{Q}(\lambda_T)} = \{\pm \tau^p \mid p \in \mathbb{Z}\} \simeq \{\pm 1\} \times \langle \tau \rangle, \tag{31}$$

where $\tau = x_1 + y_1\lambda_T \in \mathcal{O}^{\times}_{\mathbb{Q}(\lambda_T)}$ *is not of finite order.*

ϕ_B induces an isomorphism of groups between $C(B)$ and $\phi_B(C(B))$, and $\phi_B(C(B))$ is a subgroup of $\mathcal{O}^{\times}_{\mathbb{Q}(\lambda_T)}$. Moreover, $-1 \in \phi_B(C(B))$. Then Lemma 8.1 gives that $\phi_B(C(B)) \simeq \{\pm 1\} \times G$, where G is a subgroup of $\langle x_1 + y_1\lambda_T \rangle$, and thus is generated by one element gen_G. We cannot say if $C(B)$ is always isomorphic to $\mathcal{O}^{\times}_{\mathbb{Q}(\lambda_T)}$ (i.e if $g(B) = \tau$ with notations of Lemma 8.1). However there exists an unique $g(B) \in C(B)$ such that $\phi_B(g(B)) = gen_G$, and the bijection ϕ_B gives:

$$C(B) = \{\pm g(B)^p \mid p \in \mathbb{Z}\} \simeq (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}. \tag{32}$$

One can easily determine the symmetry group (i.e its commutant), or equivalently its "generator" $g(B)$, of a given matrix B of $\mathbb{S}_T$:

We first determine a fundamental unit τ of $\mathbb{Q}(\lambda_T)$. There exist methods to determine it, for example see [11]. One can also use the website

http://www.numbertheory.org/php/unit.php

that can compute the fundamental unit of each $\mathbb{Q}(\sqrt{d})$, for given $d \in \mathbb{N}_{>0}$ that is squarefree and lesser than 319086769. Then we know from (31) and the isomorphism ϕ_B that there exists an unique $P(B) = xI_2 + yB \in \mathbb{Q}(B)$ such that $\mathcal{O}^{\times}_{\mathbb{Q}(\lambda_T)} = \{\pm P(\lambda_T)^p \mid p \in \mathbb{Z}\}$. One can determine the polynom $P = x + yX$ by considering the equation $P(\lambda_T) = \tau$. Then one has to determine

$$p_0 = \min\{p \in \mathbb{N}_{>0} \mid P(B)^p \text{ has integer coefficients}\},$$

and the isomorphism ϕ_B gives $g(B) = P(B)^{p_0}$ and $C(B) = \{\pm P(B)^{p_0 p} \mid p \in \mathbb{Z}\}$.

Some examples of symmetry groups are given in Annexe C.2.

8.2. Reversing symmetries

Let $B \in \mathbb{S}_T$. If we know the reduced cycle associated to B, it is possible to see if B is reversible or not, according to Remark 7.2. If B is not reversible, by definition we get $\mathcal{R}(B) = C(B)$, which we have already determined in Section 8.1. If B is reversible, then there exists $K \in \mathrm{GL}(2,\mathbb{Z})$ such that $KB = B^{-1}K$. If we set $K = [[a,b][c,d]]$ and $B = [[b_1,b_2][b_3,b_4]]$, we have

$$KB = B^{-1}K \Longleftrightarrow [a = -d \quad \text{and} \quad a(b_4 - b_1) = bb_3 + cb_2].$$

Then:

- If $b_4 = b_1$, the matrix $K = [[1,0][0,-1]]$ is a reversing symmetry of B.
- If $b_4 \neq b_1$, then $(b_4 - b_1)$ divides $bb_3 + cb_2$, and K is a reversing symmetry of B if, and only if, there exist $b, c \in \mathbb{Z}$ such that

$$K = \begin{pmatrix} \frac{bb_3+cb_2}{b_4-b_1} & b \\ c & \frac{bb_3+cb_2}{b_1-b_4} \end{pmatrix} \text{ with } \det(K) = -\left(\frac{bb_3+cb_2}{b_4-b_1}\right)^2 - bc = \pm 1. \qquad (33)$$

  This condition is equivalent to the Pell-Fermat equation

$$b^2b_3^2 + c^2b_2^2 + \left(2b_1b_2 - (b_1 - b_4)^2\right)bc = \pm(b_4 - b_1)^2$$

  in b and c, which has a solution because B is reversible. This is possible to find a solution to this equation, for example by using continued fractions. One can also use the website
  https://www.alpertron.com.ar/QUAD.HTM
  that computes the integer solutions of an equation of this form given explicitly. Once we have determined a solution of this equation, formula (33) gives us a particular solution K of $KB = B^{-1}K$.

But we know that if K is a reversing symmetry of B, the set of all reversing symmetries of B is $C(B).K$ (see [9], pp). Since we have determined $C(B)$, we have all the reversing symmetries of B. Then if M is reversible,

$$\mathcal{R}(B) = C(B) \cup C(B).K. \qquad (34)$$

Thus $\mathcal{R}(B)$ is generated by $g(B)$ and $-I_2$, together with some matrix K in the case where B is reversible. Thus the results of Sections 8.1 and 8.2 are very useful, because if one wants to study the set $\left(\mathbb{Z}^2/L_1(B,r)\right)/\mathcal{R}(B) = \left(\mathbb{Z}^2/L_1(B,r)\right)/\mathcal{R}(B)_r$ (with notations of Section 5.5), one just has to consider the action of 2 (if B is not reversible) or 3 (if B is reversible) matrices on $\mathbb{Z}^2/L_1(B,r)$, which makes at most 3 computations.

9. Isomorphism classes of $\mathbb{L}_{\mathbb{S}_T}$

In this section we give a method to determine all the isomorphism classes of $\mathbb{L}_{\mathbb{S}_T}$ for a fixed trace $T > 2$.

1. **Determine one representative for each equivalence class of $\mathbb{S}_T$.** A method is given in subsection 7.4, that gives exactly one reduced cycle for each equivalence class of matrices of trace T. Since the reduced cycles contain a finite number of matrices, one can choose a representative for each equivalence class in the reduced cycle associated. Let us denote C the **finite** (see the last paragraph of Subsection 7.4) system of reprensentatives of $\mathcal{S}_T$ chosen.

2. **Determine the reversing symmetry group of each representative.**
   a) **Determine the commutant of each representative.** A method is given in subsection 8.1.
   b) **Determine a reversing symmetry of each representative.** We know if each $B \in C$ is reversible or not, since we have computed their reduced cycle (see Remark 7.2). Then we can compute $\mathcal{R}(B)$, using the result of Step 2a and the method given in Part 8.2.

3. **Determine the set $(\mathbb{Z}^2/L_1(B,r))/\mathcal{R}(B)_r$, for each $B \in C$ and $r \in \mathbb{N}_{>0}$.** From part 8.2 we know that we just need to compute $L_1(B,r)$ and to consider the action of at most 3 elements of $\mathcal{R}(B)_r$.

4. **Conclusion.** Since we have done steps 2 and 3 for the representative of each element of $\mathcal{S}_T$, we have described all $\mathcal{L}_B^r$ for $B \in C, r \in \mathbb{N}_{>0}$. But (17) gives the explicit bijections

$$\mathcal{L}_{\mathbb{S}_T} = \bigsqcup_{r \in \mathbb{N}^*} \bigsqcup_{\mathcal{C} \in \mathcal{S}_T} \mathcal{L}_{\mathcal{C}}^r \overset{\Psi_T}{\simeq} \bigsqcup_{r \in \mathbb{N}^*} \bigsqcup_{B \in C} (\mathbb{Z}^2/L_1(B,r))/\mathcal{R}(B)_r, \tag{35}$$

where Ψ_T is defined by $\Psi_T = \Psi_{B,r}$ on each $\mathcal{L}_{\mathcal{C}}^r$, where $B \in C$ is the representative of $\mathcal{C}$ in $\mathcal{S}_T$ we have determined. Since we have described a set $E(B,r)$ of representatives of $(\mathbb{Z}^2/L_1(B,r))/\mathcal{R}(B)$ for each $B \in C$ and $r \in \mathbb{N}_{>0}$, we get an explicit bijection:

$$\mathcal{L}_{\mathbb{S}_T} \simeq \bigsqcup_{r \in \mathbb{N}^*} \bigsqcup_{B \in C} E(B,r).$$

In Annexe C (C.1, C.2, C.3 and C.4), we use this method to describe the set $\mathcal{L}_{\mathbb{S}_T}$ for traces from 1 up to 6. The results we find are summarized in the following table :

<table>
<tr><td>Trace</td><td>3</td><td colspan="2">4</td><td colspan="2">5</td><td colspan="5">6</td></tr>
<tr><td>$n_{rc}(T)$</td><td>1</td><td colspan="2">2</td><td colspan="2">2</td><td colspan="5">6</td></tr>
<tr><td>$n_{eq}(T)$</td><td>1</td><td colspan="2">1</td><td colspan="2">1</td><td colspan="5">2</td></tr>
<tr><td>Representatives</td><td>M_3</td><td colspan="2">M_4</td><td colspan="2">M_5</td><td colspan="3">M_6</td><td colspan="2">M'_6</td></tr>
<tr><td>Reversible</td><td>yes</td><td colspan="2">yes</td><td colspan="2">yes</td><td colspan="3">yes</td><td colspan="2">yes</td></tr>
<tr><td>Symmetrizable</td><td>yes</td><td colspan="2">no</td><td colspan="2">no</td><td colspan="3">no</td><td colspan="2">no</td></tr>
<tr><td>$g(M)$</td><td>$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$</td><td colspan="2">$M_4$</td><td colspan="2">$M_5$</td><td colspan="3">$M_6$</td><td colspan="2">$\begin{pmatrix}2 & 1\\ 1 & 0\end{pmatrix}$</td></tr>
<tr><td>Rev. symmetry</td><td>$\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}$</td><td colspan="2">$\begin{pmatrix}-1 & 3\\ 0 & 1\end{pmatrix}$</td><td colspan="2">$\begin{pmatrix}-1 & 5\\ -1 & 4\end{pmatrix}$</td><td colspan="3">$\begin{pmatrix}-1 & 4\\ 0 & 1\end{pmatrix}$</td><td colspan="2">$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$</td></tr>
<tr><td rowspan="2">$E(M,r)$</td><td rowspan="2">$(0,0)$</td><td>$2\nmid r$</td><td>$2|r$</td><td>$3\nmid r$</td><td>$3|r$</td><td>$4|r$</td><td>$r\equiv\pm1[4]$</td><td>$r\equiv 2[4]$</td><td>$2|r$</td><td>$2\nmid r$</td></tr>
<tr><td>$(0,0)$</td><td>$(0,0)$, $(0,1)$</td><td>$(0,0)$</td><td>$(0,0)$, $(0,1)$</td><td>$(0,0)$, $(0,1)$, $(0,2)$</td><td>$(0,0)$</td><td>$(0,0)$, $(0,1)$</td><td>$(0,0)$, $(0,1)$, $(1,1)$</td><td>$(0,0)$</td></tr>
</table>

One can notice that for small traces, we do not get many equivalence classes in $\mathcal{L}_{\mathbb{S}_T}$. For the last step, we only used the method of the study of $(\mathbb{Z}^2/L_1(B,r))/\mathcal{R}(B)_r$ we gave in Section 9 for trace 6, because for traces from 3 up to 5 there were faster methods. However the advantage of the method of Section 9 is that we know that we will always find a system of representatives $E(B,r)$ of $\left(\mathbb{Z}^2/L_1(B,r)\right)/\mathcal{R}(B)_r$ in a finite number of computations (see the last paragraph of Section 8.2).

10. Commensurability

In this section we develop another classification of $\mathbb{L}$, the classification up to commensurability.

Definition 10.1. *Two abstract groups G and G' are said commensurable if there exist two subgroups of finite index $H \subset G$ and $H' \subset G'$, and an isomorphism of groups between H and H'.*

Definition 10.2. *Two lattices $L_1, L_2 \in \mathbb{L}$ are said to be abstract commensurable if they are commensurable as abstract groups. They are said to be commensurable as lattices if this isomorphism is the restriction of an automorphism of Lie groups of* $\mathrm{Osc}_{1,0}$.

As for the equivalence and isomorphism classes, we can deduce from Theorem 5.1 that two lattices of $\mathrm{Osc}_{1,0}$ are abstract commensurable if, and only if, they are commensurable as lattices. Thus we will say that two lattices are **commensurable** if they are abstract commensurable, or equivalently, commensurable as lattices. The relation "being commensurable" is an equivalence relation. It is clear that being isomorphic implies being commensurable, which means that the commensurability classes are greater than the isomorphism classes. This equivalence relation induces an equivalence relation on $\mathcal{P}_1$, defined by:

$$(r,B,P,x,\delta,z) \sim (r',B',P',x',\delta',z') \Leftrightarrow L(r,B,P,x,\delta,z) \text{ and } L(r',B',P',x',\delta',z') \text{ are commenurable.} \quad (36)$$

We will denote $\mathcal{LC} = \mathcal{P}_1/\sim$.

Proposition 10.1. * *Let $\Gamma(r,B,k,l)$ and $\Gamma(r',B',k',l')$ be two discrete oscillator groups. Then $\Gamma(r,B,k,l)$ and $\Gamma(r',B',k',l')$ are commensurable if, and only if there exist $n,m \in \mathbb{Z}^*$ and $M \in$* $\mathrm{GL}(2,\mathbb{Q})$ *such that $B'^n = MB^mM^{-1}$.*

Let $\overset{c}{\sim}$ be an equivalence relation defined on $\mathbb{S}$ by :

$$B \overset{c}{\sim} B' \iff \exists m, n \in \mathbb{Z}^*, \exists M \in \mathrm{GL}(2,\mathbb{Q}), \quad B^m = M B'^n M^{-1}.$$

In the following, we will denote $\sim$ the relation $\overset{c}{\sim}$ (which is not the same as the relation $\sim$ of Sections from 2 up to 9).

Lemma 10.1. * *Let $B, B' \in \mathbb{S}$. Let λ, λ' be eigenvalues of B and B' respectively. $B \sim B'$ if, and only if, $\mathbb{Q}(\lambda) = \mathbb{Q}(\lambda')$.*

Let $\mathcal{C}_{\mathbb{R}}$ denote the set of all quadratic real number fields. If $K \in \mathcal{C}_{\mathbb{R}}$, the Dirichlet unit Theorem gives that $\mathcal{O}_K^\times = \{\pm\tau^p \mid p \in \mathbb{Z}\}$ (see section 5.4, equation (30)), where τ is a fundamental unit of K and is not of finite order. Then it is clear that $\mathcal{O}_K^\times$ is generated by -1 and τ. Even if it means considering τ^{-1}, $-\tau$ or $-\tau^{-1}$ instead of τ, we can assume $\tau > 1$. Let's set

$$\lambda_K = \tau^2.$$

Then $\lambda_K \notin \mathbb{Q}$ because otherwise we would have $\tau^2 \in \mathbb{Q}$, but the norm $N(\tau) = 1$ means $\tau^2 = \pm 1$, and τ is of finite order, which is not possible. Thus $\lambda_K \in \mathbb{R}\backslash\mathbb{Q}$, $K = \mathbb{Q}(\lambda_K)$ and the norm $N(\lambda_K) = 1$. Let's consider the $\mathbb{Q}$-basis $e = (1, \lambda_K)$ of K and the matrix B_K of multiplication by λ_K in e. Then B_K is of the form $[[0,a][1,b]]$ with $a, b \in \mathbb{Q}$. Since $\det(B_K) = N(\lambda_K) = 1$, we deduce that $a = -1$.

Moreover, the minimal polynomial over $\mathbb{Q}$ of λ_K (which is also its characteristical polynomial because $\lambda_K \in \mathbb{R}\backslash\mathbb{Q}$) is $\Pi_{\lambda_K} = (X - \sigma_1(\lambda_K))(X - \sigma_2(\lambda_K))$, where σ_1 and σ_2 are the two embeddings of number fields from K to $\mathbb{C}$. But $\lambda_K \in \mathcal{O}_K$ implies by definition that there exist $a_1, a_2 \in \mathbb{Z}$ such that $a_1 + a_2\lambda_K + \lambda_K^2 = 0$. If we compose this equation by σ_1 and σ_2, we get

$$a_1 + a_2\sigma_1(\lambda_K) + \sigma_1(\lambda_K)^2 = 0 \text{ and } a_1 + a_2\sigma_2(\lambda_K) + \sigma_2(\lambda_K)^2 = 0 \quad \Longrightarrow \quad \sigma_1(\lambda_K), \sigma_2(\lambda_K) \in \overline{\mathbb{Z}}.$$

Since $tr(\lambda_K) \in \mathbb{Q}$, we have $tr(\lambda_K) \in \overline{\mathbb{Z}} \cap \mathbb{Q} = \mathbb{Z}$. Notice that $\sigma_1(\lambda_K) \neq \sigma_2(\lambda_K)$ because otherwise one would have $\lambda_K \in \mathbb{Q}$. B_K has same characteristical polynomial than λ_K, and thus B_K is diagonalizable with two distinct real eigenvalues, and its characteristic polynomial $X^2 - tr(B_K)X + 1$ has two roots, and this means $tr(B) > 2$. Thus

$$B_K = \begin{pmatrix} 0 & -\det(B_K) \\ 1 & tr(B_K) \end{pmatrix} \in \mathbb{S}.$$

For $K \in \mathcal{C}_{\mathbb{R}}$, the matrix B_K is uniquely defined, and λ_K is an eigenvalue of B_K (more precisely the one that is greater than 1).

Let's define $\Gamma_K = \Gamma(1, B_K, 0, 0)$ for each $K \in \mathcal{C}_{\mathbb{R}}$.

Proposition 10.2. * *Each lattice of $\mathrm{Osc}_{1,0}$ is commensurable to Γ_K for some $K \in \mathcal{C}_{\mathbb{R}}$. More precisely, for $\Gamma = \Gamma(r, B, k, l)$ being a discrete oscillator group, if λ is an eigenvalue of B, then Γ is commensurable to $\Gamma_{\mathbb{Q}(\lambda)}$. Moreover, Γ_K is not commensurable to $\Gamma_{K'}$ if $K \neq K'$.*

Let's define the map:

$$\Psi_{\mathcal{P}_1} : \begin{cases} \mathcal{P}_1 \to \mathcal{C}_{\mathbb{R}} \\ (r, B, P, x, \delta, z) \mapsto \mathbb{Q}(\lambda) \text{ where } \lambda \text{ is an eigenvalue of } B. \end{cases}$$

$\Psi_{\mathcal{P}_1}$ is well-defined because if λ is an eugenvalue of B, then the other eigenvalue of B is $\frac{1}{\lambda}$, and $\mathbb{Q}(\lambda) = \mathbb{Q}(\frac{1}{\lambda})$.

Corollary 10.1. * *The map $\Psi_{\mathcal{P}_1}$ is invariant under $\sim$, and the quotient map $\overline{\Psi_{\mathcal{P}_1}} : \mathcal{LC} \to \mathcal{C}_{\mathbb{R}}$ is a bijection.*

11. Conclusion

This internship was very enriching. While I thought I would need strong prerequisites in solvable Lie groups theory, it turned out the study required tools in diverse domains. It was an occasion for me to deepen my knowledge in algebraic number theory, in Lie groups theory and in dynamical systems, and to see how abstract results coming from very different domains can be applied to the same research work.

My task was to parametrize and classify modulo $\mathrm{Aut}(\mathrm{Osc}_{1,0})$ the set $\mathbb{L}$. The study of the equivalence classes showed that there was no bijection between $\mathcal{L}$ and a set that is simple to study (see section 6). However, writing it as the union of the $(\mathcal{L}_{\mathbb{S}_T})_{T>2}$ allowed us to focus on these sets. They raised two problems, both linked to the action of $\mathrm{GL}(2,\mathbb{Z})$ by conjugation on $\mathbb{S}_T$, $T > 2$. With these two problems, among others, I discovered the issues a researcher has to face when they are stuck on a question, and how to work with a researcher as a team. It led me to read litterature in order to find ideas of solutions to the problem, which taught me to select the main information of an article. We tried to adopt many points of view, including the one of continued fractions that we developped in Annexe E.

However, it was the idea of cycles introduced in [6] by Aicardi that fitted most with a description, not only of $\mathbb{S}_T/\,\mathrm{SL}(2,\mathbb{Z})$, but of $\mathcal{S}_T$. Then, although one cannot determine a system of representatives of $\mathcal{S}_T$ in terms of T, we found a method to determine a system of representatives of $\mathcal{S}_T$ when $T > 2$ is given explicitly. Then, by describing the set $\mathcal{R}(B)_r$ for B being a class representative in $\mathbb{L}_{\mathbb{S}_T}$, we managed to describe the set $\left(\mathbb{Z}^2/L_1(B,r)\right)/\mathcal{R}(B)_r$ and give a method to give a more detailed description of the set $\mathcal{L}_{\mathbb{S}_T}$ and thus to simplify (25) for each explicit $T > 2$. This method appears to be efficient for small traces T.

It was also interesting to study another classification of $\mathbb{L}$ in Section 10, the classification up to commensurability, because we got a complete description of the commensurability classes. Even if this classification appeared to be simpler than the one up to isomorphism, they were both related to the quadratic number fields theory. If one wants to go further, one could study classification up to inner automorphism: two lattices L and L' of $\mathrm{Osc}_{1,0}$ are said to be equal up to inner automorphism if there exist $S \in \mathrm{GL}(2,\mathbb{Z})$, $SA = \mu AS$ ($\mu \in \{-1,1\}$) and $\eta \in \mathbb{R}^2$ such that $L = F_S \circ F_\eta(L')$. Contrary to the commensurability classes, the classes for this relation are smaller than the isomorphism classes, and for now this study looks more complicated. Maybe one could use the tool of normalized lattices we developped in Annexe D, as in [4].

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A. Proofs

A.1. Proofs of Section 2

Proof of Proposition 2.1: in [7], the necessary and sufficient condition for a map α to be in $\mathcal{A}$ is that there exist $\delta \in \mathbb{R}^2$, $\lambda \in \mathbb{R}^*$, $S \in \mathrm{GL}(2,\mathbb{R})$, such that $S^*\omega = \lambda\omega$ and $\alpha = \begin{pmatrix} \lambda & {}^t\delta \\ 0 & S \end{pmatrix}$ (we have denoted $S^*\omega$ the map defined by $S^*\omega(\xi,\xi') = \omega(S\xi, S\xi'))$. But a simple computation shows that $\forall S \in \mathrm{GL}(2,\mathbb{R})$, $S^*\omega =^t S\omega S = \det(S)\omega$. Thus, for an $\alpha \in \mathcal{A}$ defined as is proposition 2.1, we have $\lambda = \det(S)$. □

Proof of Corollary 2.1: According to proposition 2.1, $\bar{F}_{P^{-1}} = \begin{pmatrix} \det(P^{-1}) & 0 \\ 0 & P^{-1} \end{pmatrix}$ is an automorphism of H. But The image $\bar{F}_{P^{-1}}(\Gamma_r)$ of a lattice Γ_r of H by an automorphism $\bar{F}_{P^{-1}}$ of H is still a lattice of H. Then $\Gamma_P^r = \bar{F}_{P^{-1}}(\Gamma_r)$ is a lattice of H. □

Proof of Proposition 2.2 : It is known that H is a connected and nilpotent Lie group. It is clear from (37) that H is a normal subgroup of $\mathrm{Osc}_{1,0}$. If there is a subgroup F of $\mathrm{Osc}_{1,0}$ satisfying these properties, then in particular, F is a normal subgroup of $\mathrm{Osc}_{1,0}$. Suppose that $F \notin H$. Then there exists $(a,b,l) \in F\backslash H$. Then $l \neq 0$. Then $\langle (a,b,l) \rangle = \{(a,b,l)^n = (*,*,nl) \mid n \in \mathbb{N}\}$ is included in L. In particular, $\{nl \mid n \in \mathbb{N}\}$ is included the projection $p(F)$ of F on $F/(F \cap H)$ (this quotient is a group because $F \cap H$ is the commutator subroup of F). But F is connected, and so is $p(F)$. Thus $p(F) = \mathbb{R}$. But since F is nilpotent, $p(F)$ should be nilpotent too, which is is not the case. Hence the contradiction. Thus $F \subset H$. This proves that H is the maximal nilpotent connected normal Lie subgroup of $\mathrm{Osc}_{1,0}$. Then, using Coroallry 3.5 of [8], $L \cap H$ is a lattice of $\mathrm{Osc}_{1,0}$. □

Proof of Proposition 2.3: It is clear that $\det(B_{P,s}) = 1$, since $\det(e^{sA}) = e^{sTr(A)} = 1$. A is diagonalizable :

$$A = T\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}T^{-1}, T = T^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Thus e^{sA} is diagonalizable and so is $B_{P,s}$. Thus its discriminant is positive, which means that $|a+d| \geq 2$. Its eigenvalues are $\lambda, \frac{1}{\lambda}$ with $\lambda = \frac{1}{2}\left(a+d+\sqrt{(a+d)^2-4}\right) \geq 1$. Then

$$B_{P,s} = KDK^{-1}, \quad D = diag\left(\lambda, \frac{1}{\lambda}\right) \in \mathrm{SO}(1,1) \text{ and } K \in \mathrm{GL}(2,\mathbb{R})$$

and by identification, $\lambda = e^s$. Then $\lambda > 0$, which is the case if, and only if $a+d > 0$. But we already have $|a+d| \geq 2$. Thus $a+d \geq 2$ and we have $s = \ln\lambda$. Since $s \neq 0$, this shows that $\lambda \neq 1$, and then $a+d > 2$. □

Proof of Propositions 2.4: Let's first notice that for all $(z,\xi,t) \in \mathrm{Osc}_{1,0}$, $(v,\eta,0) \in H$, we have

$$(z,\xi,t)(v,\eta,0)(z,\xi,t)^{-1} = \left(v + \omega(\xi, e^{tA}\eta), e^{tA}\eta, 0\right). \tag{37}$$

We will show the proposition for $r = 1$. The proof is similar for any r. Denote $L = L^1_{B,P,x}$ and $s = s_B$. Suppose that L is a lattice of $\mathrm{Osc}_{1,0}$ such that $L \cap H = \Gamma^1_P$. Applying (37) to $(z, \xi, t) = (0, P^{-1}x, s)$, $(v, \eta, 0) \in \Gamma_P$, we have :

$$(v, \eta, 0) \in \Gamma^1_P \Rightarrow \left(v + \omega\left(P^{-1}x, e^{sA}\eta\right), e^{sA}\eta, 0\right) \in \Gamma^1_P. \tag{38}$$

This is equivalent to:

$$(v', \eta', 0) \in \Gamma_1 \Rightarrow \left(v + \det(P)\omega\left(P^{-1}x, P^{-1}B\eta\right), B\eta, 0\right) \in \Gamma_1. \tag{39}$$

Applying (39) to $(0, e_1, 0) \in \Gamma_1$ (resp. $(0, e_2, 0) \in \Gamma_1$), we have that

$$\left(\det(P)\omega(P^{-1}x, P^{-1}(a, c)), (a, c), 0\right) \in \Gamma_1$$

$$(\text{resp. } \left(\det(P)\omega(P^{-1}x, P^{-1}(b, d)), (b, d), 0\right) \in \Gamma_1).$$

Since ${}^tP^{-1}\omega P^{-1} = \det(P^{-1})\omega$, this is equivalent to

$$\left(\omega\left(x, (a, c)\right), (a, c), 0\right) \in \Gamma_1$$

$$(\text{resp. } (\omega(x, (b, d)), (b, d), 0) \in \Gamma_1).$$

This is equivalent to:

$$cx_1 - ax_2 \in \frac{1}{2}ac + \mathbb{Z} \text{ and } dx_1 - bx_2 \in \frac{1}{2}bd + \mathbb{Z}, \tag{40}$$

with $x = (x_1, x_2)$.
Finally, this is equivalent to:

If ab and cd are even, then $(x_1, x_2) \in \mathbb{Z}^2$;
If ab is even and cd is odd, then $(x_1, x_2) \in \mathbb{Z} \times (\frac{1}{2} + \mathbb{Z})$;
If ab is odd and cd is even, then $(x_1, x_2) \in (\frac{1}{2} + \mathbb{Z}) \times \mathbb{Z}$.

The case "ab and cd are odd" is impossible because $B \in \mathrm{GL}(2, \mathbb{Z})$. Indeed, if this was the case, then a, b, c, d would be odd, and then ad and bc would be odd. But then we can't have $\det(B) = ad - bc = \pm 1$.

We have proven that conditions of Proposition 2.4 are necessary. Let's show that they are sufficient. Suppose that B and x satisfy the conditions of the proposition. Then, according to the equivalences above, $\left(\omega(\xi_0, P^{-1}Be_i), Be_i, 0)\right) \in \Gamma_1 \quad \forall i \in \{1, 2\}$. Since Γ_1 is generated by $(1, 0, 0)$, $(0, e_1, 0)$ and $(0, e_2, 0)$, we deduce that (39) is satisfied. Since (39) is equivalent to (38), we have :

$$(v, \eta, 0) \in \Gamma^1_P \Rightarrow (v + \omega\left(P^{-1}x, e^{sA}\eta\right), e^{sA}\eta, 0) \in \Gamma^1_P.$$

Then $\forall (v, \eta, 0) \in \Gamma^1_P$, $(0, P^{-1}\xi_0, s)(v, \eta, 0)(0, P^{-1}x, s)^{-1} = (v + \omega\left(P^{-1}x, e^{sA}\eta\right), e^{sA}\eta, 0) \in \Gamma^1_P$, and by induction, $\forall n \in \mathbb{N}$,

$$(0, \sum_{j=1}^{n} P^{-j}x, ns)(v, \eta, 0)(0, \sum_{j=1}^{n} P^{-j}x, ns)^{-1} = (v + \omega(\sum_{j=1}^{n} P^{-j}x, e^{nsA}\eta), e^{nsA}\eta, 0) \in \Gamma^1_P. \tag{41}$$

We see by a computation that $(B^{-1}, x) \in \mathcal{B}_1$. Thus (41) is actually satisfied $\forall n \in \mathbb{Z}$. But $L \cap H$ is generated by elements of the form $(v + \omega(\sum_{j=1}^{n} P^{-j}x, e^{nsA}\eta), e^{nsA}\eta, 0)$. Thus $L \cap H = \Gamma^1_P$. $\square$

A.2. Proofs of Section 3

Proof of Theorem 3.1: Let's first show that for $u \in \mathbb{R}$, $\eta \in \mathbb{R}^2$ and $S \in \mathrm{GL}(2, \mathbb{R})$ such that $SA = \mu AS$ ($\mu \in \{-1, 1\}$), F_u, F_η and F_S are automorphisms of $\mathrm{Osc}_{1,0}$. It is clear that these three applications are invertible, with $F_\eta^{-1}(z, \xi, t) = F_{-\eta}$, $F_u^{-1}(z, \xi, t) = F_{-u}$ and $F_S^{-1}(z, \xi, t) = F_{S^{-1}}$.

It is clear that F_η is an atomorphism of $\mathrm{Osc}_{1,0}$ (it is an inner automorphism). Let's show that F_S is a morphism of $\mathrm{Osc}_{1,0}$. Let $(z, \xi, t), (z', \xi', t') \in \mathrm{Osc}_{1,0}$.

$$\begin{aligned}
F_S((z,\xi,t)(z',\xi',t')) &= F_S\left(z + z' + \frac{1}{2}\omega(\xi, e^{tA}\xi'), \xi + e^{tA}\xi', t + t'\right) \\
&= \left(det(S)(z + z' + \frac{1}{2}\omega(\xi, e^{tA}\xi')), S(\xi + e^{tA}\xi'), \mu_S(t + t')\right) \\
&= \left(det(S)z + det(S)z' + \frac{1}{2}\omega(S\xi, Se^{tA}\xi'), S\xi + Se^{tA}\xi', \mu t + \mu_S t'\right) \\
&= \left(det(S)z + det(S)z' + \frac{1}{2}\omega(S\xi, e^{\mu_S tA}S\xi'), S\xi + e^{\mu_S tA}S\xi', \mu t + \mu t'\right) \\
&= (det(S)z, S\xi, \mu t)\left(det(S)z', S\xi', \mu t'\right) = F_S(z, \xi, \mu_S \mu t)F_S(z', \xi', \mu t').
\end{aligned}$$

Thus F_S is a morphism. A computation gives that F_u is also morphisms of $\mathrm{Osc}_{1,0}$. Thus F_S, F_η and F_u are automorphisms of $\mathrm{Osc}_{1,0}$. Then by composition, all maps of the form of Proposition 3.1 are automorphisms of $\mathrm{Osc}_{1,0}$.

Let F be an automorphism of $\mathrm{Osc}_{1,0}$. Let's first determine the application f defined by $f(t) = F(0, 0, t) = (\alpha(t), \beta(t), \gamma(t))$. Notice that f is a morphism from $\mathbb{R}$ to $\mathrm{Osc}_{1,0}$. $\forall s, t \in \mathbb{R}$,

$$f(s + t) = (\alpha(s + t), \beta(s + t), \gamma(s + t)).$$

Besides,

$$\begin{aligned}
f(s + t) = f(s).f(t) &= (\alpha(s), \beta(s), \gamma(s))(\alpha(t), \beta(t), \gamma(t)) \\
&= \left(\alpha(s) + \alpha(t) + \frac{1}{2}\omega(\beta(s), e^{\mu sA}\beta(t)), \beta(s) + e^{\mu sA}\beta(t), \gamma(s) + \gamma(t)\right).
\end{aligned}$$

Thus γ is a morphism of $\mathbb{R}$. It can't be null, because if this was the case, applying (37) to $(z, \xi, t) = (0, 0, t)$, we would have for all $(v, \eta, 0) \in H$: $F(v + \omega(\xi, e^{tA}\eta), e^{tA}\eta, 0) = 0$. This is impossible because F is injective. Thus γ is an automorphism of $\mathbb{R}$, and there exists $\mu \in \mathbb{R}^*$ such that $\gamma(t) = \mu t \quad \forall t \in \mathbb{R}$.

Besides, according to (A.2), $\beta(s + t) = \beta(s) + e^{\mu sA}\beta(t)$. Differentiating with respect to s and setting $s = 0$, we have:

$$\beta'(t) = \mu A\eta + \mu A\beta(t) \quad \forall t \in \mathbb{R},$$

with b the unique element of $\mathbb{R}^2$ such that $\beta'(0) = \mu A\eta$. Setting $t = 0$, we find the initial condition $\beta(0) = 0$. The solution of this differential equation is

$$\beta(t) = e^{\mu tA}\eta - \eta \quad \forall t \in \mathbb{R}.$$

Set $F_1 = F_{-\eta} \circ F$. Then $F_1(0, 0, t) = (\alpha_1(t), 0, \mu t)$. But F_1 is an automorphism of $\mathrm{Osc}_{1,0}$, according to the beginning of the Proof. Thus $F_1(0, 0, t + s) = (\alpha_1(t), 0, \mu t)(\alpha_1(s), 0, \mu s) = (\alpha_1(t) + \alpha_1(s), 0, \mu(t + s))$. By identification, $\alpha_1(t + s) = \alpha_1(t) + \alpha_1(s)$ and there exists an $m \in \mathbb{R}$ such that $\alpha_1(t) = mt, \forall t \in \mathbb{R}$. Thus $\forall (z, \xi, t) \in \mathrm{Osc}_{1,0}$,

$$F_1(z, \xi, t) = (mz, 0, \mu t).$$

Let $F_2 = F_{\frac{-m}{\mu}} \circ F_1$. For all $t \in \mathbb{R}$,

$$F_2(0,0,t) = F_{\frac{-m}{\mu}}(0,0,\mu t) = (0,0,\mu t) \tag{42}$$

Moreover, since H is the commutator subgroup of $\mathrm{Osc}_{1,0}$, F_2 maps H onto itself. Therefore, F_2 induces an automorphism of H. According to Proposition 2.1, there exists a $\delta \in \mathbb{R}^2$, a matrix $S \in \mathrm{GL}(2,\mathbb{R})$ such that $\forall (z,b,0) \in H$,

$$F_2(z,b,0) = \left(\det(S)z +^t \delta b, Sb, 0\right). \tag{43}$$

Since F_2 is a morphsim , we have : $F_2(0,0,t)F_2(z,\eta,0)F_2(0,0,t)^{-1} = F_2(z, e^{tA}b, 0)$. According to (42) and (43), this implies:

$$(0,0,\mu t)\left(\det(S)z +^t \delta\eta, Sb, 0\right)(0,0,\mu t) = \left(\det(S)z +^t \delta e^{tA}b, Se^{tA}b, 0\right)$$

$$\Longleftrightarrow$$

$$(\det(S)z +^t \delta b, e^{\mu tA}Sb, 0) = (\det(S)z +^t \delta e^{tA}b, Se^{tA}b, 0).$$

This is true $\forall (z,b,0) \in H, \forall t \in \mathbb{R}$. Hence $\delta = 0$ and $e^{\mu tA}S = Se^{tA}$ $\forall t$. Differentiating this last formula and setting $t = 0$ gives $\mu AS = SA$. Then

$$F_2(z,b,t) = F_2((z,b,0)(0,0,t)) = (\det(S)z, Sb, \mu t)$$

and $F_2 = F_S$. Thus, $F_{\frac{-m}{\mu}} \circ F_{-\eta} \circ F = F_S$, and $F = F_\eta \circ F_u \circ F_S$ with $SA = \mu AS$ and $u = \frac{m}{\mu}$. □

A.3. Proofs of Section 4

Proof of Proposition 4.1: We have $F_{-P^{-1}\delta}\left(\det(P)^{-1}z, P^{-1}x, s\right) = \left(a(\det(P)^{-1}z, P^{-1}x, s), b(\det(P)^{-1}z, P^{-1}x, s), s\right)$. Let's set

$$u = -\frac{\det(P)^{-1}z}{s}, F = F_u \circ F_{-P^{-1}\delta}.$$

Then according to Theorem 3.1, F is an automorphism of $\mathrm{Osc}_{1,0}$ and $F(\bar{F}_\eta(\Gamma^r_P)) = \Gamma^r_P$ and $F \circ F_{P^{-1}\delta}\left(\det(P)^{-1}z, P^{-1}x, s\right) = (0, P^{-1}x, s)$. Thus L is mapped to $\langle \Gamma^r_P, (0,x,s)\rangle = L^r_{B,P,x}$ via an automorphism of $\mathrm{Osc}_{1,0}$. Then L is a lattice of $\mathrm{Osc}_{1,0}$ such that $L \cap H = F_{P^{-1}\delta}(\Gamma^r_P)$ if, and only if, $L^r_{B,P,x}$ is a lattice of $\mathrm{Osc}_{1,0}$ such that $L^r_{B,P,x} \cap H = \Gamma^r_P$ if, and only if, $B \in \mathbb{S}$ and $(B,x) \in \mathcal{B}_r$ (Proposition 2.4). □

Proof of Lemma 4.1: Since H is the commutator subgroup of $\mathrm{Osc}_{1,0}$, α induces an automorphism on H. We have, for all $(z,\eta) \in \Gamma_r$,

$$\alpha_{\|H}(z,\eta) = (\det(S)z + \omega(\delta, S\eta), S\eta) = \bar{F}_\delta \circ \bar{F}_S(z,\eta).$$

Suppose that $\alpha_{\|H}$ maps Γ_r onto itself. Then S maps $\mathbb{Z}^2$ onto itself, and $S \in \mathrm{GL}(2,\mathbb{Z})$. Moreover, if we apply α to $(0,e_1)$ and $(0,e_2)$, we find the result on δ. Reciprocally, it is easy to see that such a map maps Γ_r onto itself. □

Poof of Proposition 4.4: We will denote $s = s_B$ and $s' = s_{B'}$. If $L^r_{B,P,x}$ and $L^r_{B',P',x'}$ are isomorphic as lattices, the automorphism induces an isomorphism between Γ^r_P and $\Gamma^{r'}_{P'}$. Thus Γ_r and $\Gamma_{r'}$ are ismomrphic via an element of $\mathcal{A}$, and thus they have same covolume in H. Thus $r = r'$.

Let F be the automorphism of $\mathrm{Osc}_{1,0}$ mapping $L^r_{B,P,x}$ onto $L^{r'}_{B',P',x'}$. Then F is of the form of Theorem 3.1. In particular, $F(0,x,s) = (*,*,\mu s) \in H \times \mathbb{Z}.s'$. This implies that $s \in \mathbb{Z}.s'$. Considering F^{-1}, we find $s' \in \mathbb{Z}.s$. Thus $s = s'$.

$F_{||H}$ induces an automorphism of H (because H is stable under autommorphism), and

$$\exists (S,\mu,\eta) \in \mathrm{GL}(2,\mathbb{Z}) \times \{\pm 1\} \times \mathbb{R}^2, \quad AS = \mu SA, \quad F_{||H} = F_{P^{-1}\eta} \circ F_{S||H} = \overline{F}_\eta \circ \overline{F}_S.$$

Thus $\Gamma_r = \bar{F}_P \circ \bar{F}_{P^{-1}\eta} \circ \bar{F}_{SP'^{-1}}(\Gamma_r) = \bar{F}_\eta \circ \bar{F}_{PSP'^{-1}}(\Gamma_r)$. Then, according to Lemma 4.1,

$$\left(PSP'^{-1}, \eta\right) \in \mathcal{B}_r.$$

In particular, $E = PSP'^{-1} \in \mathrm{GL}(2,\mathbb{Z})$ we deduce $B = Pe^{sA}P^{-1} = PSe^{\mu sA}S^{-1}P^{-1}$ because $AS = \mu SA$. Then if $\mu = 1$, $B = EB'E^{-1}$, and if $\mu = -1$, $B^{-1} = EB'E^{-1}$. □

Proof of Proposition 4.5: We will denote $s = s_B > 0$ and $s' = s_{B'} > 0$. Suppose $\tilde{B} = EB^\mu E^{-1}$, with $E \in \mathrm{GL}(2,\mathbb{Z})$ and $\mu = \pm 1$. Let's set $S = \tilde{P}^{-1}EP$. Then $Se^{sA} = e^{\mu s' A}S$, which means that $s = s'$ and $AS = \mu SA$. Thus F_S is an automorphism of $\mathrm{Osc}_{1,0}$. Now Let's set $\eta \in \mathbb{R}^2$ such that $(\tilde{P}SP^{-1}, \eta) \in \mathcal{B}_r$.

According to the proof of proposition 4.4 and to Lemma 4.1, $\tilde{F} = F_{P^{-1}\eta} \circ F_S$ is an automorphism of $\mathrm{Osc}_{1,0}$ such that $\tilde{F}(\Gamma^r_P) = \Gamma^r_{\tilde{P}}$. Thus $\tilde{F}(L) = \langle \Gamma^r_{\tilde{P}}, (a,b,c)\rangle$, with

$$(a,b,c) = F_{P^{-1}\eta} \circ F_S(0, P^{-1}x, s) = \Big(\det(P^{-1})\big[\omega(B^\mu \eta, PSP^{-1}x) + \omega(\eta, PSP^{-1}x - B^\mu\eta)\big], SP^{-1}x - e^{\mu sA}P^{-1}\eta + P^{-1}\eta, \mu s\Big).$$

Let's set

$$u = -\frac{a}{s}, \quad x' = \tilde{P}[SP^{-1}x - e^{\mu sA}P^{-1}\eta + P^{-1}\eta].$$

Then $F_u \circ \tilde{F}(L) = \langle \Gamma^r_{\tilde{P}}, (0, \tilde{P}^{-1}x', s)\rangle$ and L is isomorphic to $L_{\tilde{B},\tilde{P},x'}$. Moreover, $L^r_{\tilde{B},\tilde{P},x'}$ is a lattice of $\mathrm{Osc}_{1,0}$ such that $L^r_{\tilde{B},\tilde{P},\tilde{x}} \cap H = F(L^r_{B,P,x}) \cap H = F(L^r_{B,P,x}) \cap F(H) = F(L^r_{B,P,x} \cap H) = F(\Gamma^r_P) = \Gamma^r_{\tilde{P}}$. Thus $(\tilde{B}, x') \in \mathcal{B}_r$. □

A.4. Proofs of Section 5

Proof of Proposition 5.1: Let's define the map ϕ from $\Gamma(r,B,k,l)$ to $L^r_{B,P,x}$ by the following properties : ϕ is a morphism such that

$$\phi(\alpha) = (0, P^{-1}e_1, 0); \quad \phi(\beta) = (0, P^{-1}e_2, 0); \quad \phi(\gamma) = (\frac{\det(P^{-1})}{r}, 0, 0); \quad \phi(\delta) = (0, P^{-1}x, s).$$

Let's show that such a morphism is well defined. It is unique because it is defined on a set of generators of $\Gamma(r,B,k,l)$. The morphism $\phi_{||H}$ is clearly well defined. Thus we just have to show that $\phi(\delta\alpha\delta^{-1}) = \phi(\delta)\phi(\alpha)\phi(\delta)^{-1}$ and $\phi(\delta\beta\delta^{-1}) = \phi(\delta)\phi(\beta)\phi(\delta)^{-1}$. But

$$\phi(\delta\alpha\delta^{-1}) = (ac + k, (a,c), 0) \text{ and } \phi(\delta)\phi(\beta)\phi(\delta)^{-1} = (cx_1 - ax_2, (a,c), 0)),$$

and by definition of k these two quantities are equal. A similar comuptation gives $\phi(\delta\beta\delta^{-1}) = \phi(\delta)\phi(\beta)\phi(\delta)^{-1}$ by definition of l. Thus ϕ is a morphism. It is a bijection because it maps a minimal set of generators of $\Gamma(r, B, k, l)$ onto a set of generators of $L^r_{B,P,x}$. □

Proof of Proposition 5.2: It is straightforward, according to Proposition 5.1, Remark 5.1 and the fact that two lattices are equivalent if, and only if their underlying discrete oscillator groups are isomorphic. □

Proof of Proposition 5.3 : The isomorphism K defined by $K_{\|H} = I_3$ and $K(\delta) = \delta'\alpha$ maps $\Gamma(k, l)$ onto $\Gamma(k+r, l)$, and the isomorphism defined K defined by $K_{\|H} = I_3$ and $K(\delta) = \delta'\beta$ maps $H^r \rtimes_{S_{k,l}} \mathbb{Z}.\delta$ onto $\Gamma(k, l+r)$. Hence (i).

The isomorphism K defined by $K_{\|H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_1 & t_2 & 1 \end{pmatrix}$ and $K(\delta) = \delta'$ maps $\Gamma(k, l)$ onto $\Gamma(k', l')$. Hence (ii).

The isomorphism K defined by $K_{\|H} = \begin{pmatrix} \bar{K} & 0 \\ 0 & 1 \end{pmatrix}$ and $K(\delta) = \delta'$ maps $\Gamma(k, l)$ onto $\Gamma((k, l).\bar{K})$.

Let $\bar{K} \in \mathcal{R}(B)$. If $\bar{K}B = B\bar{K}$, the isomorphism K defined by $K_{\|H} = \begin{pmatrix} \bar{K} & 0 \\ 0 & \det(\bar{K}) \end{pmatrix}$ and $K(\delta) = \delta'$ maps $\Gamma(k, l)$ onto $\Gamma\big(\det(\bar{K}).(k, l).\bar{K} + r(D, D')\big)$, $D, D' \in \mathbb{Z}$. According to (i), $\Gamma(k, l)$ and $\Gamma(\det(\bar{K}).(k, l).\bar{K})$ are isomorphic. If $\bar{K}B = B^{-1}\bar{K}$, one can use the same arguments as in the previous case, using the map K that is equal to $\begin{pmatrix} \bar{K} & 0 \\ 0 & \det(\bar{K}) \end{pmatrix}$ on H and that maps δ on δ'^{-1}. Hence (iii). □

Proof of Theorem 5.2: Suppose that (k, l) is equivalent to (k', l'). Then there exists an isomorphism K mapping $\Gamma(k, l)$ onto $\Gamma(k', l')$. Since

$$K(\gamma) \in Z(\Gamma(k', l')) = \langle \gamma' \rangle \quad \text{and} \quad K^{-1}(\gamma') \in Z(\Gamma(k, l)) = \langle \gamma \rangle,$$

there exists $\mu \in \{\pm 1\}$ such that $K(\gamma) = \gamma'^{\mu}$. Moreover, the equality $K(\alpha\beta\alpha^{-1}\beta^{-1}) = K(\alpha)K(\beta)K(\alpha)^{-1}K(\beta)^{-1}$ gives $\mu = \det(\bar{K})$. First suppose $K(\delta) = \delta$. Then $K_{\|H}$ can be written $\begin{pmatrix} \bar{K} & & 0 \\ t_1 & t_2 & \det(\bar{K}) \end{pmatrix}$, with $\bar{K} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$. We have

$$S_{k',l'}(K(\alpha)) = \delta' K(\alpha) \delta'^{-1} = K(\delta)K(\alpha)K(\delta)^{-1} = K(\delta\alpha\delta^{-1}) = K(S_{k,l}(\alpha)).$$

By developping, we find that it implies

$$\alpha'^{at_{11}+bt_{21}} \beta'^{ct_{11}+dt_{t_{21}}} \gamma'^{t_1+k't_{11}+l't_{21}+rD} = \alpha'^{at_{11}+bt_{12}} \beta'^{at_{21}+bt_{t_{22}}} \gamma'^{t_1+k't_{11}+l't_{21}+rD'}$$

with $D, D' \in \mathbb{Z}$. This implies $at_{11} + bt_{21} = at_{11} + bt_{12}$ and $ct_{11} + dt_{t_{21}} = at_{21} + bt_{t_{22}}$. Similarly, $S_{k',l'}(K(\beta)) = K(S_{k,l}(\beta))$ gives $at_{12} + bt_{22} = bt_{11} + dt_{12}$ and $ct_{12} + dt_{t_{22}} = bt_{21} + dt_{t_{22}}$. Thus

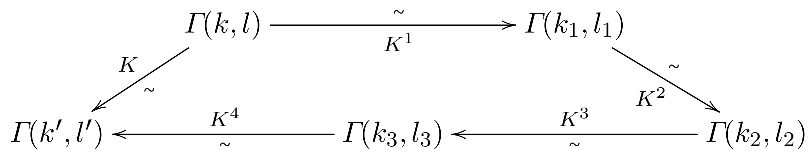
$\bar{K}B = B\bar{K}$.
Now we can write

$$K_{\|H} = \begin{pmatrix} \bar{K} & 0 \\ 0 & \det(\bar{K}) \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ t_1 \quad t_2 & 1 \end{pmatrix} = K^2_{\|H} \circ K^1_{\|H},$$

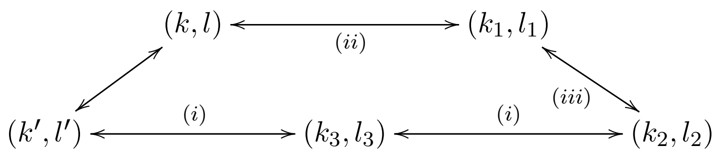
with K^2 defined by $K^2_{|H} = \begin{pmatrix} \bar{K} & 0 \\ 0 & \det(\bar{K}) \end{pmatrix}$, and $K^2(\delta) = \delta'$, and K^1 defined by $K^1_{|H} = \begin{pmatrix} I_2 & 0 \\ t_1 \quad t_2 & 1 \end{pmatrix}$, and $K^1(\delta) = \delta'$. More precisely, $K = K^4 \circ K^3 \circ K^2 \circ K^1$, with K^3 and K^4 defined by $K^3_{\|H} = K^4_{\|H} = I_3$ and $K^3(\delta) = \delta' \alpha'^{D'-D}$ and $K^4(\delta) = \delta' \beta'^{D''}$ for a suitable D''. According to the proof of Proposition 5.3:

- K^1 is an isomorphism between $\Gamma(k,l)$ and $\Gamma(k_1,l_1)$, with $(k_1,l_1) = (k,l) + (t_1,t_2).(B - I_2)$.
- Then K^2 is an isomorphism between $\Gamma(k_1,l_1)$ and $\Gamma(k_2,l_2)$, with $(k_2,l_2) = \det(\bar{K}).(k_1,l_1).\bar{K}$.
- Then K^3 is an isomorphism between $\Gamma(k_2,l_2)$ and $\Gamma(k_3,l_3)$, with $(k_3,l_3) = (k_2 + r(D'-D), l_2)$, and K^4 is an isomorphism between $\Gamma(k_3,l_3)$ and $\Gamma(k_4,l_4)$, with $(k_4,l_4) = (k_3, l_3 + rD'')$.

But then, necessarily, $(k_4,l_4) = (k',l')$ (because $K = K^4 \circ K^3 \circ K^2 \circ K^1$), and we have the following commutative diagram (where the arrow are isomorphisms) :



which gives the commutative diagram (where the rows are equivalences) :



and the equivalence $(k,l) \sim (k',l')$ can be rewritten

$$(k,l) \overset{(ii)}{\sim} (k_1,l_1) \overset{(iii)}{\sim} (k_2,l_2) \overset{(i)}{\sim} (k_3,l_3) \overset{(i)}{\sim} (k',l'), \tag{44}$$

where we denote $\overset{(i)}{\sim}$, $\overset{(ii)}{\sim}$ and $\overset{(iii)}{\sim}$ if the equivalence is respectively of the form of (i), (ii) and (iii).

If we do not assume $K(\delta) = \delta'$, we have $K(\delta) = \delta'^{\pm 1} \alpha'^x \beta'^y \gamma'^z$. One can compose K by isomorphisms of the form of (i), and then we can get back to the case where $K(\delta) = \gamma^z \delta^{\pm 1}$, via relations of the form of (i). But the application which is equal to I_3 on H and maps δ onto $\delta\gamma^{-z}$ is an automorphism of $H_r \rtimes_{S_{k',l'}} \mathbb{Z}.\delta$, and thus we can get back to the case where $K(\delta) = \delta'^{\pm 1}$. We have already seen the case where $K(\delta) = \delta'$. If $K(\delta) = \delta'^{-1}$, then an similar analysis as above gives $\bar{K}B = B^{-1}\bar{K}$ and we can rewrite K the same ways as above (we get the same diagrams), with $K^2(\delta) = \delta'^{-1}$ instead of δ. Then we get a sequence of equivalences similar to (44). □

Proof of Corollary 5.1: It is a consequence of the proof of Theorem 5.2. □

Proof of Proposition 5.4 : Let $K \in \mathcal{R}(B)$. If $K \in C(B)$, then

$$\det(K)L_1.K = \det(K)\mathbb{Z}^2.(B-I_2).K + r\det(K)\mathbb{Z}^2.K = \det(K)\mathbb{Z}^2.K.(B-I_2) + r\det(K)\mathbb{Z}^2.K.$$

But since $\det(K)K \in \mathrm{GL}(2,\mathbb{Z})$,

$$\det(K)\mathbb{Z}^2.K = \mathbb{Z}^2 \quad \text{and} \quad \det(K)L_1.K = L_1.$$

If $K \in \mathcal{R}(B)\backslash C(B)$,

$$\det(K)L_1.K = \det(K)\mathbb{Z}^2.K.(B^{-1}-I_2)+r\det(K)\mathbb{Z}^2.K = \det(K)\mathbb{Z}^2.K.(-B^{-1}).(B-I_2)+r\det(K)\mathbb{Z}^2.K.$$

But $-\det(K)KB^{-1} \in \mathrm{GL}(2,\mathbb{Z})$, hence

$$-\det(K)\mathbb{Z}^2.KB^{-1} = \mathbb{Z}^2 \quad \text{and} \quad \det(K)L_1.K = L_1.$$

Thus L_1 is stable under the action of $\mathcal{R}(B)$ on $\mathbb{Z}^2$. Then the action $\bar{\rho}$ of $\mathcal{R}(B)$ defined on $\mathbb{Z}^2/L_1$ by $\bar{\rho}(K)(\Pi_{L_1}(k,l)) = \Pi_{L_1}(\rho(K)(k,l))$ is well defined and is induced by ρ via the quotient $\mathbb{Z}^2 \to \mathbb{Z}^2/L_1$.

Let's show that $\Psi^{(3)}_{B,r}$ is well defined. If $\omega_{\mathcal{R}(B)}[\Pi_{L_1}(k,l)] = \omega_{\mathcal{R}(B)}[\Pi_{L_1}(k',l')]$,

$$\exists K \in \mathcal{R}(B), \quad \Pi_{L_1}(k,l) = \bar{\rho}(K)\left(\Pi_{L_1}(k',l')\right) = \Pi_{L_1}\left(\det(K)(k',l').K\right)$$

$$\Longleftrightarrow$$

$$(k,l) - \det(K).(k',l').K \in L_1$$

$$\Longleftrightarrow$$

$$\exists D, D', t_1, t_2 \in \mathbb{Z} \quad (k,l) - \det(K).(k',l').K = (t_1,t_2).(B-I_2) + r(D,D').$$

Thus $(k,l) = \det(K).(k',l').K + (t_1,t_2).(B-I_2) + r(D,D')$ and according to Proposition 5.3, $(k,l) \overset{B,r}{\sim} (k',l')$. Thus

$$\Psi^{(3)}_{B,r}\left(\omega_{\mathcal{R}(B)}\left(\Pi_{L_1}(k,l)\right)\right) = \Pi_{B,r}(k,l) = \Pi_{B,r}(k',l') = \Psi^{(3)}_{B,r}\left(\omega_{\mathcal{R}(B)}\left(\Pi_{L_1}(k',l')\right)\right)$$

and $\Psi^{(3)}_{B,r}$ is well defined.

$\Psi^{(3)}_{B,r}$ is clearly surjective : one just has to take $\omega_{\mathcal{R}(B)}(\Pi_{L_1}(k,l)]$ as an antecedent of $\Pi_{B,r}(k,l) \in \mathbb{Z}^2/\overset{B,r}{\sim}$. Let's show that it is injective. Let $(k,l),(k',l') \in \mathbb{Z}^2$ such that $\Pi_{B,r}(k,l) = \Pi_{B,r}(k',l')$. This means that $(k,l) \overset{B,r}{\sim} (k',l')$. According to Theorem 5.2, there exists a finite sequence (k_1,l_1), ...(k_{p-1},l_{p-1}) ($p \geq 1$) such that

$$(k,l) \overset{(1)}{\sim} (k_1,l_1) \overset{(2)}{\sim} \ldots \overset{(p-1)}{\sim} (k_{p-1},l_{p-1}) \overset{(p)}{\sim} (k',l'),$$

where each of the equivalences $\overset{(i)}{\sim}$ is of the form

$$(k_{i-1},l_{i-1}) \overset{(i),(ii)}{\sim} (k'_{i-1},l'_{i-1}) \overset{(iii)}{\sim} (k''_{i-1},l''_{i-1}) \overset{(i),(ii)}{\sim} (k_i,l_i).$$

Denote $(k,l) = (k_0, l_0)$ and $(k', l') = (k_p, l_p)$. For each $i \in \{1, ..., p\}$, we have $\Pi_{L_1}(k_{i-1}, l_{i-1}) = \Pi_{L_1}(k'_{i-1}, l'_{i-1})$ and $\Pi_{L_1}(k_i, l_i) = \Pi_{L_1}(k''_{i-1}, l''_{i-1})$. Moreover, there exists $K_i \in \mathcal{R}(B)$ such that

$$(k''_{i-1}, l''_{i-1}) = \rho(K_i)(k'_{i-1}, l'_{i-1}). \tag{45}$$

Thus in $\mathbb{Z}^2/L_1$,

$$\omega\left(\Pi_{L_1}(k_{i-1}, l_{i-1})\right) = \omega\left(\Pi_{L_1}(k'_{i-1}, l'_{i-1})\right) \overset{\text{equation (45)}}{=} \omega\left(\Pi_{L_1}(k''_{i-1}, l''_{i-1})\right) = \omega\left(\Pi_{L_1}(k_i, l_i)\right).$$

Then by transitivity, $\omega\left(\Pi_{L_1}(k,l)\right) = \omega[(\Pi_{L_1}(k', l'))$. Since this is true for all $(k,l), (k',l') \in \mathbb{Z}^2$ such that $\Pi_{B,r}(k,l) = \Pi_{B,r}(k',l')$, $\Psi^{(3)}_{B,r}$ is injective. □

Proof of Proposition 5.5: Let's show that $\rho_{B,r}$ is well defined. Let $K, K' \in \mathcal{R}(B)$ such that $(K \mod r) = (K' \mod r)$. Then r divides each entry of $\det(K).K - \det(K').K'$, and

$$\exists K'' \in \mathrm{M}_2(\mathbb{Z}), \quad \det(K).K - \det(K').K' = rK''.$$

Let $(k,l) \in \mathbb{Z}^2$. Then $\det(K)(k,l).K - \det(K').(k,l).K' = r(k,l).K''$.Thus

$$\det(K)(k,l).K = \det(K').(k,l).K' + r(k,l).K'' = \det(K').(k,l).K' + r(D,D') \overset{B,r}{\sim} \det(K').(k,l).K'. \tag{46}$$

Then

$$\begin{aligned} \rho_{B,r}\left(K \mod r\right)\left(\Pi_{L_1}(k,l)\right) = \bar{\rho}(K)\left(\Pi_{L_1}(k,l)\right) &\overset{\text{definition of } \bar{\rho}}{=} \Pi_{B,r}\left(\rho(K)(k,l)\right) \\ &\overset{(46)}{=} \Pi_{B,r}\left(\rho(K')(k,l)\right) \\ &\overset{\text{definition of } \bar{\rho}}{=} \rho_{B,r}\left(K' \mod r\right)\left(\Pi_{L_1}(k,l)\right) \end{aligned} \tag{47}$$

and $\rho(K \mod r)$ does not depend on the representative of the class of K modulo r. Then $\rho_{B,r}$ is well defined. It is clearly a group action, because $\bar{\rho}$ and the reduction modulo r are morphisms.

Let $(k,l) \in \mathbb{Z}^2$. According to the definition of $\rho_{B,r}$, it is clear that the orbit of $\Pi_{L_1}(k,l)$ for $\bar{\rho}$ is equal to its orbit for $\rho_{B,r}$. Hence $(\mathbb{Z}^2/L_1)/\mathcal{R}(B) = (\mathbb{Z}^2/L_1)/\mathcal{R}(B)_r$. □

A.5. Proofs of Section 7

Proof of Proposition 7.1: Let $B_1 = [[a,b][c,d]] \in H^0_T$ be a representative of the class of B lying in H^0_T. Then we know that $[B_1, ..., B_t] := \gamma_B(B_1, T_1, ..., T_t) = \gamma_{B_1}(T_1, ..., T_t)$.

First suppose that $a < b$. Then $\det(B_1) = 1$ implies $d > c$, and there exists a unique couple $(k,r) \in \mathbb{N}^* \times \{1, ..., c-1\}$ such that $d = kc + r$ ($k > 0$ because $d > c$, and $r > 0$ because $d \wedge c = 1$). For all $i \in \{1, ..., k+1\}$ we set $B_i = N^{i-1} B_1 N^{-(i-1)} := [[a_i, b_i][c_i, d_i]]$. Then $a_i = a + (i-1)c > 0$, $c_i = c > 0$, $d_i = d - (i-1)c \geq r > 0$ and $b_i > 0$ (because $c_i b_i = a_i d_i - 1 > 0$ and thus c_i and b_i have same sign), and $B_i \in H^0_T$. Thus, according to the construction of the cycle of B, $(B_1, ..., B_{k+1})$ is the beginning of the cycle of B. In particular $B_{k+1} = [[a + kc, b_k][c, r]]$ is in the cycle of B, with $a_{k+1} = a + kc > c$ (because $a > 0$ and $k > 0$), $a_k > c_k > r = d_k$ and $a_k > b_k$ (because $d_k < c_k$ and $\det(B_{k+1}) = 1 = a_{k+1}d_{k+1} - b_{k+1}c_{k+1}$). Then $a_{k+1} = \max(a_{k+1}, b_{k+1}, c_{k+1}, d_{k+1})$ and B_{k+1} satisfies the conditions of the Proposition.

Now suppose $a = b$. Since $a \wedge b = 1$, it implies $a = b = 1$, and the condition $\det(B_1) = 1$ gives

$B_1 = [[1,1][c,c+1]]$. Then as in the case where $a < b$, for all $i \in \{1,...,c+1\}$, $N^{i-1}B_1N^{-(i-1)}$ is in the cycle of B and $B_{c+1} = N^cBN^{-c}$ satisfies $a_{c+1} = 1+c > c_{c+1} = c$, $a_{c+1} > d_{c+1} = 1$ and from the same arguments as in the case where $a > b$, $a_{c+1} > b_{c+1}$. Then B_{c+1} satisfies the conditions of the Proposition.

Now suppose that $a > b$. There exists a unique couple $(k,r) \in \mathbb{N}^* \times \{1,...,b-1\}$ such that $a = kb+r$. For all $i \in \{1,...,k+1\}$, set $B_i = S^{i-1}B_1S^{-(i-1)} := [[a_i,b_i][c_i,d_i]]$. Then according to (28), $a_i = a-(i-1)b \geq r > 0$, $b_i = b > 0$, $d_i = d+(i-1)b > 0$ and $c_i > 0$ (because $c_ib_i = a_id_i - 1 > 0$ and thus c_i and b_i have same sign). Thus, according to the construction of the cycle of B, $(B_1,...,B_{k+1})$ is the beginning of the cycle of B. In particular $B_{k+1} = [[r,b][c_k,d+bk]]$ is in the cycle of B, with $a_{k+1} = r < b = b_{k+1}$. Then we are brought back to the case where $a < b$, by replacing the matrix B by B_{k+1} being in the cycle of B.
The Proof is similar for the existence of B_j. □

Proof of Lemma 7.1: Suppose that $a_i = \max(a_i,b_i,c_i,d_i)$. Then a computation gives that all the coefficients of SB_iS^{-1} and $N^{-1}B_iN$ are positive, and according to the construction of $\gamma_B(T_1,...,T_t)$, we get $B_{i+1} = SB_iS^{-1}$ and $B_{i-1} = NB_iN^{-1}$.

Reciprocally, suppose that $B_{i+1} = SB_iS^{-1}$ and $B_{i-1} = NB_iN^{-1}$. Then $0 < a_{i+1} = a_i - b_i \Rightarrow a_i > b_i$. Similarly, $0 < a_{i-1} = a_i - c_i \Rightarrow a_i > c_i$. Moreover, since $a_i > b_i$ and $\det(B_i) = 1$, we get $d_i < c_i < a_i$. Thus $a_i = \max(a_i,b_i,c_i,d_i)$. The reasonning is the same for the Proof of the second equivalence. □

Proof of Theorem 7.1: Let's show the existence of such a reduced cycle. Let $B \in \mathbb{S}$. According to Proposition 7.1, there exists $B_1 \in \gamma_B$ satisfying $a_1 = \max(a_1,b_1,c_1,d_1)$. Then there exist $T_1,...,T_t$ $(t \geq 1)$ such that $\gamma_B(B_1,T_1,...,T_t) = \gamma_{B_1}(T_1,...,T_t)$.
Let's set $i_0 = 1$ and let's construct by induction one part of the reduced cycle $\gamma_B^R(B_1,T_1,...,T_n)$. Suppose that $i_0 < ... < i_k$ are defined such that $\forall j \in \{1,...,k\}$, $a_j = \max(a_j,b_j,c_j,d_j)$. If the set $K = \{i > i_k \mid a_i = \max(a_i,b_i,c_i,d_i)\}$ is empty, the construction stops. If it is not, set $i_{k+1} = \min K$. This construction stops at one point because the cycle $\gamma_B(B_1,T_1,...,T_t)$ has a finite number of elements. We have built a cyclic sequence $(B_{i_1},...,B_{i_n})$ of all the elements of the cycle associated to B satisfying $a_i = \max(a_i,b_i,c_i,d_i)$.
Now let's construct the reduced cycle $\gamma_B^R(B_1,T_1,...,T_t)$. Let $k \in \{1,...,n\}$ (we will identify $n+1$ with 1 according to Remark 7.1). According to Lemma 7.1, we have $B_{i_{k+1}} = NB_{i_{k+1}-1}N^{-1}$ and $B_{i_k+1} = SB_{i_k}S^{-1}$. Still according to Lemma 7.1, there can't exist a $p \in \{i_k+1,...,i_{k+1}-1\}$ such that $B_p = NB_{p-1}N^{-1}$ and $B_{p+1} = SB_pS^{-1}$, because we would have $a_p = \max(a_p,b_p,c_p,d_p)$ and it would contradict the fact that $i_{k+1} = \min\{i > i_k \mid a_i = \max(a_i,b_i,c_i,d_i)\}$. This implies that

$$(T_{i_k},...,T_{i_{k+1}-1}) = (S,...,S,N,...,N) \quad (\text{ with p times } N \text{ and q times } S,\ p+q = i_{k+1} - i_k),$$

or, equivalently,

$$T_{i_{k+1}-1}...T_{i_k} = N^pS^q, \quad p+q = i_{k+1} - i_k.$$

Let's set $j_k = i_k + q$. Then according to the form of $(T_{i_k},...,T_{i_{k+1}-1})$, $B_{j_k} = SB_{j_k-1}S^{-1}$ and $B_{j_k+1} = NB_{j_k}N^{-1}$. Then according to Lemma 7.1, $d_{j_k} = \max(a_{j_k},b_{j_k},c_{j_k},d_{j_k})$. Moreover, $i_k < j_k < i_{k+1}$.

Now let's set the finite sequence of integers $(k_p)_{p \in \{0,...,2n+1\}}$ defined by $k_{2p} = i_p$ and $k_{2p+1} = j_p$. It gives a subsequence $(B_{k_0},...,B_{k_{2n+1}})$ of $(M_1,...,M_t) = \gamma_{M_1}(T_1,...,T_n)$ satisfying all the conditions of Proposition 7.1 by construction.

Moreover, by construction, there are no other matrices $[[a,b][c,d]]$ in the cycle associated to B satisfying $a = \max(a,b,c,d)$ or $d = \max(a,b,c,d)$, and thus there are no other matrices in H_T^0 that are conjugate to B by N or S and such that $a = \max(a,b,c,d)$ or $d = \max(a,b,c,d)$. A fortiori, if there exists another reduced cycle $C = [B'_{p_0}, ..., B'_{p_l}]$ associated to $B \in H_T^0$, then $C \subset \gamma_B^R$. Let $(T'_{p_0}, ..., T'_{p_l})$ be a sequence of operators associated to C. If $C \neq \gamma_B^R$, it means that there exists $j \in \{0, ..., 2n+1\}$ such that $B_{k_j} \in C$ and $B_{k_{j+1}} \notin C$. One can guess $B_{k_j} = B_{p_0}$. Suppose for example that $d_{k_j} = \max(a_{k_j}, b_{k_j}, c_{k_j}, d_{k_j})$. Then $T_{k_j} = N^{q_j}$ and $T'_{p_0} = N^{q'_{p_0}}$. Suppose $q'_{p_0} > q_j$. A quick computation gives that if $[[a,b][c,d]] := N^{q'_{p_0}-q_j} B_{k_{j+1}} N^{q'_{p_0}-q_j-1}$, then $d < 0$ and $N^{q'_{p_0}-q_j} B_{k_{j+1}} N^{q'_{p_0}-q_j-1} \notin H_T^0$. But then $B'_{p_1} \notin H_T^0$, which is absurd. If $q'_{p_0} < q_j$, one can do the same reasoning by exchanging B_{k_j} and B_{p_0}. □

Proof of Proposition 7.2: One can suppose that $B \in H_T^0$ without loss of generality. Since $\gamma_B^R(T_1, ..., T_t) = (B_1, ..., B_t)$, is cyclic, one just has to study the case of $i = 1$ to deduce the result for all $i \in \{1, ..., t\}$. Suppose that B_1 satisfies $a_1 = \max(a_1, b_1, c_1, d_1)$, which is equivalent to $T_1 = S^{q_1}$. Moreover it gives that $d_2 = \max(a_2, b_2, c_2, d_2)$. We have $a_1 = \max(a_1, b_1, c_1, d_1)$, which means that $a_1 \geq b_1$. Moreover, we have $d_2 \geq c_2$. If $d_2 = c_2$, we have $d_2 = c_2 = 1$, and thus $a_2 = b_2 + 1$, with $b_2 > 0$, which means that $a_2 > 1 = d_2$, which is impossible because $d_2 = \max(a_2, b_2, c_2, d_2)$. Thus $d_2 > c_2$, and thus $a_2 \leq b_2 = b_1$. But $a_2 = a_1 - q_1 b_1$. This means that $a_1 = q_1 b_1 + a_2$, with $a_2 \leq b_1$. Thus, if $a_2 < b_1$, with the notations of the Proposition and by uniqueness of the Euclidian division, $(q_1, a_2) = (k_1, r_1)$. If $a_2 = b_1 = b_2$, this means that $a_2 = b_2 = 1$, which means that $b_1 = b_2 = 1$. In this case, $(q_1, a_2) = (a_1 - 1, 1)$.
The case where B_1 satisfies $d_1 = \max(a_1, b_1, c_1, d_1)$ is similar (it is equivalent to $T_1 = N^{q_1}$). Hence the Proof. □

Proof of Proposition 7.3: Let's first recall one result:

Lemma A.1. *Let $A, B \in \mathrm{SL}(2,\mathbb{Z})$. A and B are conjugated in $\mathrm{GL}(2,\mathbb{Z})$ via a matrix M with determinant -1 (i.e $A = MBM^{-1}$) if, and only if, A and $B' = UBU^{-1}$ are conjugate in $\mathrm{SL}(2,\mathbb{Z})$ via $M' = MU^{-1}$ (i.e $A = M'B'M'^{-1}$).*

A direct consequence of this Lemma is that $B, A \in \mathbb{S}_T$ are conjugate in $\mathrm{SL}(2,\mathbb{Z})$ if, and only if, UBU^{-1} and UAU^{-1} are.

Let's first show the Proposition for the sequence $[B'_1, .., B'_t]$. By assumption, each of the matrices $B_i = [[a_i, b_i][c_i, d_i]] \in H_T^0$ satisfies $a_i = \max(a_i, b_i, c_i, d_i)$ or $d_i = \max(a_i, b_i, c_i, d_i)$. Then $B'_i = UB_iU^{-1} = [[d_i, c_i][b_i, a_i]]$ and each of the matrices $B'_i = [[a'_i, b'_i][c'_i, d'_i]]$ is in H_T^0 and satisfies

$$a'_i = \max(a'_i, b'_i, c'_i, d'_i) \quad \text{or} \quad d'_i = \max(a'_i, b'_i, c'_i, d'_i).$$

Moreover, noticing that $N = USU^{-1}$ and $S = UNU^{-1}$, we get, $\forall i \in \{1, ..., t\}$:

$$T'_i = UT_iU^{-1} = \begin{cases} S^{q_i} \text{ if } T_i = N^{q_i}, \\ N^{q_i} \text{ if } T_i = S^{q_i}, \end{cases}$$

and it is clear that we have $B'_{i+1} = T'_i B'_i T_i'^{-1}$, and the condition on the operators associated to the cycle is satisfied. Then $\bar{\gamma}$ is a reduced cycle.

It is clear that this reduced cycle is associated to the matrix $UBU^{-1} \in \mathbb{S}_T$: indeed, B_1 is in the conjugacy class of B in $\mathrm{SL}(2,\mathbb{Z})$ by definition of the reduced cycle associated to a matrix, and then, according to Lemma A.1, B'_1 is in the conjugacy class of UBU^{-1} in $\mathrm{SL}(2,\mathbb{Z})$. Then by uniqueness of the reduced cycle, $\bar{\gamma}$ is the unique reduced cycle associated to UBU^{-1}.

Finally, by uniqueness of the reduced cycle, γ and $\bar{\gamma}$ are equal or disjoint.

Now let's show Proposition 7.3 for the sequence $[B''_1, .., B''_t]$. By assumption, each of the matrices $B_i = [[a_i, b_i][c_i, d_i]] \in H_T^0$ satisfies $a_i = \max(a_i, b_i, c_i, d_i)$ or $d_i = \max(a_i, b_i, c_i, d_i)$. Then

$$B''_i =^t B_{t-i+1} = \begin{pmatrix} a_{t-i+1} & c_{t-i+1} \\ b_{t-i+1} & d_{t-i+1} \end{pmatrix}$$

(see (29)) and each of the matrices $B''_i = [[a''_i, b''_i][c''_i, d''_i]]$ is in H_T^0 and satisfies

$$a''_i = \max(a''_i, b''_i, c''_i, d''_i) \quad \text{or} \quad d''_i = \max(a''_i, b''_i, c''_i, d''_i).$$

Moreover, noticing that $N = WS^{-1}W^{-1}$ and $S = WN^{-1}W^{-1}$, we get, $\forall i \in \{1, ..., t\}$:

$$T''_i = WT^{-1}_{t-i+1}W^{-1} = \begin{cases} S^{q_{t-i+1}} \text{ if } T_{t-i+1} = N^{q_{t-i+1}}, \\ N^{q_{t-i+1}} \text{ if } T_{t-i+1} = S^{q_{t-i+1}}. \end{cases}$$

Then we have

$$B''_{i+1} =^t B_{t-i} =^t (T^{-1}_{t-i+1} B_{t-i+1} T_{t-i+1}) =^t T^t_{t-i+1} B^t_{t-i+1} T^{-1}_{t-i+1} = T''_i B''_i T''^{-1}_i,$$

and $(T''_i)_{i\in\{1,..,t\}}$ satisfies the conditions on the sequence of operators associated to the cycle. Then ${}^t\gamma$ is a reduced cycle.

It is clear that this reduced cycle is associated to the matrix $B^{-1} \in \mathbb{S}_T$: indeed, B_1 is in the conjugacy class of B in $\mathrm{SL}(2,\mathbb{Z})$ by definition of the reduced cycle associated to a matrix, and then, according to (29), $B''_t = WB_1^{-1}W^{-1}$ is in the conjugacy class of B_1^{-1} and then of B^{-1} in $\mathrm{SL}(2,\mathbb{Z})$. Then by uniqueness of the reduced cycle, $\bar{\gamma}$ is the unique reduced cycle associated to B^{-1}.

Finally, by uniqueness of the reduced cycle, γ and ${}^t\gamma$ are equal or disjoint. □

Proof of Lemma 7.2: If $\gamma^R_B = \gamma^R_{B'}$, it is clear that B and B' are in the same equivalence class, because they are conjugate in $\mathrm{SL}(2,\mathbb{Z})$. If $\gamma^R_B = \bar{\gamma}^R_{B'} = \gamma^R_{UB'U^{-1}}$, B and $UB'U^{-1}$ are conjugate in $\mathrm{SL}(2,\mathbb{Z})$, thus B and B' are conjugate in $\mathrm{GL}(2,\mathbb{Z})$ (according to Lemma A.1) and B and B' are *a fortiori* in the same equivalence class. If $\gamma^R_B =^t \gamma^R_{B'} = \gamma^R_{B^{-1}}$, then B and B^{-1} are conjugate in $\mathrm{SL}(2,\mathbb{Z})$, which implies that B and B' are in the same equivalence class. Finally, if $\gamma^R_B = {}^t\bar{\gamma}^R_{B'} = \gamma^R_{UB'^{-1}U^{-1}}$, then B and $UB'^{-1}U^{-1}$ are conjugate in $\mathrm{SL}(2,\mathbb{Z})$, and B and B'^{-1} are conjugate in $\mathrm{SL}(2,\mathbb{Z})^-$ (Lemma A.1), which implies that B and B' are in the same equivalence class.

Reciprocally, if B and B' are in the same equivalence class, there are four possibilities :

- B and B' are conjugate in $\mathrm{SL}(2,\mathbb{Z})$, in which case, by uniqueness of the reduced cycle, $\gamma^R_B = \gamma^R_{B'}$;
- B and B' are conjugate in $\mathrm{SL}(2,\mathbb{Z})^-$, in which case, B and $UB'U^{-1}$ are conjugate in $\mathrm{SL}(2,\mathbb{Z})$ (Lemma A.1), and $\gamma^R_B = \gamma^R_{UB'U^{-1}} = \bar{\gamma}^R_{B'}$;
- B and B'^{-1} are conjugate in $\mathrm{SL}(2,\mathbb{Z})$, in which case $\gamma^R_B = \gamma^R_{B'^{-1}} =^t \gamma^R_{B'}$;

- B and B'^{-1} are conjugate in $\mathrm{SL}(2,\mathbb{Z})^-$, in which case, B and $UB'^{-1}U^{-1}$ are conjugate in $\mathrm{SL}(2,\mathbb{Z})$ (Lemma A.1), and $\gamma_B^R = \overline{\gamma_{UB'^{-1}U^{-1}}^R} = {}^t\overline{\gamma_{B'}^R}$.

Hence the result. $\square$

Proof of Proposition 7.4: We know that $\gamma_B^R, \overline{\gamma_B^R}, {}^t\gamma_B^R$ and ${}^t\overline{\gamma_B^R}$ are contained in the class of B (see Lemma 7.2). Let's show that they are the only reduced cycles in H_T^0 that are in the equivalence class of B. If γ is a reduced cycle in H_T^0 that is in the equivalence class of B, and B' an element of this γ, then according to Lemma 7.2, we necessarilly have $\gamma = \gamma_{B'}^R \in \{\gamma_B^R, \overline{\gamma_B^R}, {}^t\gamma_B^R, \overline{{}^t\gamma_B^R}\}$.

If B is conjugate to itself in $\mathrm{SL}(2,\mathbb{Z})^-$ (i.e is symmetrizable), (i.e $\mathcal{R}(B) \cap \mathrm{SL}(2,\mathbb{Z})^- \neq \varnothing$), then Lemma 7.2 gives $\gamma_B^R = \overline{\gamma_B^R}$. In this case an element of $\mathbb{S}_T$ is equivalent to B if, and only if it is conjugate to B or B^{-1} in $\mathrm{SL}(2,\mathbb{Z})$. Then the equivalence class of B is equal to the union of the conjugacy classes of B and B^{-1} in $\mathrm{SL}(2,\mathbb{Z})$.

If B is conjugate to B^{-1} in $\mathrm{SL}(2,\mathbb{Z})$ (in particular B is reversible), then Lemma 7.2 gives $\gamma_B^R =^t \gamma_B^R$. In this case, a matrix of $\mathbb{S}_T$ is in the equivalence class of B if, and only if it is conjugate in $\mathrm{GL}(2,\mathbb{Z})$ to B or B^{-1} if, and only if, it is conjugate to B in $\mathrm{GL}(2,\mathbb{Z})$. Thus the equivalence class of B is equal to its conjugacy class in $\mathrm{GL}(2,\mathbb{Z})$.

If B is conjugate to B^{-1} in $\mathrm{SL}(2,\mathbb{Z})^-$ (in particular B is reversible), then Lemma 7.2 gives $\gamma_B^R =^t (\overline{\gamma_B^R}) = \overline{({}^t\gamma_B^R)}$. Then by conjugation of cycles, $\overline{\gamma_B^R} =^t \gamma_B^R$. In this case, an element of $\mathbb{S}_T$ is equivalent to B if, and only if, it is conjugate to B in $\mathrm{GL}(2,\mathbb{Z})$. Thus the equivalence class of B is equal to its conjugacy class in $\mathrm{GL}(2,\mathbb{Z})$.

The Proposition follows. $\square$

Proof of Corollary 7.1: $\gamma =^t \gamma$ if, and only if, $\gamma_B^R = \gamma_{B^{-1}}^R = \gamma_{{}^tB}^R$ because $B^{-1} = W^{-1t}BW$ is conjugate to ${}^tB$ in $\mathrm{SL}(2,\mathbb{Z})$. Moreover, ${}^tB$ is a reduced matrix of H_T^0, which means that it lies in its reduced cycle, and ${}^tB \in^t \gamma$. Since two reduced cycles are equal if and only if they have one common matrix, it is clear that $\gamma =^t \gamma$ if, and only if, ${}^tB \in \gamma$.

The same reasonning gives the second assertion. $\square$

A.6. Proofs of Section 8

Proof of Lemma 8.1: According to the Dirichlet unit Theorem, there exist distincts $a_1, a_2, ..., a_n \in \mathcal{O}_{\mathbb{Q}(\lambda_T)}^\times$ such that $\mathcal{O}_{\mathbb{Q}(\lambda_T)}^\times$ is the set of all elements $\xi^k a_{i_1}...a_{i_p}$, for ξ being a root of 1 in $\mathcal{O}_{\mathbb{Q}(\lambda_T)}$, $k, i_1, ..., i_p \in \mathbb{N}$. Moreover, $n = r_1 + r_2 - 1$, with r_1 the number of morphisms from $\mathbb{Q}(\lambda_T)$ to $\mathbb{R}$, and r_2 the number of conjugate pairs of morphisms from $\mathbb{Q}(\lambda_T)$ to $\mathbb{C}$. A more precise result gives that $\mathcal{O}_{\mathbb{Q}(\lambda_T)}^\times$ is isomorphic to the product of the group of unit roots of K and an abelan free group on rank $n = r_1 + r_2 - 1$.

In our case, the roots of 1 are 1 and -1, and the injection $\mathbb{Q}(\lambda_T) \subset \mathbb{R}$ gives $r_1 \geq 1$. But we also have $2 = dim(\mathbb{Q}(\lambda_T)) = r_1 + 2r_2$ (classical result) which implies $r_2 = 0$ and $r_1 = 2$. Then $n = 1$ and there exists $\tau \in \mathcal{O}_{\mathbb{Q}(\lambda_T)}^\times$ with infinite order such that

$$\mathcal{O}_{\mathbb{Q}(\lambda_T)}^\times = \{\pm\tau^p \mid p \in \mathbb{Z}\} \simeq \{\pm 1\} \times \langle\tau\rangle. \quad \square$$

A.7. Proofs of Section 10

Proof of Proposition 10.1: Let's recall a classical result of group theory (see for example [12]):

Lemma A.2. *Let G be a finitely generated group. For a fixed $n \in \mathbb{N}_{>0}$, G has a finite number of subgroups of index n.*

Let's set $[[a,b][c,d]] := B$ and $[[a',b'][c',d']] := B'$. Notice that H_1 is clearly a subgroup of index r of H_r, generated by α, β, γ^r. For $q \in \mathbb{N}_{>0}$, $S_B^{k,l}(\delta^q)$ maps H_1 onto a normal subgroup of index r of H_r. Then according to Lemma A.2, there exists $q \in \mathbb{N}_{>0}$ such that $S_B^{k,l}(\delta^q)$ maps H_1 onto itself. Then $H_1 \rtimes_{S_B^{k,l}} \mathbb{Z}.q\delta \simeq \Gamma(1, B^q, k_1, l_1)$ is a subgroup of $\Gamma(r, B, k, l)$ of finite index, with

$$(S_B^{k,l})^q = \begin{pmatrix} B^q & & 0 \\ k_1 & l_1 & 1 \end{pmatrix}.$$

But we know from Proposition 5.2, (i) (that we apply to $r = 1$), that $\Gamma(1, B^q, k_1, l_1)$ is isomorphic to $\Gamma(1, B^q, 0, 0)$. Thus $\Gamma(1, B^q, 0, 0)$ is commensurable to $\Gamma(r, B, k, l)$. Similarly, there exists q' such that $\Gamma(1, B'^{q'}, 0, 0)$ and $\Gamma(r', B', k', l')$ are commensurable.

Thus $\Gamma(r, B, k, l)$ and $\Gamma(r', B', k', l')$ are commensurable if, and only if, $\Gamma(1, B'^{q'}, 0, 0)$ and $\Gamma(1, B^q, 0, 0)$ are, and we may assume $r = r' = 1$ and $k = k' = l = l' = 0$. We will denote $\Gamma(B)$ instead of $\Gamma(1, B, 0, 0)$ and S_B instead of $S_B^{0,0}$.

First suppose that $\Gamma(B)$ and $\Gamma(B')$ are abstractly commensurable. Let Γ be the finite index subgroup of $\Gamma(B)$ that can be embedded as a finite index subgroup into $\Gamma(B')$. Let L be a lattice of $\mathrm{Osc}_{1,0}$ such that there exists an isomorphism

$$\phi : \Gamma(B) \longrightarrow L.$$

L and ϕ exist according to Remark 5.1. Then $\phi(\Gamma)$ is a finite index subgroup of L. This implies that $\phi(\Gamma)$ lattice of $\mathrm{Osc}_{1,0}$. This lattice is isomorphic to $\Gamma(\tilde{r}, \tilde{B}, \tilde{k}, \tilde{l})$ for some $\tilde{k}, \tilde{l}$ (see Proposition 5.1). Thus Γ is a discrete oscillator group $\Gamma(\tilde{r}, \tilde{B}, \tilde{k}, \tilde{l})$, that can be embeded in both $\Gamma(B)$ and $\Gamma(B')$. But there exists an injective morphsim $\Gamma(\tilde{B}) = \Gamma(1, \tilde{B}, 0, 0) \longrightarrow \Gamma(\tilde{r}, \tilde{B}, \tilde{k}, \tilde{l})$. Recall the notations

$$\Gamma(B) = \langle \alpha, \beta, \gamma, \delta \rangle, \quad \Gamma(B') = \langle \alpha', \beta', \gamma', \delta' \rangle, \quad \Gamma(\tilde{B}) = \langle \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta} \rangle.$$

Then there exist two injective morphisms $\phi_1 : \Gamma(\tilde{B}) \to \Gamma(B)$ and $\phi_2 : \Gamma(\tilde{B}) \to \Gamma(B')$. Then classical arguments give that $\phi_1(\tilde{\gamma}) \in \langle \gamma \rangle$, i.e there exists $p \in \mathbb{Z}$ such that $\phi(\tilde{\gamma}) = \gamma^p$, and $p \neq 0$ because ϕ_1 is injective. One can write $\phi_1{}_{|H_{\tilde{r}}}^{|H^r} = \begin{pmatrix} \bar{T} & & 0 \\ t_1 & t_2 & p \end{pmatrix}$ with $\bar{T} = [[t_{11}, t_{12}][t_{21}, t_{22}]] \in \mathrm{GL}(2, \mathbb{R}) \cap \mathrm{M}_2(\mathbb{Z})$ and $t_1, t_2 \in \mathbb{Z}$, and $\phi_1(\delta) = \delta^m \alpha^x \beta^y \gamma^z$. Then we have:

$$(\alpha^{t_{11}} \beta^{t_{21}})^a (\alpha^{t_{12}} \beta^{t_{22}})^c = \phi_3(\alpha)^a \phi(\beta)^c = \phi_3(\delta \alpha \delta^{-1}) =$$
$$\beta^y \alpha^x \delta^m \phi_3(\alpha) \delta^{-m} \beta^{-y} \alpha^{-x} = \beta^y \alpha^x \delta^m \alpha^{t_{11}} \beta^{t_{21}} \delta^{-m} \beta^{-y} \alpha^{-x}.$$

Developping the left factor and the right factor of this formula and setting $[[a_m, b_m][c_m, d_m]] := B^m$, we get

$$\alpha^{at_{11}+ct_{12}} \beta^{at_{21}+ct_{22}} \gamma^D = \alpha^{a_m t_{11}+b_m t_{12}} \beta^{a_m t_{21}+b_m t_{22}} \gamma^{D'},$$

with $D, D' \in \mathbb{Z}$. Considering $\phi_3(\alpha)^b \phi(\beta)^d = \phi_3(\delta \beta \delta^{-1})$, we get

$$\alpha^{bt_{11}+dt_{12}} \beta^{bt_{21}+dt_{22}} \gamma^C = \alpha^{b_m t_{11}+c_m t_{12}} \beta^{b_m t_{21}+c_m t_{22}} \gamma^{C'},$$

with $C, C' \in \mathbb{Z}$. This means that $B^m \bar{T} = \bar{T} \tilde{B}$. Similarly, $B'^m \bar{T}' = \bar{T}' \tilde{B}$ for some $\bar{T}' \in \mathrm{GL}(2, \mathbb{R}) \cap \mathrm{M}_2(\mathbb{Z})$. Let's set $M = \bar{T}'^{-1} \bar{T} \in \mathrm{GL}(2, \mathbb{Q})$, then $B^m = M B'^m M^{-1}$.

Reciprocally, if there exist $m, n \in \mathbb{Z}^*$ and $M \in \mathrm{GL}(2,\mathbb{Q})$ such that $B^m = MB'^n M^{-1}$, even if it means multiplying M by an appropriate integer, one can assume that M has integer coefficients. Let's show that $\Gamma(B)$ and $\Gamma(B')$ are commensurable.

Let's define the map f by $\begin{pmatrix} M & & 0 \\ 0 & 0 & s \end{pmatrix}$, with $s = \det(M)$. Then f is an injective morphism from H_1 to itself. Using Lemma A.2, one can chose $p \in \mathbb{N}_{>0}$ such that the image of f is invariant under $(S_{B'})^{pn}$. Set $a = (S_{B'})^{pn}$ and $b = f^{-1}af$, with f^{-1} being the inverse of $f^{|Imf}$. b is well defined and is an isomorphism of H_1. We have $b(\gamma) = f^{-1}af(\gamma) = \gamma$, and $b = \begin{pmatrix} \tilde{B} & & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $\tilde{B} = M^{-1}B'^{np}M \in \mathrm{SL}(2,\mathbb{Z})$.

Let's extend f to an isomorphism from $H_1 \rtimes_b \mathbb{Z}\delta$ to $H_1 \rtimes_a \mathbb{Z}\delta \simeq H_1 \rtimes_{S_{B'}} \mathbb{Z}\delta^{pn}$ by setting $f(\delta) = \delta$. Then $\Gamma(\tilde{B})$ can be injected in $\Gamma(B')$ and they are commensurable. Moreover, $b = f^{-1}af$ implies that $\tilde{B} = M^{-1}B'^{np}M = B^{mp}$, which implies $(S_B)^{mp} = S_{\tilde{B}}$, and $\Gamma(\tilde{B}) = \Gamma(B^{mp}) \simeq H_1 \rtimes_{S_B} \mathbb{Z}.\delta^{mp}$ is a subgroup of $\Gamma(B)$ of finite index. Thus $\Gamma(B')$ and $\Gamma(B)$ are commensurable. $\square$

Proof of Lemma 10.1: For each $B \in \mathbb{S}$, B is cyclic, i.e there exists $x \in \mathbb{Q}^2$ such that (x, Bx) is a basis of $\mathbb{Q}^2$ (take $x = (1,0)$ for example). Then a typival result gives that B is conjugate over $\mathbb{Q}$ to the compagnon matrix $\mathcal{C}(\pi_B)$ of its minimal polynomial π_B (see for example the lesson of Marc Sage, "Invariants de similitudes et réduction de Frobenius"). To prove this, we could also have computed that for each $B = [[a,b][c,d]]$, if we set $P = [[1,a][0,c]] \in \mathrm{GL}(2,\mathbb{Q})$, we have

$$P^{-1}BP = \begin{pmatrix} 0 & -\det(B) \\ 1 & tr(B) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & tr(B) \end{pmatrix},$$

and have deduced that B is conjugate to the compagnon matrix of π_B. Then π_B is the invariant factor of B in the Frobenius decomposition of B. But it is known that two matrices are conjugate if, and only if, they have the same sequence of invariant factors. Here, it means that two matrices of $\mathbb{S}$ are conjugate over $\mathbb{Q}$ if, and only if, they have the same minimal polynomial if, and only if, they have the same trace.

Thus if $B, B' \in \mathbb{S}$ and λ (resp. λ') is an eigenvalue of B (resp. B'), we have: $B \sim B'$ if, and only if there exist $n, m \in \mathbb{Z}^*$ such that $\lambda^m + \frac{1}{\lambda^m} = tr(B^m) = tr(B'^n) = \lambda'^n + \frac{1}{\lambda'^n}$, which is equivalent to

$$\lambda^m = \lambda'^n \text{ or } \lambda^m = \lambda'^{-n}. \tag{48}$$

Suppose that $B \sim B'$. Then (48) gives that there exist $n, m \in \mathbb{Z}^*$ such that $\lambda^m = \lambda'^n$. Then $\lambda^m \in \mathbb{Q}(\lambda')$, and there exist $a, b \in \mathbb{Q}$ such that $\lambda^m = a\lambda' + b$. Then $a \neq 0$ (because λ is irrational) and $\lambda' = \frac{\lambda^m - b}{a} \in \mathbb{Q}(\lambda)$. Thus $\mathbb{Q}(\lambda') \subset \mathbb{Q}(\lambda)$. Since they have same dimension, $\mathbb{Q}(\lambda) = \mathbb{Q}(\lambda')$. .

Reciprocally, suppose that $\mathbb{Q}(\lambda) = \mathbb{Q}(\lambda')$. λ and λ' are both elements of $\mathcal{O}^\times_{\mathbb{Q}(\lambda)}$, as we saw in subsection 5.4, equation (30). But we know from equation (31) that $\mathcal{O}^\times_{\mathbb{Q}(\lambda)} = \{\pm\tau^p \mid p \in \mathbb{Z}\}$, with $\tau \in \mathcal{O}^\times_{\mathbb{Q}(\lambda)}$. Then there exist $\delta, \delta' \in \{\pm 1\}$ and $k, k' \in \mathbb{Z}^*$ ($k, k' \neq 0$ because λ and λ' are irrational) such that $\lambda = \delta\tau^k$ and $\lambda' = \delta'\tau^{k'}$. Set $m = k'$ and $n = k$. Then $\lambda^m = \delta^{m-n}\lambda^n$. Since λ^m and λ'^n have same sign, $\delta^{m-n} = 1$ and $\lambda^m = \lambda'^n$. From (48), we deduce that $B \sim B'$. $\square$

Proof of Proposition 10.2: Let $K, K' \in \mathcal{C}_\mathbb{R}$. Suppose that Γ_K and $\Gamma_{K'}$ are commensurable and let's show that $K = K'$. Since Γ_K and $\Gamma_{K'}$ are commensurable, according to Proposition 10.1, $B_K \sim B_{K'}$. By construction we have $K = \mathbb{Q}(\lambda_K)$ and $K' = \mathbb{Q}(\lambda_{K'})$, where λ_K (resp. $\lambda_{K'}$)

is an eigenvalue of B_K (resp. of $B_{K'}$). Then according to Lemma 10.1, $K = K'$.

Let $\Gamma = \Gamma(r, B, k, l)$. Let λ be an eigenvalue of B and K be the set $\mathbb{Q}(\lambda)$. Then $\mathbb{Q}(\lambda) = K = \mathbb{Q}(\lambda_K)$, where λ_K is the eigenvalue of B_K that is greater than 1 by construction. Then according to Lemma 10.1, $B \sim B_K$, and according to Proposition 10.1, Γ is commensurable to Γ_K. □

Proof of Corollary 10.1: Let's show that $\Psi_{\mathcal{P}_1}$ is well defined. Let $a = (r, B, P, x, \delta, z)$ and $a' = (r', B', P', x', \delta', z') \in \mathcal{P}_1$ satisfy $a \sim a'$. Then $L(a)$ and $L(a')$ are commensurable. $L(a)$ (resp. $L(a')$) is isomorphic to $\Gamma(r, B, k, l)$ for suitable k and l (resp. to $\Gamma(r', B', k', l')$ for suitable k' and l'). Thus $\Gamma(r, B, k, l)$ and $\Gamma(r', B', k', l')$ are commensurable, which means that $B \sim B'$ according to Proposition 10.1. Then according to Lemma 10.1, $\Psi_{\mathcal{P}_1}(a) = \Psi_{\mathcal{P}_1}(a')$. Thus $\Psi_{\mathcal{P}_1}$ is invariant under $\sim$. Thus $\Psi_{\mathcal{P}_1}$ is well-defined.

Moreover, $\Psi_{\mathcal{P}_1}$ is clearly a surjection : let $K \in \mathcal{C}_{\mathbb{R}}$, . Then $\Gamma_K = \Gamma(1, B_K, 0, 0)$ is a discrete oscillator group and thus is isomorphic to some $L = L(a) \in \mathbb{L}$, $a = (r, B_K, P, x, \delta, z) \in \mathcal{P}_1$. Then we have $\Psi_{\mathcal{P}_1}(a) = K$.

Finally, $\overline{\Psi_{\mathcal{P}_1}}$ is an injection : indeed, let $a = (r, B, P, x, \delta, z)$ and $a' = (r', B', P', x', \delta', z')$ such that $\Psi_{\mathcal{P}_1}(a) = \Psi_{\mathcal{P}_1}(a')$. Then $\mathbb{Q}(\lambda) = \mathbb{Q}(\lambda')$ where λ and λ' are eigenvalues of B and B' respectively. Then according to Lemma 10.1, $B \sim B'$. Then $\Gamma(1, B, 0, 0)$ and $\Gamma(1, B', 0, 0)$ are commensurable (according to Proposition 10.1). But $\Gamma(1, B, 0, 0)$ (resp. $\Gamma(1, B', 0, 0)$) is commensurable to $\Gamma(r, B, k, l)$ (resp. $\Gamma(r', B', k, l)$) for each $k, l \in \mathbb{Z}$, and in particular for k, l such that $\Gamma(r, B, k, l)$ (resp. $\Gamma(r', B', k, l)$) is isomorphic to $L(a)$ (resp. $L(a')$). Then $\Gamma(1, B, 0, 0)$ (resp. $\Gamma(1, B', 0, 0)$) is commensurable to $L(a)$ (resp. $L(a')$). Then $L(a)$ and $L(a')$ are commensurable and $a \sim a'$. Thus $\overline{\Psi_{\mathcal{P}_1}}$ is injective. □

B. Two explicit examples of lattices

Let's give two explicit examples of lattices of $\mathrm{Osc}_{1,0}$ that illustrate Proposition 2.4. Let's recall that if we diagonalize A, we find $A = T diag(1, -1) T^{-1}$ with $T = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$.

Let's set $B_1 = [[2, 3][1, 2]] \in \mathbb{S}$. If we diagonalize B_1, we have :

$$B_1 = K_1 D_1 K_1^{-1}, \text{ with } D_1 = diag(\lambda, \frac{1}{\lambda}), \quad \lambda = 2 + \sqrt{3}, \quad K_1 = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -\sqrt{3} \\ 1 & 1 \end{pmatrix}.$$

Then $P_1 = K_1 T^{-1} = [[-\sqrt{3}, 0][0, 1]]$ is in C_{B_1}. We have

$$\Gamma^1_{P_1} = \left\{ (z, (\xi_1, \xi_2), 0) \mid (\xi_1, \xi_2) \in \frac{\mathbb{Z}}{\sqrt{3}} \times \mathbb{Z}, z \in \frac{1}{2}\xi_1\xi_2 + \mathbb{Z} \right\}.$$

Since $(B_1, 0) \in \mathcal{B}_1$, and $L_1 = L^1_{B_1, P_1, 0}$ is a lattice of $\mathrm{Osc}_{1,0}$ such that $L_1 \cap H = \Gamma^1_{P_1}$, i.e $L_1 \in \mathbb{L}^1_{B_1} \subset \mathbb{L}_{\mathbb{S}}$.

Let's set $B_2 = [[4, 5][-1, -1]] \in \mathbb{S}$. Notice that $(B_2, 0) \notin \mathcal{B}_1$. Then for each $P_2 \in C_{B_2}$, $L^1_{B_2, P_2, 0}$ is not a lattice such that $L^1_{B_2, P_2, 0} \cap H = \Gamma^1_{P_2}$. However we are in the case where *ab* is even and *cd* is odd, and thus

$$\left(B_2, x := (1, \frac{3}{2})\right) \in \mathcal{B}_1$$

Then for all $P_2 \in C_{B_2}$, $L_2 = L^1_{B_2,P_2,x}$ is a lattice of $\mathrm{Osc}_{1,0}$ such that $L_2 \cap H = \Gamma^1_{P_2}$, i.e $L_2 \in \mathbb{L}^1_{B_2} \subset \mathbb{L}_{\mathbb{S}}$. One can check that $P_2 = K_2 T^{-1} \in C_{B_2}$, with

$$K_2 = \begin{pmatrix} \frac{-5-\sqrt{5}}{2\sqrt{1+\frac{1}{4}(5+\sqrt{5})^2}} & \frac{\sqrt{5}-5}{2\sqrt{1+\frac{1}{4}(5-\sqrt{5})^2}} \\ \frac{1}{2\sqrt{1+\frac{1}{4}(5+\sqrt{5})^2}} & \frac{1}{2\sqrt{1+\frac{1}{4}(5-\sqrt{5})^2}} \end{pmatrix}.$$

In this case, $\Gamma^1_{P_2} = \left\{(det({P_2}^{-1})z, {P_2}^{-1}\xi, 0) \in \mathrm{Osc}_{1,0} \mid \xi \in \mathbb{Z}^2, z \in \frac{1}{2}\xi_1\xi_2 + \mathbb{Z}\right\}$.

C. Study of $\mathcal{L}_{\mathbb{S}_T}$ for small T

C.1. Computation of $\mathcal{S}_T$ for small T (Section 7.4)

Let's set $M_3 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathbb{S}_3$, $M_4 = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \in \mathbb{S}_4$, $M_5 = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \in \mathbb{S}_5$, $M_6 = \begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix} \in \mathbb{S}_6$ and $M'_6 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \in \mathbb{S}_6$.

T = 3. We get exactly one reduced cycle in H^0_3, equal to $\gamma^R_M(M_3, S, N) = [M_3, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}]$, which is thus conjugate and transposed to itself (this gives that M_3 is reversible). We choose M_3 as representative of the unique equivalence class of $\mathbb{S}_3$ for $\sim$.

T = 4. We find two disjoint reduced cycles transposed to each other in H^0_4: $\gamma^R_{M_4}(M_4, S, N^2) = [M_4, \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}]$ and ${}^t\gamma^R_{M_4}$. This gives only one equivalence class, for which we choose M_4 as a representative. Notice that $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \in \gamma^R_{M_4}$, which means that M_4 is reversible (Remark 7.2).

T=5. We find two distinct reduced cycles that are transposed to each other: $\gamma^R_{M_5}(M_5, S, N^3) = [M, \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}]$ and ${}^t\gamma^R_{M_5}$. This gives only one equivalence class, for which we choose M_5 as a representative. Notice that M_5 is reversible, because $\begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \in \gamma^R_{M_5}$.

T = 6. We get exactly 6 reduced cycles in H^0_6, which are $\gamma^R_{M_6} = \gamma^R_{M_6}(M, S, N^4) = [M_6, \begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix}]$, $\gamma^R_{M'_6} = \gamma^R_{M'_6}(M'_6, S^2, N^2) = [M'_6, \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}]$, together with their transposed and conjugate reduced cycles (this gives that M_6 and M'_6 are reversible). Then there are exactly two equivalence classes in $\mathbb{S}_6$, represented by M_6 and M'_6.

C.2. Examples of Symmetry Groups (see Section 8.1)

We take the notations of Annexe C.1. Recall that for $M \in \mathbb{S}_T$, $g(M)$ determines completely $C(M)$, because

$$C(M) = \{\pm g(M)^p \mid p \in \mathbb{Z}\}.$$

T = 3. We get $\lambda_3 = \frac{3}{2} + \frac{1}{2}\sqrt{5}$, and the fundamental unit of $\mathbb{Q}(_3) = \mathbb{Q}(\sqrt{5})$ is $\tau = \frac{1}{2}(1+\sqrt{5}) \neq \lambda_3$.

Then, if $P = x + yX$, $P(\lambda_3) = \tau$ if, and only if $P = -1 + X$. Then $P(M_3) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $p_0 = 1$ and

$$g(M_3) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{49}$$

T = 4. We get $\lambda_4 = 2 + \sqrt{3}$, where λ is the fundamental unit of $\mathbb{Q}(\lambda_4)$. Then $C(B) = \{\pm B^p \mid p \in \mathbb{Z}\}$. Thus

$$\forall B \in \mathbb{S}_4` \quad \phi_B(B) = \lambda_4 \quad \text{and} \quad g(B) = B \text{ (in particular for } M_4). \tag{50}$$

T = 5. We get $\lambda_5 = \frac{1}{2}(5 + \sqrt{21})$, which is the fundamental unit of $\mathbb{Q}(\sqrt{21}) = \mathbb{Q}(\lambda_5)$. Thus

$$\forall B \in \mathbb{S}_5` \quad \phi_B(B) = \lambda_5 \quad \text{and} \quad g(B) = B \text{ (in particular for } M_5). \tag{51}$$

T = 6. We have $\lambda_6 = 3 + 2\sqrt{2}$. The fundamental unit of $\mathbb{Q}(\lambda_6)$ is $\tau = 1 + \sqrt{2}$. Then $P = x + yX$ satisfies $P(\lambda_6) = \tau$ if, and only if, $P = \frac{-1}{2} + \frac{1}{2}\sqrt{2}$.
For M_6, this gives $P(M_6) \notin \mathrm{GL}(2, \mathbb{Z})$ and $P(M_6)^2 = M_6 \in \mathrm{GL}(2, \mathbb{Z})$, thus

$$g(M_6) = M_6. \tag{52}$$

For M_6', we get $P(M_6') = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$, and

$$g(M_6') = P(M_6'). \tag{53}$$

T = 7. We have $\lambda_7 = \frac{1}{2}(7 + 3\sqrt{5})$. A fundamental unit of $\mathbb{Q}(\lambda_7) = \mathbb{Q}(\sqrt{5})$ is $\tau = \frac{1}{2}(1 + \sqrt{5})$. For all $B = [[a, b][c, d]] \in \mathrm{SL}(2, \mathbb{Z})$ with trace 7, the unique element $P(B) = x + yB$ of $\mathbb{Q}(B)$ such that $\phi(P(B)) = P(\lambda) = \tau$ satisfies

$$y = \frac{1}{3} \quad \text{and} \quad x = \frac{-1}{6}.$$

But $\frac{2a-1}{6}$ cannot be in $\mathbb{Z}$ because $2a - 1$ is odd and 6 is even. Then

$$P(B) \notin \mathrm{GL}(2, \mathbb{Z}) \Longrightarrow \tau = \phi(P(B)) \notin \phi(C(B)) \Longrightarrow \{\pm 1\} \times \langle \tau \rangle \not\subset \phi(C(B))$$

and for a matrix B of $\mathbb{S}_7$, we never have $\phi(C(B)) = \mathcal{O}^\times_{\mathbb{Q}(\lambda_7)}$.

T = 8. We have $\lambda_8 = 4 + \sqrt{15}$ and the fundamental unit of $\mathbb{Q}(\lambda_8) = \mathbb{Q}(\sqrt{15})$ is λ_8. Thus for every matrix $B \in \mathbb{S}_8$, $\phi(B) = \lambda_8$ implies that $\phi(C(B)) = \mathcal{O}^\times_{\mathbb{Q}(\sqrt{15})}$, and

$$g(B) = B. \tag{54}$$

General case. λ_T is the root of $\Pi_T = X^2 - TX + 1$ that is greater than 1, ie $\lambda = \frac{T + \sqrt{T^2 - 4}}{2}$, and $\mathbb{Q}(\lambda) = \mathbb{Q}(\sqrt{T^2 - 4})$. Let d be a squarefree positive integer such that $T^2 - 4 = m^2 d$, $m \in \mathbb{Z}$. Then

$$\mathcal{O}_{\mathbb{Q}(\lambda_T)} = \mathbb{Z}(\alpha),$$

with $\alpha = \sqrt{d}$ if $d \equiv 2, 3 \quad [4]$ and $\alpha = \frac{1+\sqrt{d}}{2}$ if $d \equiv 1 \quad [4]$. Then for $d \equiv 2, 3 \quad [4]$ the fundamental unit of $\mathbb{Q}(\lambda)$ is equal to the minimal solution of the Pell Fermat equation $x^2 - dy^2 = \pm 1$.

C.3. Examples of reversing symmetry groups (Section 8.2)

We use the notations of Annexe C.1.

T = 3. M_3 is reversible (see Section C.1). Resolving the equation $K_3M_3 = M_3^{-1}K_3$ as in the method of Section 9, we get a solution $K_3 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$. The computation of Section C.2, equation (49), and formula 34 give

$$\mathcal{R}(M_3) = \left\{ \pm \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^p, \pm \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^p \cdot \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \mid p \in \mathbb{Z} \right\}. \tag{55}$$

T = 4. M_4 is reversible (see Section C.1). We get $K_4 = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}$ as a solution of $K_4M_4 = M_4^{-1}K_4$. The computation of Section C.2, equation (50), and the formula 34 give

$$\mathcal{R}(M_4) = \left\{ \pm M_4^p, \pm M_4^p K_4 \mid p \in \mathbb{Z} \right\}. \tag{56}$$

T = 5. M_5 is reversible (see Section C.1). We get $K_5 = \begin{pmatrix} -1 & 3 \\ -1 & 4 \end{pmatrix}$ as a solution of $K_5M_5 = M_5^{-1}K_5$. The computation of Section C.2, equation (51) and formula (34) give

$$\mathcal{R}(M_5) = \left\{ \pm M_5^p, \pm M_5^p K_5 \mid p \in \mathbb{Z} \right\}. \tag{57}$$

T = 6. M_6 is reversible (see Section C.1). We find that $K_6 = \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}$ satisfies $K_6M_6 = M_6^{-1}K_6$. The computation of Section C.2, equation (52), and the formula (34) give

$$\mathcal{R}(M_6) = \left\{ \pm M_6^p \cdot \begin{pmatrix} -1 & 4 \\ 0 & 1 \end{pmatrix}, \pm M_6^p \mid p \in \mathbb{Z} \right\}. \tag{58}$$

M_6' is reversible (see Section C.1). We find that $K_6' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ satisfies $K_6'M_6' = M_6'^{-1}K_6'$. The computation of Section C.2, equation (53) and the formula (34) give

$$\mathcal{R}(M_6') = \left\{ \pm \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^p \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^p \mid p \in \mathbb{Z} \right\}. \tag{59}$$

C.4. Computation of $\mathcal{L}_{\mathbb{S}_T}$ for small T (Section 9)

In this section we use the method given in Section 9 in order to determine the set $\mathcal{L}_{\mathbb{S}_T}$ for small values of $T > 2$.

T = 3. We have seen in Annexe C.1 that there is only one equivalence class in $\mathbb{S}_3$ for $\sim$, represented by M_3. Let $r \in \mathbb{N}_{>0}$. Let us determine $\mathcal{L}^r_{M_3}$. Let $(k,l) \in \mathbb{Z}^2$. Applying relation (ii) of Proposition 5.2, we get $(k,l) \sim (k + t_1 + t_2, l + t_1) \sim (k + t_1, l + t_2)$ for all $t_2, t_3 \in \mathbb{Z}$. Thus there is one equivalence class in $\mathbb{Z}^2$ for $\overset{M_3,r}{\sim}$, for any $r \in \mathbb{N}^*$, represented by $(0,0)$.

T = 4. In Annexe C.1, we have determined that there is only one equivalence class in $\mathbb{S}_4$, represented by M_4. Let's determine $\mathcal{L}^r_{M_4}$. Let $r \in \mathbb{N}_{>0}$. Let $(k,l) \in \mathbb{Z}^2$. Applying (ii) gives $(k,l) \overset{(ii)}{\sim} (k + 2t_1 + t_2, l + 2t_1)$ for all $(t_1, t_2) \in \mathbb{Z}$. This is equivalent to say that

$$(k,l) \overset{(ii)}{\sim} (k + (2 \wedge 1)t, l + 2t') \sim (k + t, l + 2t') \quad \forall (t,t') \in \mathbb{Z}^2.$$

Then each couple $(k,l) \in \mathbb{Z}^2$ is equivalent either to $(0,0)$ or $(0,1)$. Let $L = \Gamma(r, M_4, 0, 0)$ and $L' = \Gamma(r, M_4, 0, 1)$. In L, we have $[\delta,\alpha] = \delta\alpha\delta^{-1}\alpha^{-1} = \alpha^2\beta\gamma^r$, $[\delta,\beta] = \alpha^2$, $[\alpha,\beta] = \gamma^r$. Hence $L^{ab} := L/[L,L] = \langle \alpha,\beta,\gamma,\delta \mid \alpha,\beta,\gamma,\delta \text{ commute}, \alpha^2 = 1, \beta = 1, \gamma^r = 1\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_r \times \mathbb{Z}$ (where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$). Then the fundamental theorem of finite abelian groups gives that

$$L^{ab} \simeq \mathbb{Z}_2 \times \mathbb{Z}_r \times \mathbb{Z} \simeq \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_r \times \mathbb{Z} \text{ if } r \text{ is even ;} \\ \mathbb{Z}_{2r} \times \mathbb{Z} \text{ if } r \text{ is odd.} \end{cases}$$

On the other hand, in L', we have $[\delta,\alpha] = \delta\alpha\delta^{-1}\alpha^{-1} = \alpha^2\beta\gamma^r$, $[\delta,\beta] = \alpha^2\gamma$, $[\alpha,\beta] = \gamma^r$, thus $L'^{ab} = \langle \alpha,\beta,\gamma,\delta \mid \alpha,\beta,\gamma,\delta \text{ commute}, \alpha^2 = \beta^{-1} = \gamma^{-1}, \gamma^r = 1\rangle = \langle \alpha,\delta \mid \alpha,\delta \text{ commute}, \alpha^{2r} = 1\rangle \simeq \mathbb{Z}_{2r} \times \mathbb{Z}$. Then if r is even, $L^{ab} \simeq \mathbb{Z}_2 \times \mathbb{Z}_r \times \mathbb{Z} \not\simeq \mathbb{Z}_{2r} \times \mathbb{Z} \simeq L'^{ab}$ (by uniqueness of the decomposition in the fundamental theorem of finite abelian groups), and $L \not\simeq L'$. Thus $(0,0)$ and $(0,1)$ are not equivalent, and there are exactly two equivalence classes in $\mathbb{Z}^2$ for $\overset{M_4,r}{\sim}$, represented by $(0,0)$ and $(0,1)$.

If r is odd, applying (i) to $(0,0)$ gives $(0,0) \overset{M_4,r}{\sim} (0,r) \overset{(ii)}{\sim} (0, r - 2\frac{r-1}{2}) = (0,1)$, and there is only one equivalence class in $\mathbb{Z}^2$ for $\overset{M_4,r}{\sim}$.

T=5. In Annexe C.1, we have determined that there is only one equivalence class in $\mathbb{S}_5$, represented by M_5. Let's determine $\mathcal{L}^r_{M_5}$. Let $r \in \mathbb{N}_{>0}$. Let $(k,l) \in \mathbb{Z}$. Applying (ii) gives $(k,l) \overset{(ii)}{\sim} (k + 3t_1 + t_2, l + 3t_1)$ for all $(t_1,t_2) \in \mathbb{Z}$. This is equivalent to say that

$$(k,l) \sim (k + (3 \wedge 1)t, l + 3t') \sim (k + t, l + 3t') \quad \forall (t,t') \in \mathbb{Z}^2.$$

Therefore each couple $(k,l) \in \mathbb{Z}^2$ is equivalent either to $(0,0), (0,1)$ or $(0,-1)$. But $-I_2 \in \mathcal{R}(M_5)$ gives $(0,1) \overset{M_5,r}{\sim} (0,-1)$. Then each couple $(k,l) \in \mathbb{Z}^2$ is equivalent either to $(0,0)$ or $(0,1)$. Set $L = \Gamma(r, M_5, 0, 0)$ and $L' = \Gamma(r, M_5, 0, 1)$. In L, we have $[\delta,\alpha] = \alpha^3\beta\gamma^r$, $[\delta,\beta] = \alpha^3$, $[\alpha,\beta] = \gamma^r$, thus $L^{ab} = \langle \alpha,\beta,\gamma,\delta \mid \alpha,\beta,\gamma,\delta \text{ commute}, \alpha^3 = \gamma^r = \beta = 1\langle\alpha\rangle \simeq \mathbb{Z}_3 \times \mathbb{Z}_r \times \mathbb{Z}$. Then the fundamental theorem of finite abelian groups gives that

$$L^{ab} \simeq \begin{cases} \mathbb{Z}_3 \times \mathbb{Z}_r \times \mathbb{Z} \text{ if } r \equiv 0 \mod 3; \\ \mathbb{Z}_{3r} \times \mathbb{Z} \text{ if } r \equiv \pm 1 \mod 3. \end{cases}$$

On the other hand, in L', we have $[\delta,\alpha] = \alpha^3\beta\gamma^r$, $[\delta,\beta] = \alpha^3\gamma$, $[\alpha,\beta] = \gamma^r$, thus

$$L'^{ab} = \gamma^{-1}, \gamma^r = 1\rangle = \langle \alpha,\delta \mid \alpha,\delta \text{ commute}, \alpha^{3r} = 1\rangle \simeq \mathbb{Z}_{3r} \times \mathbb{Z}.$$

Then if 3 divides r, $L^{ab} \simeq \mathbb{Z}_3 \times \mathbb{Z}_r\mathbb{Z} \not\simeq \mathbb{Z}_{3r} \times \mathbb{Z} \simeq L'^{ab}$ (by uniqueness of the decomposition in the fundamental theorem of finite abelian groups), and $L \not\simeq L'$. Therefore $(0,0)$ and $(0,1)$ are not equivalent, and there are exactly two equivalence classes in $\mathbb{Z}^2$ for $\overset{M_5,r}{\sim}$, represented by $(0,0)$ and $(0,1)$. If $r \equiv 1 \mod 3$ (resp. $r \equiv -1 \mod 3$), applying (i) gives $(0,0) \overset{M_5,r}{\sim} (0,r) \overset{(ii)}{\sim} (0, r - 3\frac{r-1}{3}) = (0,1)$ (resp. $(0,0) \overset{M_5,r}{\sim} (0,r) \overset{(ii)}{\sim} (0, r - 3\frac{r+1}{3}) = (0,-1) \overset{M_5,r}{\sim} (0,1)$), and there is only one equivalence class in $\mathbb{Z}^2$ for $\overset{M_4,r}{\sim}$.

T = 6. In Section C.1, we have determined that there are exactly two equivalence classes in $\mathbb{S}_3$, represented by M_6 and M'_6 respectively.

Let's first determine $\mathcal{L}^r_{M_6}$. Let $r \in \mathbb{N}^*$. Let $(k,l) \in \mathbb{Z}^2$. We compute

$$L_1 = span_{\mathbb{Z}}\{(4,4), (1,0), (0,r), (r,0)\} = span_{\mathbb{Z}}\{(1,0), (0, 4 \wedge r)\}.$$

Then if r is odd, $L_1 = \mathbb{Z}^2$ and there is one equivalence class in $\mathbb{Z}^2$ for $\overset{M_6,r}{\sim}$, represented by $(0,0)$.

In the following, we suppose that $r = 2r'$ is even. If $r \equiv 2 \mod 4$, $L_1 = span_{\mathbb{Z}}\{(1,0),(0,2)\}$ and $\mathbb{Z}^2/L_1$ has for representatives $(1,0)$, $(0,1)$, and $(0,-1)$. The action of $-I_2$ on $\mathbb{Z}^2/L_1$ gives $\omega_{\mathcal{R}(M)}(0,1) = \omega_{\mathcal{R}(M)}(0,-1)$, hence $(0,1) \overset{M_6,r}{\sim} (0,-1)$. Then each couple of $\mathbb{Z}^2$ is equivalent either to $(0,0)$ or $(0,1)$. If $r \equiv 0 \mod 4$, $L_1 = span_{\mathbb{Z}}\{(1,0),(0,4)\}$, and $\mathbb{Z}^2/L_1$ has for representatives $(1,0)$, $(0,1)$, $(0,2)$ and $(0,-1)$, with again $\omega_{\mathcal{R}(M)}(0,1) = \omega_{\mathcal{R}(M)}(0,-1)$. Then each couple of $\mathbb{Z}^2$ is equivalent either to $(0,0)$, $(0,1)$ or $(0,2)$. Let's define $L = \Gamma(r, M_5, 0, 0)$, $L' = \Gamma(r, M_5, 0, 1)$ and $L'' = \Gamma(r, M_5, 0, 2)$. We find

$$\begin{cases} L^{ab} = \langle \alpha,\beta,\gamma,\delta \mid \alpha,\beta,\gamma,\delta \text{ commute}, \alpha^4 = \beta = \gamma^r = 1\rangle \simeq \mathbb{Z}_4 \times \mathbb{Z}_r \times \mathbb{Z} \simeq \begin{cases} \mathbb{Z}_4 \times \mathbb{Z}_r \times \mathbb{Z} \text{ if } r \equiv 0 \mod 4; \\ \mathbb{Z}_2 \times \mathbb{Z}_{2r} \times \mathbb{Z} \text{ if } r \equiv 2 \mod 4, \end{cases} \\ L'^{ab} = \langle \alpha,\delta \mid \alpha,\delta \text{ commute}, \alpha^{4r} = 1\rangle \simeq \mathbb{Z}_{4r} \times \mathbb{Z} \\ L''^{ab} = \langle \gamma' = \alpha\gamma, \alpha' = \alpha^2\gamma, \delta \mid \alpha',\delta,\gamma' \text{ commute}, \alpha'^2 = 1, \gamma'^{2r} = 1\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2r} \times \mathbb{Z}. \end{cases}$$

Then if $r \equiv 0 \mod 4$, L^{ab}, L'^{ab} and L''^{ab}, and thus L, L' and L'' are pairwise not isomorphic, and $(0,0)$, $(0,1)$ and $(0,2)$ are not equivalent. Then there are three equivalence classes in $\mathbb{Z}^2$ for $\overset{M_6,r}{\sim}$ for r odd, represented by $(0,0)$, $(0,1)$ and $(0,2)$. If $r \equiv 2 \mod 4$, L^{ab} and $L'ab$ are not isomorphic, therefore $(0,0)$ and $(0,1)$ are not equivalent and there are two equivalence classes in $\mathbb{Z}^2$ for $\overset{M_6,r}{\sim}$, represented by $(0,0)$ and $(0,1)$.

Now let's determine $\mathcal{L}^r_{M_6'}$. Let $(k,l) \in \mathbb{Z}^2$. We compute

$$L_1 := L_1(M_6', r) = span_{\mathbb{Z}}\{(4,2),(2,0),(0,r),(r,0)\} = span_{\mathbb{Z}}\{(r \wedge 2, 0),(0, 2 \wedge r)\}.$$

Then if r is odd, $L_1 = \mathbb{Z}^2$ and there is one equivalence class in $\mathbb{Z}^2$ for $\overset{M_6',r}{\sim}$, represented by $(0,0)$. In the following we suppose $r = 2r'$ is even. Then

$$L_1 = span_{\mathbb{Z}}\{(2,0),(0,2)\}$$

and $\mathbb{Z}^2/L_1 = \{(0,0),(0,1),(1,0),(1,1)\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Then each element of $\mathbb{Z}^2$ is equivalent either to $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$. As said in the method of Section 9, one just has to determine the action of $-I_2$, M_6' and K_6' on $\mathbb{Z}^2/L_1$ to completely determine the action of $\mathcal{R}(M)$ on $\mathbb{Z}^2/L_1$.

The action of K_6' gives $\omega_{\mathcal{R}(M)}(k,l) = \omega_{\mathcal{R}(M)}(-l,k) = \omega_{\mathcal{R}(M)}(l,k)$ (because in $\mathbb{Z}^2/L_1$, $(k,l) = (-k,l) = (k,-l) = (-k,-l)$). The action of M_6' gives $\omega_{\mathcal{R}(M)}(k,l) = \omega_{\mathcal{R}(M)}(k-2l, -2k+5l)$ and the action of $-I_2$ is trivial. Then for $(k,l),(k',l') \in \mathbb{Z}^2/L_1$,

$$\omega_{\mathcal{R}(M)}(k,l) = \omega_{\mathcal{R}(M)}(k',l') \iff (k',l') = (k,l) \quad \text{or} \quad (l,k).$$

Thus $\Pi_{L_1}(0,1)$ and $\Pi_{L_1}(1,0)$ are in the same orbit, and $(0,1)$ and $(1,0)$ are equivalent. But $\Pi_{L_1}(0,0)$, $\Pi_{L_1}(0,1)$ and $\Pi_{L_1}(1,1)$ are not in the same orbit, and $(0,0)$, $(0,1)$ and $(1,1)$ are pairwise not equivalent. Then

$$\mathbb{Z}^2/\overset{M_6',r}{\sim} \simeq \mathbb{Z}^2/L_1/\mathcal{R}(M) = \left\{\omega_{\mathcal{R}(M)}(\Pi_{L_1}(0,0)), \omega_{\mathcal{R}(M)}(\Pi_{L_1}(0,1)), \omega_{\mathcal{R}(M)}(\Pi_{L_1}(1,1))\right\}$$

and there are 3 distinct equivalence classes in $\mathbb{Z}^2$ for $\overset{M_6',r}{\sim}$ for r even, represented by $(0,0)$, $(0,1)$ and $(1,1)$.

D. Normalized Lattices

In this Section, we will use the following parametrization of $\mathbb{L}$

$$\mathcal{P}_2 = \left\{(r, P, x, \eta, z, s) \in \mathbb{N}^* \times \mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_{>0} \mid Pe^{sA}P^{-1} \in \mathrm{GL}(2,\mathbb{Z}), (Pe^{sA}P^{-1}, x) \in \mathcal{B}_r\right\}.$$

It is a parametrization via the map

$$p_2 : \begin{cases} \mathcal{P}_2 \longrightarrow \mathbb{L} \\ (r, P, x, \eta, z, s) \mapsto L(r, Pe^{sA}P^{-1}, P, x, \eta, z). \end{cases} \tag{60}$$

This parametrization is actually equivalent to the parametrisation $\mathcal{P}_1$, because $\mathcal{P}_1$ and $\mathcal{P}_2$ are in one-to-one correspondance via the map

$$\begin{cases} \mathcal{P}_2 \simeq \mathcal{P}_1 \\ (r, P, x, \eta, z, s) \mapsto (r, Pe^{sA}P^{-1}, P, x, \eta, z). \end{cases} \tag{61}$$

This gives the following parametrization of $\mathbb{L}_\mathbb{S}$ (which is equivalent to $\mathcal{P}_1^\mathbb{S}$):

$$\mathcal{P}_2^\mathbb{S} = \left\{(r, P, x, s) \in \mathbb{N}^* \times \mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}^2 \times \mathbb{R}_{>0} \mid Pe^{sA}P^{-1} \in \mathrm{GL}(2,\mathbb{Z}), (Pe^{sA}P^{-1}, x) \in \mathcal{B}_r\right\}$$

via the map

$$p_2^\mathbb{S} : \begin{cases} \mathcal{P}_2^\mathbb{S} \simeq \mathbb{L} \\ (r, P, x, s) \mapsto L(r, Pe^{sA}P^{-1}, P, x, 0, 0) = p_2(r, P, x, 0, 0, s). \end{cases} \tag{62}$$

In order to avoid confusions with the other parametrization $\mathcal{P}_1$, we will denote $L[r, P, x, \eta, z, s]$ (in brackets) the lattice $L(r, Pe^{sA}P^{-1}, x, \eta, z) = p_2(r, P, x, \eta, z, s)$, with $(r, P, x, \eta, z, s) \in \mathcal{P}_2$.

Definition D.1. *Let $L \in \mathbb{L}$. Let $\bar{L}_0$ denote the projection of $L_0 = L \cap H$ on $H/Z(H) \simeq \mathbb{R}^2$. the lattice L is called normalized if $\bar{L}_0$ is a normalized lattice of $\mathbb{R}^2$, i.e if it has covolume one with respect to the metric of $\mathbb{R}^2$.*

We will denote $\mathbb{L}_0$ the class of all normalized lattices of $\mathrm{Osc}_{1,0}$.
Let $(r, P, \eta, z, x, s) \in \mathcal{P}_2$ and $L = L[r, P, \eta, z, x, s]$. Then $\bar{L}_0 = \bar{L \cap H} = \{P^{-1}\xi \mid \xi \in \mathbb{Z}^2\}$ has covolume 1 if, and only if, $\det(P) = \pm 1$. Then the set $\mathbb{L}_0$ is parametrised by the following subset of $\mathcal{P}_2$:

$$\mathcal{P}_2^\mathcal{L} = \left\{(r, P, \eta, z, x, s) \in \mathbb{N}^* \times \mathrm{GL}(2,\mathbb{R}) \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}_{>0} \mid B_{P,s} \in \mathrm{GL}(2,\mathbb{Z}), (B_{P,s}, x) \in \mathcal{B}_r, \det(P) = \pm 1\right\},$$

via the restriction of p_2 to $\mathbb{P}_2^\mathcal{L}$.

One can wonder when two normalized lattices with different parameters are equal. It would give more information, not only on the parametrization of normalized lattices via $p_{2|\mathcal{L}}$ but also on the one of the set of lattices via p_2 (and thus via p_1 too, since they are equivalent), because for each lattice we can find a simple and explicit bijection between it and a normalized lattice (see Remark D.2). In next Proposition D.1, we aim at answering this question. Let's first recall that

$$\Gamma_P^r = \langle \alpha_P, \beta_P, \gamma_P \rangle \quad \text{with} \quad \alpha_P = (0, P^{-1}e_1, 0), \quad \beta_P = (0, P^{-1}e_2, 0) \quad \text{and} \quad \gamma_P = \left(\frac{\det(P^{-1})}{r}, 0, 0\right).$$

Thus $\langle \Gamma^r_P, (\det(P^{-1})z, P^{-1}x, s)\rangle = \langle \alpha_P, \beta_P, \gamma_P, \delta_P\rangle$ with $\delta_P = (\det(P^{-1})z, P^{-1}x, s)$, and

$$L(r,P,\eta,z,x,s) = \langle \alpha'_P, \beta'_P, \gamma'_P, \delta'\rangle \text{ with } \begin{cases} \alpha'_P = F_{P^{-1}\eta}(\alpha_P) = (\mu\omega(\eta,e_1), P^{-1}e_1, 0) = (-\mu\eta_2, P^{-1}e_1, 0), \\ \beta'_P = (\mu\eta_1, P^{-1}e_2, 0) = (\mu\omega(\eta,e_2), P^{-1}e_2, 0), \\ \gamma'_P = (\frac{\mu}{r}, 0, 0), \\ \delta' = \left(\mu z + \frac{\mu}{2}\left[\omega(B\eta,\eta) + \omega((B+I_2)\eta, x)\right], P^{-1}(x-(B-I_2)\eta), s\right). \end{cases} \tag{63}$$

where we have denoted $B = B_{P,s}$ and $\mu = \det(P)$.

Proposition D.1. * *Let $(r,P,x,\eta,z,s) \in \mathcal{P}_2$ and $\tilde r \in \mathbb{N}^*, \tilde P \in \mathrm{SL}(2,\mathbb{R})^{\pm 1}, \tilde s \in \mathbb{R}_{>0}, \in \mathbb{R}^2, \tilde\eta \in \mathbb{R}^2$ such that $(B_{\tilde P,\tilde s}, \tilde x) \in \mathcal{B}_{\tilde r}$ and $B_{\tilde P,\tilde s} \in \mathbb{S}$ and $z \in \mathbb{R}$. There exists $\tilde z \in \mathbb{R}$ such that $L[r,P,x,\eta,z,s] = L[\tilde r,\tilde P,\tilde x,\tilde \eta,\tilde z,\tilde s]$ if, and only if,*

$$\begin{cases} r = \tilde r, s = \tilde s, \\ P\tilde P^{-1} = K \in \mathrm{GL}_2(\mathbb{Z}), \\ (K, K\tilde\eta - \eta) \in \mathcal{B}_r, \\ (x - (B_{P,s} - I_2)\eta) - K\left(\tilde x - (B_{\tilde P,s} - I_2)\tilde\eta\right) \in \mathbb{Z}^2. \end{cases} \tag{64}$$

Proof of Proposition D.1: Suppose that such a $\tilde z$ exists. Let's denote $L = L[r,P,x,\eta,z,s]$ and $\tilde L = L[\tilde r,\tilde P,\tilde x,\tilde\eta,\tilde z,\tilde s]$, $B = B_{P,s}$, $\tilde B = B_{\tilde P,\tilde s}$. We will also denote $\mu = \det(P) = \det(P)^{-1}$ and $\tilde\mu = \det(\tilde P) = \det(\tilde P)^{-1}$ (because $\det(P)$, $\det(\tilde P) \in \{-1,1\}$). Recall that by definition

$$L = F_{P^{-1}\eta}\left(\langle \Gamma^r_P, (\mu z, P^{-1}x, s)\rangle\right) \quad \text{and} \quad \tilde L = F_{\tilde P^{-1}\tilde\eta}\left(\langle \Gamma^{\tilde r}_{\tilde P}, (\tilde\mu\tilde z, \tilde P^{-1}\tilde x, \tilde s)\rangle\right).$$

Proposition 4.4 gives that $r = \tilde r$ and $s = \tilde s$. Moreover, $F_{P^{-1}\eta}(\Gamma^r_P) = L \cap H = \tilde L \cap H = F_{\tilde P^{-1}\tilde\eta}(\Gamma^{\tilde r}_{\tilde P})$, which means that

$$\Gamma_r = \bar F_P \circ \bar F_{\tilde P^{-1}\tilde\eta - P^{-1}\eta} \circ \bar F_{\tilde P^{-1}}(\Gamma_r) = \bar F_{P\tilde P^{-1}\tilde\eta - \eta} \circ F_{P\tilde P^{-1}}(\Gamma_r).$$

Thus, according to Lemma 4.1, $\left(P\tilde P^{-1}, \left(P\tilde P^{-1}\tilde\eta - \eta\right)\right) \in \mathcal{B}_r$. Set $K = P\tilde P^{-1} \in \mathrm{GL}(2,\mathbb{Z})$.

The equality between L and $\tilde L$ implies that in particular $F_{P^{-1}\eta}(\mu z, P^{-1}x, s) \in L = \tilde L$. By definition of $\tilde L$, this means that there exists $(k,l,0) \in \Gamma_r$ such that

$$(\mu z, P^{-1}x, s) = F_{\tilde P^{-1}\tilde\eta - P^{-1}\eta}(\tilde\mu k, \tilde P^{-1}l, 0) F_{\tilde P^{-1}\tilde\eta}(\tilde\mu\tilde z, \tilde P^{-1}\tilde x, s) = F_{P^{-1}D}(\tilde\mu Z, \tilde P^{-1}X, \tilde s)$$

with $D = \tilde\eta - K^{-1}\eta \in \mathbb{R}^2$, $Z = (k + \tilde z + \frac{1}{2}\omega(l,\tilde x)) \in \mathbb{R}$ and $X = l + \tilde x \in \mathbb{R}^2$. Notice that $(k,l,0) \in \Gamma_r$ implies that $l \in \mathbb{Z}$, and since $K \in \mathrm{GL}(2,\mathbb{Z})$, $Kl \in \mathbb{Z}^2$. From the same computation as the one of δ' from $_P$ in (63), we have:

$$F_{\tilde P^{-1}D}(\tilde\mu Z, \tilde P^{-1}X, \tilde s) = \left(\tilde\mu Z + \frac{\tilde\mu}{2}\left[\omega(\tilde BD, D) + \omega([\tilde B + I_2]D, X)\right], \tilde P^{-1}[X - (\tilde B - I_2)D], \tilde s\right). \tag{65}$$

Thus by identification, $x = \tilde P^{-1}[X - (\tilde B - I_2)D]$, which gives :

$$x - (B - I_2)\eta - K(\tilde x - (\tilde B - I_2)\tilde\eta) = Kl \in \mathbb{Z}^2.$$

Notice that in this case, by identification, we have

$$\begin{aligned}\tilde{\mu}\mu z - \tilde{z} &= k + \frac{1}{2}\left[\omega(\tilde{B}D, D) + \omega((\tilde{B}+I_2)D, \tilde{x}+l)\right] \\ &\in \frac{1}{2}\left[\omega(\tilde{B}D, D) + \omega((\tilde{B}+I_2)D, \tilde{x}+l)\right] + \frac{l_1 l_2}{2} + \frac{\mathbb{Z}}{r}.\end{aligned} \tag{66}$$

Reciprocally, we suppose that (64) is satisfied by (r, P, x, η, z, s) and $\tilde{r}, \tilde{P}, \tilde{\eta}, \tilde{s}$. Let's show that $L = \tilde{L}$. According to Lemma 4.1, since $(K, K\tilde{\eta} - \eta) \in \mathcal{B}_r$, we have $F_{P^{-1}\eta}(\Gamma_P^r) = F_{\tilde{P}^{-1}\tilde{\eta}}(\Gamma_{\tilde{P}}^r) \subset \tilde{L}$. Thus we only need to show that

$$F_{P^{-1}\eta}(\mu z, P^{-1}x, s) \in \tilde{L} \quad \text{and} \quad F_{\tilde{P}^{-1}\tilde{\eta}}(\tilde{\mu}\tilde{z}, \tilde{P}^{-1}\tilde{x}, s) \in L. \tag{67}$$

Let's set $l = K^{-1}(x - (B - I_2)\eta) - (\tilde{x} - (\tilde{B} - I_2)\tilde{\eta}) \in \mathbb{Z}^2$ and $k \in \frac{l_1 l_2}{2} + \frac{\mathbb{Z}}{r}$ and $D = \tilde{\eta} - K^{-1}\eta$. Then one can find $\tilde{z} \in \mathbb{R}$ satisfying (66). Then

$$F_{P^{-1}\eta}(\mu z, P^{-1}x, s) = F_{\tilde{P}^{-1}\tilde{\eta}}(\tilde{\mu}k, \tilde{P}^{-1}l, 0) F_{\tilde{P}^{-1}\tilde{\eta}}(\tilde{\mu}\tilde{z}, \tilde{P}^{-1}\tilde{x}, s) \in \tilde{L}. \tag{68}$$

Since $F = F_P \circ F^{-1}_{P^{-1}\eta} \circ F_{\tilde{P}^{-1}\tilde{\eta}} \circ F_{\tilde{P}^{-1}}$ is an automorphism of H that maps Γ_r onto itself, there exists $(\tilde{k}, \tilde{l}, 0) \in \Gamma_P^r$ such that $F(\tilde{k}, \tilde{l}, 0) = (k, l, 0)$. Then $F_{\tilde{P}^{-1}\tilde{\eta}}(\tilde{\mu}k, \tilde{P}^{-1}l, 0) = F_{P^{-1}\eta}(\mu\tilde{k}, P^{-1}\tilde{l}, 0)$ and

$$F_{\tilde{P}^{-1}\tilde{\eta}}(\tilde{\mu}\tilde{z}, \tilde{P}^{-1}\tilde{x}, s) = F_{P^{-1}\eta}(\mu\tilde{k}, P^{-1}\tilde{l}, 0) F_{P^{-1}\eta}(\mu z, P^{-1}x, s) \in L. \tag{69}$$

Then according to (67), (68) and (69), $L = \tilde{L}$. □

Remark D.1. *The proof of Proposition D.1 actually gives a more precise result:*
Let $(r, P, x, \eta, z, s), (\tilde{r}, \tilde{P}, \tilde{x}, \tilde{\eta}, \tilde{z}, \tilde{s}) \in \mathcal{P}_2$. $L[r, P, x, \eta, z, s] = L[\tilde{r}, \tilde{P}, \tilde{x}, \tilde{\eta}, \tilde{z}, \tilde{s}]$ *if, and only if,*

$$\begin{cases} r = \tilde{r}, s = \tilde{s}, \\ P\tilde{P}^{-1} = K \in \mathrm{GL}_2(\mathbb{Z}), \\ (K, KD) \in \mathcal{B}_r \quad \text{with} \quad D = \tilde{\eta} - K^{-1}\eta, \\ l =^t (l_1, l_2) := K^{-1}\left(x - (B_{P,s} - I_2)\eta\right) - \left(\tilde{x} - (B_{\tilde{P},s} - I_2)\tilde{\eta}\right) \in \mathbb{Z}^2, \\ \tilde{\mu}\mu z - \tilde{z} \in \frac{1}{2}\left[\omega(\tilde{B}D, D) + \omega((\tilde{B}+I_2)D, \tilde{x}+l)\right] + \frac{l_1 l_2}{2} + \frac{\mathbb{Z}}{r}, \end{cases} \tag{70}$$

where $\mu = \det(P)$ *and* $\tilde{\mu} = \det(\tilde{P})$.

We denote $\mathrm{Aut}^1(\mathrm{Osc}_{1,0})$ the subgroup of $\mathrm{Aut}(\mathrm{Osc}_{1,0})$ of all the automorphisms of $\mathrm{Osc}_{1,0}$ which map all normalized lattices on normalized ones. These maps are described in next Proposition.

Proposition D.2. * *Let* $F = F_\eta \circ F_u \circ F_S$ *be in* $\mathrm{Aut}(\mathrm{Osc})$. F *is in* $\mathrm{Aut}^1(\mathrm{Osc}_{1,0})$ *if, and only if,* $\det(S) = \pm 1$.

Proof of Proposition D.2: Let's set $F = F_u \circ F_\delta \circ F_S$. For all $L = L[r, P, x, \eta, z, s]$ being a normalized lattice, $F(L) = \langle F \circ F_\eta(\Gamma_P^r), F \circ F_\eta(z, P^{-1}x, s)\rangle$, and:

$$F \circ F_\eta(\Gamma_P^r) = \bar{F}_\delta \circ \bar{F}_S \circ \bar{F}_\eta(\Gamma_P^r) = \bar{F}_K \circ \bar{F}_{SP^{-1}}(\Gamma_r) = \bar{F}_K(\Gamma^r_{PS^{-1}}),$$

with $K = \delta + PS^{-1}\eta$. Thus $F(L) = L[r, PS^{-1}, x', PS^{-1}K, z', s]$ (where z' and x' satisfy $F_\eta(z, P^{-1}x, s) = F_K(z', SP^{-1}x', s)$), and we have seen that $F(L)$ is normalized if, and only if, $\det(PS^{-1}) = \pm 1$ if, and only if, $\det(S) = \pm 1$. □

For fixed $s \in \mathbb{R}_{>0}$ and $r \in \mathbb{N}_{>0}$, set $\mathcal{P}_{r,s} = \{(P,\eta,z,x) \mid (r,P,x,\eta,z,s) \in \mathcal{P}_2\}$ and $P_0^{r,s} = \mathcal{P}_{r,s}/\sim$, where $\sim$ is defined by

$$(P,\delta,z,x) \sim (P',\delta',z',x') \text{ if, and only if } L[r,P,x,\delta,z,s] = L[r',P',x',\delta',z',s'].$$

$P_0^{r,s}$ is in one-to-one-correspondence with the set of all normalized lattices $L[\tilde{r},P,x,\eta,\tilde{s}]$ of $\mathrm{Osc}_{1,0}$ such that $\tilde{r} = r$ and $\tilde{s} = s$. Let $[P,x,\eta,z]$ denote the projection of $(P,x,\eta,z) \in \mathcal{P}_{r,s}$ on $P_0^{r,s}$. We define the action of $\mathrm{Aut}^1(\mathrm{Osc}_{1,0})$ on $\mathcal{L}_{r,s}$ by

$$F.[P,x,\eta,z] = [P',x',\eta',z'],$$

where (r,P',x',η',z',s') parametrizes $F(L[r,P,x,\delta,z,s])$. This action is well defined by definition of $P_0^{r,s}$ and $\mathrm{Aut}^1(\mathrm{Osc}_{1,0})$.

Proposition D.3. * *For $u \in \mathbb{R}$, F_u acts on $P_0^{r,s}$ by*

$$F_u.\,[P,x,\delta,z] = \left[P,x,\delta,z + \frac{u}{\det(P^{-1})}s\right].$$

For $S \in \mathrm{GL}(2,\mathbb{R})$ such that $SA = AS$ and $\det(S) = \pm 1 = \nu$, F_S acts on $\mathbb{P}_0^{r,s}$ by

$$F_S.\,[P,x,\eta,z] = \left[PS^{-1},x,\eta,\nu z\right].$$

For $S \in \mathrm{GL}(2,\mathbb{R})$ such that $SA = -AS$ and $\det(S) = \pm 1 = \nu$, F_S acts on $P_0^{r,s}$ by

$$F_S.\,[P,x,\eta,z] = \left[PS^{-1},B_{P,s}x,\eta,-\nu z\right].$$

For $\xi \in \mathbb{R}^2$, F_ξ acts on $P_0^{r,s}$ by

$$F_\xi.[P,x,\eta,z] = [P,x,\eta + P\xi,z].$$

Proof of Proposition D.3: Let $L = L[r,P,x,\eta,z,s] = \langle \alpha'_P, \beta'_P, \gamma'_P, \delta' \rangle$, with generators defined in (63). We have:

- $F_u(L) = \langle F_u(\alpha'_P), F_u(\beta'_P), F_u(\gamma'_P), F_u(\delta')\rangle = \langle \alpha'_P, \beta'_P, \gamma'_P, \delta''\rangle$, with $\delta'' = F_{P^{-1}\eta}(\det(P^{-1})(z + \frac{u}{\det(P^{-1})}s), P^{-1}x, s)$. Thus $F_u(L) = F_{P^{-1}\eta}(\langle \Gamma^r_P, (z+us, P^{-1}x, s)\rangle)$. Hence the result on F_u.
- $F_\xi(L) = F_{P^{-1}(\eta+P\xi)}(\langle \Gamma^r_P, (z, P^{-1}x, s)\rangle)$. Hence the result on F_ξ.
- $F_S(\alpha'_P) = (\mu\nu\omega(\eta,e_1), SP^{-1}e_1, 0) = F_{SP^{-1}\eta}(\alpha_{PS^{-1}})$, and similarly $F_S(\beta'_P) = F_{SP^{-1}\eta}(\beta_{PS^{-1}})$ and $F_S(\gamma'_P) = F_{SP^{-1}\eta}(\gamma_{PS^{-1}})$. Finally, we have

  $$F_S(\delta') = F_{SP^{-1}\eta}(\det(S)z, SP^{-1}x, \mu s), \quad \text{with} \quad \mu = \pm 1 \quad \text{such that} \quad AS = \mu SA.$$

  If $AS = SA$, then $F_S(\delta') = F_{SP^{-1}\eta}(\nu z, SP^{-1}x, s) \Rightarrow F_S(L) = F_{SP^{-1}\eta}(\langle \Gamma^r_{PS^{-1}}, (\nu z, SP^{-1}x, s)\rangle)$. Hence the result.
  If $SA = -AS$, then

  $$F_S(\delta') = F_{SP^{-1}\eta}(\nu z, SP^{-1}x, -s) = F_{SP^{-1}\eta}(-\nu z, -e^{sA}SP^{-1}x, s)^{-1} = F_{SP^{-1}\eta}(-\nu z, SP^{-1}Bx, s)^{-1}$$
  $$\Rightarrow F_S(L) = F_{SP^{-1}\eta}(\langle \Gamma^r_{PS^{-1}}, (-\nu z, SP^{-1}Bx, s)^{-1}\rangle) = F_{SP^{-1}\eta}(\langle \Gamma^r_{PS^{-1}}, (-\nu z, SP^{-1}B_P x, s)\rangle).$$

  Hence the result on F_S.

□

Remark D.2. *If $L \in \mathbb{L}$, then L is equivalent to a normalized lattice. In particular, if $L = L[r,P,x,\eta,z,s]$ with $(r,P,x,\eta,z,s) \in \mathcal{P}_2$, and if we set*

$$S_P = \sqrt{|\det(P)|}I_2,$$

then $S_P A = AS_P$, and Theorem 3.1 gives that F_{S_P} is an automorphism of $\mathrm{Osc}_{1,0}$. Thus

$$F = F_{S_P P^{-1}\eta} \circ F_{S_P} \circ F_{-P^{-1}\eta},$$

is an automorphism of $\mathrm{Osc}_{1,0}$ such that $F(L) = L[r, PS_P^{-1}, x, \eta, z, s]$. Since $\det(PS_P^{-1}) = sgn(\det(P)) = \pm 1$, $F(L)$ is a normalized lattice, and L is isomorphic to this normalized lattice via F. If we denote $\tilde{P} = PS_P^{-1}$, then $F(L) = L[\tilde{P}, x, \eta, z]$.

With all of it, it is possible to prove :

Proposition D.4. * *Let $L = L^r_{B,P,x}$ and $L' = L^{r'}_{B,P,x'}$ be two lattices of $\mathrm{Osc}_{1,0}$. $L \sim L'$ if, and only if, $r = r'$ and there exist $\mu \in \{-1,1\}$, $K_\mu \in \mathrm{GL}(2,\mathbb{Z})$ such that $K_\mu B = B^\mu K_\mu$ and $\eta \in \mathbb{R}^2$ such that $(K_\mu, \eta) \in \mathcal{B}_r$ and $K_\mu x - x' \in (B^\mu - I_2)\eta + \mathbb{Z}^2$.*
Moreover, there exists $u \in \mathbb{R}$ such that $F = F_u \circ F_{\tilde{P}^{-1}\eta} \circ F_{P^{-1}K_\mu B_\mu^{-1} P}$ (with $\tilde{P} = PS_P^{-1}$, and $B_\mu = B$ if $\mu = -1$ and I_2 if $\mu = 1$) maps L onto L' if, and only if, $(K_\mu, \eta) \in \mathcal{B}_r$ and $K_\mu x - x' \in (B^\mu - I_2)\eta + \mathbb{Z}^2$.

Proof of Proposition D.4 : We will denote $s = s_B$. We have seen in Proposition 4.4 that if $r \neq r'$, then L and L' are not isomorphic. Now suppose that $r = r'$. With notations of Remark D.2, $F_{S_P}(L) = L[\tilde{P}, x, 0, 0]$ and $F_{S_P}(L') = L[\tilde{P}, x', 0, 0]$ are normalized lattices, with $(\tilde{P}, x, 0, 0), (\tilde{P}, x', 0, 0) \in \mathcal{P}_{r,s}$. L and L' are isomorphic if, and only if, the normalized lattices associated are. Let $F \in \mathrm{Aut}(\mathrm{Osc})$. F can always be written

$$F = F_u \circ F_{\tilde{P}^{-1}\eta} \circ F_S,$$

with $SA = \mu AS$, $\eta \in \mathbb{R}^2$ and $u \in \mathbb{R}$. A computation gives $F_{S_P} \circ F \circ F_{S_P^{-1}} = F_{\det(S_P)u} \circ F_{\tilde{P}^{-1}\eta} \circ F_S$. L and L' are isomorphic via F if, and only if $\tilde{F} = F_{\det(PS)u} \circ F_{\tilde{P}^{-1}\eta} \circ F_S$ maps a normalized lattice onto a normalized one (which is equivalent to $\det(S) = \pm 1$), and in particular maps $L[\tilde{P}, 0, x, 0]$ onto $L[\tilde{P}, 0, x', 0]$. But a computation gives

$$\tilde{F}.(\tilde{P}, x, 0, 0) = \left(\tilde{P}S^{-1}, B_\mu x, \tilde{P}S^{-1}\tilde{P}^{-1}\eta, \frac{u}{\det(P) sgn(P)} s \right),$$

with $B_\mu = B$ if $\mu = -1$ and I_2 if $\mu = 1$. Then, according to Proposition D.1, there exists u such that $\tilde{F}.[\tilde{P}, x, 0, 0] = [\tilde{P}, x, 0', 0]$ if, and only if,

$$\begin{cases} K = PSP^{-1} \in \mathrm{GL}(2,\mathbb{Z}), \\ (K,\eta) \in \mathcal{B}_r \\ KB_\mu x - K(B - I_2)K^{-1}\eta) - x' \in \mathbb{Z}^2. \end{cases}$$

Let's set $K_\mu = PSP^{-1}B_\mu$. It is clear that $PSP^{-1} \in \mathrm{GL}(2,\mathbb{Z})$ if, and only if $K_\mu \in \mathrm{GL}(2,\mathbb{Z})$, and a computation gives $SA = \mu AS$ if, and only if, $K_\mu B = B^\mu K_\mu$. Then the condition $KB_\mu x - K(B - I_2)K^{-1}\eta) - x' \in \mathbb{Z}^2$ becomes $K_\mu x - x' \in (B^\mu - I_2)\eta + \mathbb{Z}^2$. □

Proposition D.4 actually gives the same result as Theorem 5.2. However, since the point of view is different, we have an explicit form of the automorphism, while Theorem 5.2 allows us to translate the result in terms of group actions and is more useful if one wants to describe $\mathcal{L}$ via an explicit bijection with a simpler set.

To finish with this Section on normalized lattices, let us give another Proof of the fact that the equivalence class of $L^r_{B,P,x}$ does not depend on $P \in C_B$. This proof is interesting because it gives the explicit form of an automorphism of $\mathrm{Osc}_{1,0}$ mapping $L^r_{B,P,x}$ on $L^r_{B,\tilde{P},x}$.

Proposition D.5. * *If $P, \tilde{P} \in C_B$, then $L^r_{B,P,x}$ and $L^r_{B,\tilde{P},x}$ are isomorphic as lattices.*

Proof of Proposition D.5: Let's make an observation: For $B \in \mathcal{G}$, $P, \tilde{P} \in C_B$, we have $Pe^{sA}P^{-1} = \tilde{P}e^{sA}\tilde{P}^{-1}$. Then $K := \tilde{P}^{-1}P$ commutes with e^{sA}, which is diagonalizable with two distinct eigenvalues. A classical result then gives $K = Q(e^{sA})$ with $Q \in \mathbb{R}[X]$. This implies $P = \tilde{P}Q(e^{sA})$. Reciprocally, if $P = \tilde{P}Q(e^{sA})$ with $\tilde{P} \in C_B$ and $Q \in \mathbb{R}[X]$, then it is clear that $P \in C_B$.

With this observation and the tools of Appendix D, we are able to show D.5: one can assume $|\det(P)| = |\det(\tilde{P})| = 1$, from the same arguments as these of Proof of Proposition D.4. There exists $Q \in \mathbb{R}[X]$ such that $P = \tilde{P}Q(e^{sA})$.

Let's set $S = Q(e^{sA})$. Then it is clear that $SA = AS$ and that F_S is an automorphism of Osc. $F_S.[P, x, 0, 0] = [PQ(e^{sA})^{-1}, x, 0, 0] = [\tilde{P}, x, 0, 0]$, with $(P, 0, x, 0), (\tilde{P}, x, 0, 0) \in \mathcal{P}_{r,s}$. Thus $L^r_{B,P,x}$ and $L^r_{B,\tilde{P},x}$ are isomorphic as lattices via F_S. □

E. Another point of vew using the Continued Fractions

Another point of vew for the classification of $\mathrm{SL}(2,\mathbb{Z})$ up to conjugation by $\mathrm{GL}(2,\mathbb{Z})$ was developped by Karpenkov in "Geometry of Continued Fractions", Chapter 7. Here we use a different definition of **reduced matrix**, which is given in [5]:

Definition E.1. *A matrix $B = [[a, b][c, d]]$ is said to be* ***reduced*** *if $d > c \geq a \geq 0$.*

Karpenkov links the conjugacy class of a matrix of $\mathrm{SL}(2,\mathbb{Z})$ by $\mathrm{GL}(2,\mathbb{Z})$ to its *LLS period*, defined in his chapter 4 for angles and in Chapter 7 for matrices. We refer to these Chapters for more explanations on these notions. Recall some of the result he proves in Chapter 7:

- Two real spectrum $\mathrm{SL}(2,\mathbb{Z})$ matrices M_1 and M_2 with positive eigenvalues are integer conjugate if and only if their *LLS periods* coincide, i.e are equal up to cyclic permutation of the elements of the sequence. (Proposition 7.11, [5], Chapter 7)

- For every real spectrum matrix A in $\mathrm{SL}(2,\mathbb{Z})$ either A or $-A$ is integer conjugate to a reduced matrix. (Theorem 7.13, [5], Chapter 7)

- Consider a reduced matrix $M = [[a, b], [c, d]]$ and suppose that $\frac{c}{a} = [a_1; a_2 : ... : a_{2n-1}]$ and $\lambda = \left\lfloor \dfrac{d-1}{c} \right\rfloor$. Then its *LLS* period is

$$(a_1, a_2, ..., a_{2n-1}, \lambda).$$

  If $a = 0$, we have $M = [[0, -1], [1, \lambda + 2]]$. Then the *LLS* period for M is $(1, \lambda)$. (Theorem 7.14, [5], Chapter 7)

- For an arbitrary reduced matrix M we associate the corresponding sequence $(a_1, a_2, ..., a_{2n-1}, \lambda)$ (or $(1, \lambda)$ respectively) as in Theorem 7.14 of [5]. (Remark 7.16, [5], Chapter 7)
- The set of real spectrum reduced matrices is in one-to-one correspondence (defined in Remark 7.15) with the set of finite sequences consisting of an even number of positive integer elements. (Corollary 7.16, [5], Chapter 7).

In this section we try to describe the set $\mathbb{S}/\operatorname{GL}(2,\mathbb{Z})$ via a bijection with a set easier to study. Denote $\mathcal{R}$ the set of all real spectrum reduced matrices and $\mathcal{N}$ the set of finite sequences consisting in an even number positive integers. Two elements $(a_1, ..., a_{2n}), (b_1, ..., b_{2p}) \in \mathcal{N}$ are said equivalent if, and only if, they are equal up to cyclic permutation of the elements of the sequence, i.e if $n = p$ and there exists $k \in \mathbb{Z}/2n\mathbb{Z}$ such that

$$(a_{1+k}, ..., a_{2n+k}) = (b_1, ..., b_{2n}).$$

We will write $a \equiv b$ if $a = (a_1, ..., a_{2n})$ and $b = (b_1, ..., b_{2n})$ are equivalent for this relation. The set $\mathcal{N}/\equiv$ is actually the set of finite cyclic sequences consisting in an even number positive integers. The direct consequence of Proposition 7.11 and Corollary 7.16 of [5] is that $\mathcal{R}$ is in one-to-one correspondence with $\mathcal{N}/\equiv$, via the map being the composition of the map of Remark 7.16 and of the projection of $\mathcal{N}$ on $\mathcal{N}/\equiv$.

Theorem E.1. *Each matrix $A \in \mathbb{S}$ is integer conjugate to a reduced matrix.*

Proof of Theorem E.1: Let $A \in \mathbb{S}$. According to Theorem 7.13 of [5], Chapter 7, A or $-A$ is conjugate to a reduced matrix. But it can't be $-A$ because $tr(-A) < -2 < 0$, and a reduced matrix always has a positive trace. Thus it is A which is conjugate to a reduced matrix. □

Proposition E.1. *(Almost all reduced matrices are in $\mathbb{S}$). Let* $M = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$. *Then* $\mathcal{R}\backslash\{M\} \subset \mathbb{S}$.

Proof of Proposition E.1: The spectrum of a real spectrum reduced matrix M can't be included in $\mathbb{R}_{<0}$, because in this case its trace would be negative, which is not possible according to the definition of a reduced matrix. Moreover if a reduced matrix M has a Spectrum equal to $\{1\}$, then a computation gives $M = [[0,1][-1,2]]$. Thus except this matrix M, every reduced matrix has a positive real spectrum different from $\{1\}$, and thus is in $\mathbb{S}$. □

Let's consider $\mathcal{R}'$ the set $\mathcal{R}\backslash\{M\}$. Then $\mathcal{R}' \subset \mathbb{S}$. We denote $L_{1,1}$ the LLS of the matrix M defined in Proposition E.1.

Proposition E.2. *The set of conjugacy classes of $\mathbb{S}$ by $\operatorname{GL}(2,\mathbb{Z})$ is in one-to-one correspondence with $(\mathcal{N}/\equiv)\backslash[L_{1,1}] = (\mathcal{N}\backslash L_{1,1})/\equiv$.*

Proof of Proposition E.2: Let's first show that $(\mathcal{N}/\equiv)\backslash[L_{1,1}] = (\mathcal{N}\backslash L_{1,1})/\equiv$. This is equivalent to show that $[L_{1,1}] = \{L_{1,1}\}$. If there exists $L \in [L_{1,1}]\backslash\{L_{1,1}\}$, then according to Corollary 7.16 of [5] there exists a reduced matrix A with LLS sequence equal to L. Since $L \neq L_{1,1}$, $A \neq \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ and according to Proposition E.1, $A \in \mathbb{S}$. But then, A and $M = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ do not have the same spectrum, but their LLS period coïncide, which is impossible according to Proposition 7.11. Thus $(\mathcal{N}/\equiv)\backslash[L_{1,1}] = (\mathcal{N}\backslash L_{1,1})/\equiv$.

We know from Corollary 7.16 of [5] that there is a one-to-one correspondence between $\mathcal{R}$ and

$\mathcal{N}$, which induces a one-to-one correspondence between $\mathcal{R}\backslash\{M\}$ and $\mathcal{N}\backslash\{L_{1,1}\}$. Let's denote ϕ this one-to-one correspondence, and define $\psi : \mathbb{S}/\operatorname{GL}(2,\mathbb{Z}) \to (\mathcal{N}\backslash L_{1,1})/\equiv$ by $\psi([B]) = [a]$, where the brackets denote the projections on the quotient sets and a is the LLS period of a reduced matrix conjugate to B. This map is well-defined, since if two matrices are conjugate their LLS periods are in the same equivalence class in $\mathcal{N}/\equiv$, and since there exists a reduced matrix in every conjugacy class of $\mathbb{S}$. This map is a one-to-one correspondence because it's injective (for a given element of $(\mathcal{N}\ [L_{1,1}])/\equiv$, the reduced matrices having their LLS period in this equivalence class are conjugate) and surjective (for a given element of $[a] \in (\mathcal{N}\ [L_{1,1}])/\equiv$, according to Corollary 7.16 there exists a reduced matrix having its LLS period in $[a]$, and this matrix is in $\mathcal{R}\backslash\{M\} \in \mathbb{S}$ according to Proposition E.1). $\square$

What is interesting with Section E is that we get a description of $\mathbb{S}/\operatorname{GL}(2,\mathbb{Z})$, and not only of $\mathbb{S}_T/\operatorname{GL}(2,\mathbb{Z})$. However, the set we get $\mathcal{N}/\equiv\ [L_{1,1}]$ is not simple to study. Moreover we have to take into account the fact that we want to study $\mathcal{S}_T$ and not only $\mathbb{S}_T/\operatorname{GL}(2,\mathbb{Z})$. It turns out the point of view developped in Section 7 fits better and gives us more results.