Prophet Inequalities

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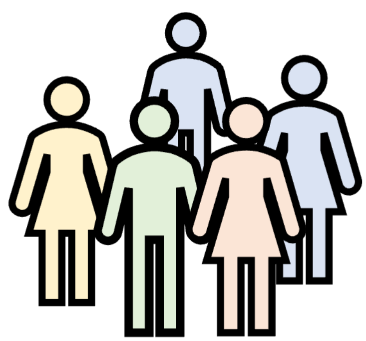
Motivation

Online platforms, e-commerce, etc

Flexible Model:

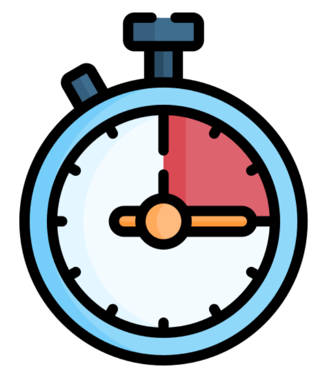
Multiple Goals

Incentives



Limited data

Sequential decisions



Course Overview

1. Classic single-choice problems:

    The classic prophet inequality, secretary problem, prophet secretary problem, etc

2. Data driven prophet inequalities:

    How can limited amount of data be nearly as useful as full distributional knowledge

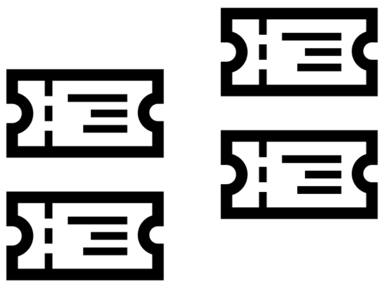
3. **Combinatorial Prophet Inequalities**

    Many ideas for single choice problems, extend to combinatorial contexts such as k-choice, Matching, hyper graph matching, and beyond

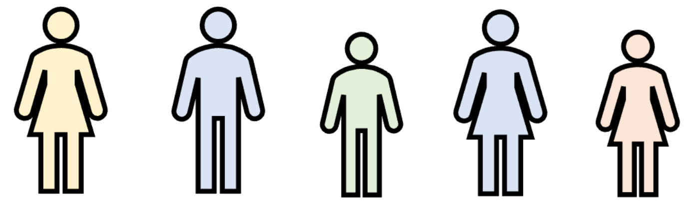
4. Online Combinatorial Auctions

    General Model that ecompasses many online selection/allocation problems

3. Combinatorial Prophet Inequalities



k identical tickets



Sequence of n agents with independent valuations (for one copy)
$v_1 \sim F_1, v_2 \sim F_2, \ldots, v_n \sim F_n$

We sell to at most k agents, we compare against the expectation of the sum of the largest k valuations

Static price policy

Set a price $T > 0$

Recall the proof for $k = 1$:

Exp. no. of sold items

Prob. item is available

Revenue

Utility

$$\mathbb{E}(ALG) \geq \underbrace{T \cdot \mathbb{P}(\max v_i \geq T)}_{\text{Revenue}} + \underbrace{\mathbb{P}(\max v_i < T) \cdot \sum_{i=1}^{n} \mathbb{E}(v_i - T)_+}_{\text{Utility}}$$

For $k > 1$, denote as $N(T)$ the number of items sold, i.e., $N(T) = |\{i: v_i \geq T\}|$.

Revenue = $T \cdot \mathbb{E}(N(T))$

Utility = $\sum_{i=1}^{n} \mathbb{E}\left((v_i - T)_+\right) \cdot P(\text{item available for } i)$

$$\geq \mathbb{P}(N(T) < k) \cdot \sum_{i=1}^{n} \mathbb{E}\left((v_i - T)_+\right)$$

$$\mathbb{E}(OPT) = \mathbb{E}\left(\sum_{i \in OPT} v_i\right) = k \cdot T + \mathbb{E}\left(\sum_{i \in OPT} (v_i - T)\right) \leq k \cdot T + \mathbb{E}\left(\sum_{i=1}^{n} (v_i - T)_+\right)$$

Then,

$$\frac{\mathbb{E}(ALG)}{\mathbb{E}(OPT)} \geq \frac{\mathbb{E}\big(N(T)\big) \cdot T + \mathbb{P}(N(T) < k) \cdot \mathbb{E}(\sum_{i=1}^{n} (v_i - T)_+)}{k \cdot T + \mathbb{E}\big(\sum_{i=1}^{n} (v_i - T)_+\big)}$$

$$\geq \min\left\{\frac{\mathbb{E}\big(N(T)\big)}{k}, \mathbb{P}(N(T) < k)\right\}$$

If we set T such that $\mathbb{E}\big(N(T)\big) = k - \sqrt{2k\log k}$

Then, a Chernoff bound implies that

$$\mathbb{P}(N(T) \geq k) \leq O\left(\frac{1}{k}\right)$$

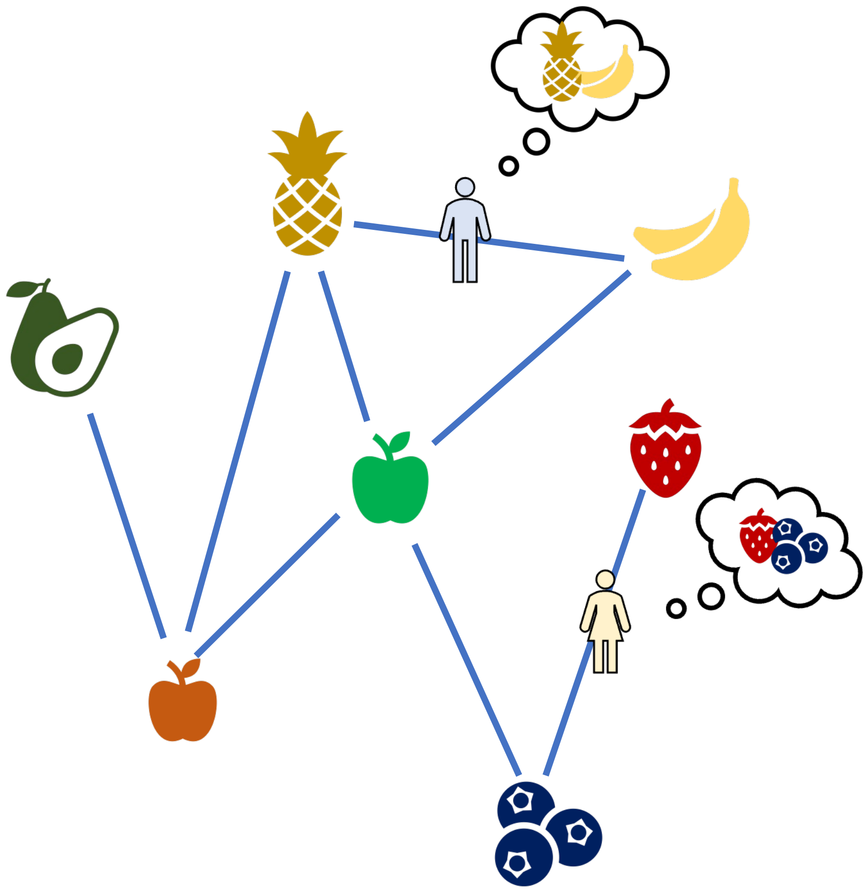
Therefore,

$$\min\left\{\frac{\mathbb{E}\big(N(T)\big)}{k}, \mathbb{P}(N(T) < k)\right\} \geq \min\left\{\frac{k-\sqrt{2k\log k}}{k}, 1 - O\left(\frac{1}{k}\right)\right\} \geq 1 - O\left(\sqrt{\frac{\log k}{k}}\right)$$

The best possible guarantee with single price when $\frac{\mathbb{E}(N(T))}{k} = \mathbb{P}(N(T) < k)$ (asymptotically it's the same bound) [Chawla, Devanur, Lykouris WINE'21] [Jiang, Ma, Zhang, 23+]

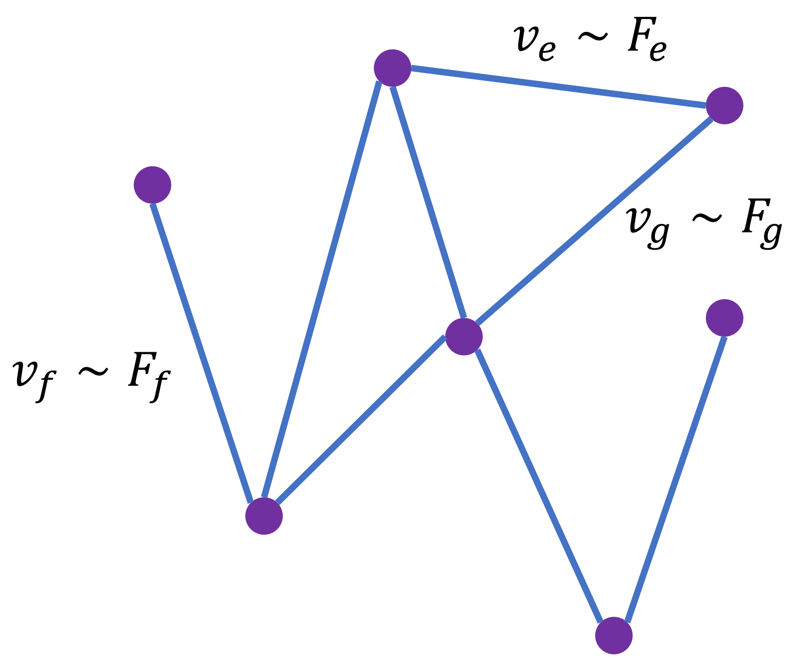
With multiple prices we can get $1 - O\left(\frac{1}{\sqrt{k}}\right)$. [Alaei FOCS'11] [Jiang, Ma, Zhang SODA'22]

Matching Prophet Inequality



Natural to think of price-based algorithms

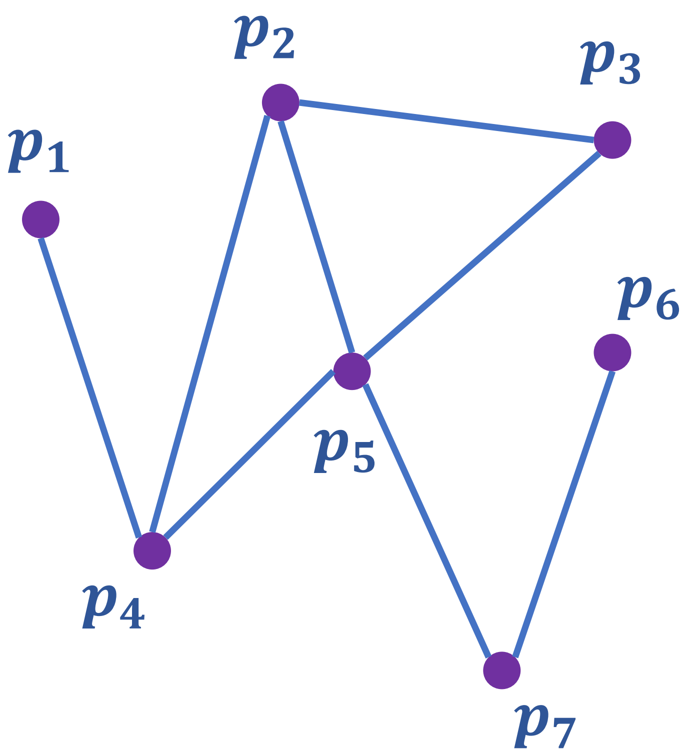
Matching Prophet Inequality



Independent edge weights come **one by one** in an arbitrary fixed order

Select **matching** on the fly

Maximize expectation



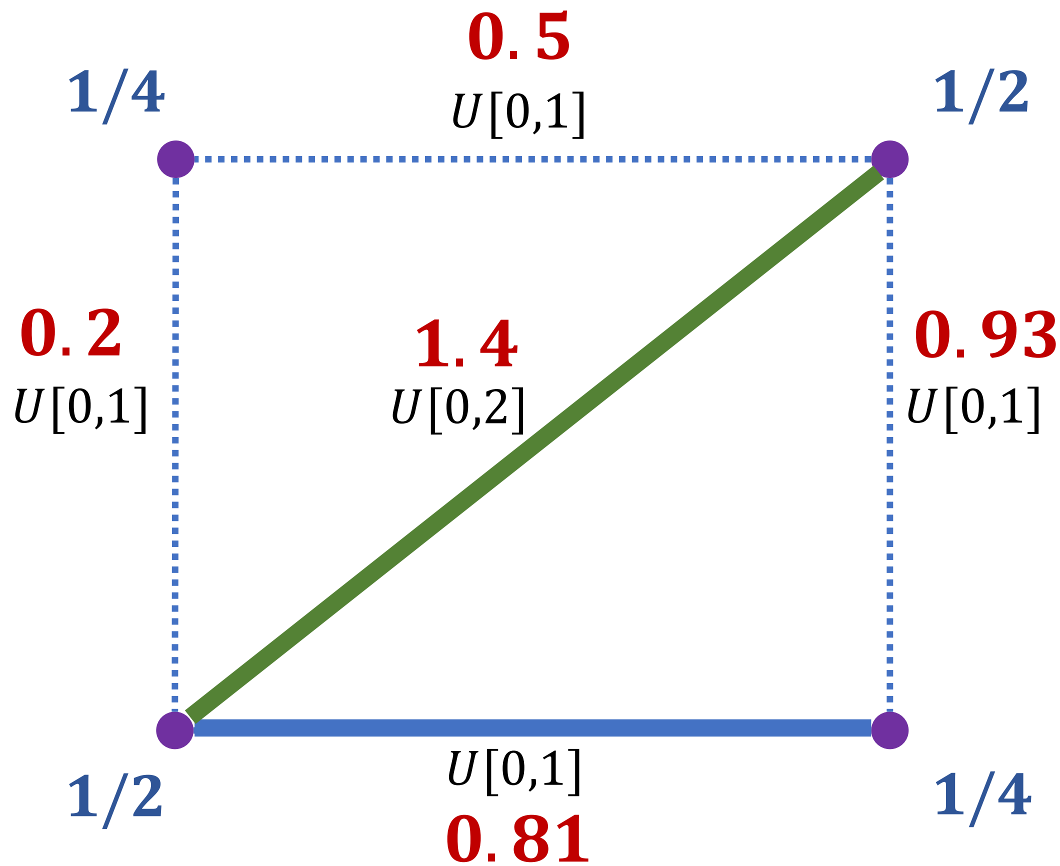
Algorithm:

$e = (u, w)$ arrives:

e buys u and w as long as they are not sold yet and $v_e \geq \boldsymbol{p_u} + \boldsymbol{p_w}$

$\boldsymbol{ALG(p)}$ resulting matching

$\boldsymbol{OPT}$ optimal matching


0.5
U[0,1]
1/4
1/2
0.2
U[0,1]
1.4
U[0,2]
0.93
U[0,1]
U[0,1]
1/2
1/4
0.81

Theorem. [Gravin and Wang, EC'19][Correa, Cristi, Fielbaum, Pollner, Weinberg, IPCO'22]
There is a vector of prices $\boldsymbol{p} \in \mathbb{R}_+^V$ s.t. for any arrival order,

$$\mathbb{E}\big(\boldsymbol{ALG}(\boldsymbol{p})\big) \geq \frac{1}{3} \cdot \mathbb{E}\left(\boldsymbol{OPT}\right)$$

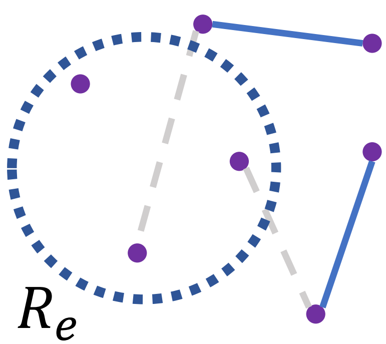
$$\mathbb{E}\left(\boldsymbol{ALG}(\boldsymbol{p})\right) = \text{revenue} + \text{utility}$$

$$= \mathbb{E}\left(\sum_{u \in V\left(ALG(p)\right)} p_u\right) + \mathbb{E}\left(\sum_{e \in ALG(p)} (v_e - p_u - p_w)\right)$$

We want balanced prices:
"high enough" so we get good revenue, yet "low enough" so buyers buy (and get good utility)

To lower bound $\mathbb{E}(ALG(p))$, utility is the tricky part:

$$\mathbb{E}\left(\sum_{e\in ALG(p)}(v_e - p_u - p_w)\right) = \sum_{e\in \mathrm{E}} \mathbb{E}\left(I_{\{e\in ALG(p)\}} \cdot (v_e - p_u - p_w)\right)$$



Recall that $\boldsymbol{ALG(p)}$ takes $e = (u, w)$ iff

- the two nodes are free, and
- $v_e \geq p_u + p_w$

R_e = set of remaining vertices when e arrives

$\boldsymbol{R_e}$ is independent of $\boldsymbol{v_e}$

$$
\begin{aligned}
\text{Utility} &= \sum_{e=(u,w)\in \mathrm{E}} \mathbb{E}\left(I_{\{u,w\in R_e\}} \cdot [v_e - p_u - p_w]_+\right) \\
&= \sum_{e=(u,w)\in E} \mathbb{P}(u,w \in R_e) \cdot \mathbb{E}([v_e - p_u - p_w]_+) \\
&\geq \sum_{e=(u,v)\in E} \mathbb{P}\left(u,w \notin V\left(ALG(p)\right)\right) \cdot \boldsymbol{z}_e(\boldsymbol{p}) \\
&= \mathbb{E}\left(\sum_{u,w\notin V\left(ALG(p)\right)} \boldsymbol{z}_e(\boldsymbol{p})\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\big(\boldsymbol{ALG(p)}\big) &= \text{revenue} + \text{utility} \\
&= \mathbb{E}\left(\sum_{u \in V(ALG(p))} p_u\right) + \mathbb{E}\left(\sum_{e \in ALG(p)} (v_e - p_u - p_w)\right) \\
&\geq \mathbb{E}\left(\sum_{u \in V(ALG(p))} p_u\right) + \mathbb{E}\left(\sum_{e=(u,w):u,w \notin V(ALG(p))} \boldsymbol{z}_e(p)\right) \\
&\geq \min_{X \subseteq V}\left\{\sum_{u \notin X} p_u + \sum_{e \in E(X)} \boldsymbol{z}_e(p)\right\}
\end{aligned}
$$

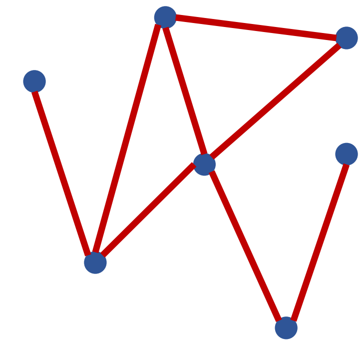
To bound $\boldsymbol{OPT}$, imagine that edges in $\boldsymbol{OPT}$ had to pay the prices

$$\mathbb{E}(\boldsymbol{OPT}) = \mathbb{E}\left(\sum_{u \in V(OPT)} p_u + \sum_{e \in OPT} (v_e - p_u - p_w) \right)$$

$$\leq \sum_{u \in V} p_u + \sum_{e \in E} \mathbb{E}([v_e - p_u - p_w\,]_+)$$

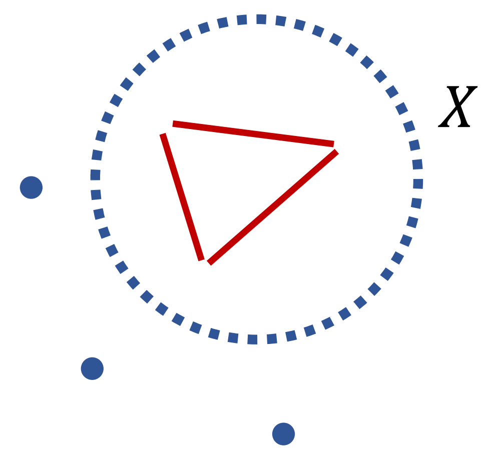
$$:= \sum_{u \in V} p_u + \sum_{e \in E} \boldsymbol{z}_e(\boldsymbol{p})$$

$$\mathbb{E}(\boldsymbol{OPT}) \leq \sum_{u \in V} p_u + \sum_{e \in E} \boldsymbol{z}_e(\boldsymbol{p})$$

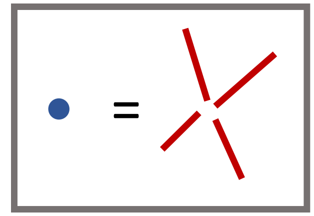


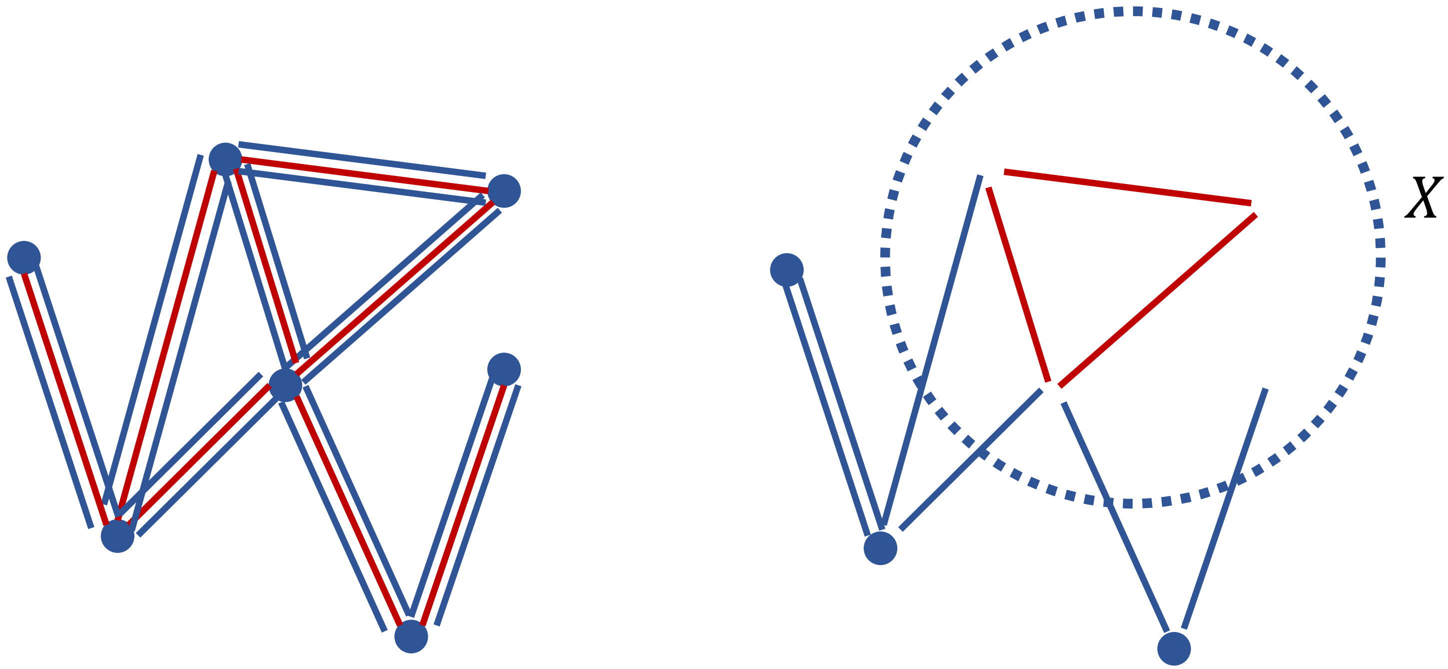
vs.

$$\mathbb{E}\big(\boldsymbol{ALG}(\boldsymbol{p})\big) \geq \min_{X \subseteq V} \left\{ \sum_{u \notin X} p_u + \sum_{e \in E(X)} \boldsymbol{z}_e(\boldsymbol{p}) \right\}$$



What if we set prices $p_u = \sum_{e \in \delta(u)} \boldsymbol{z}_e(\boldsymbol{p})$





We want prices

$$p_u = \sum_{e \in \delta(u)} \boldsymbol{z}_e(\boldsymbol{p})$$

Define the operator: $\psi_u(\boldsymbol{p}) = \sum_{e \in \delta(u)} \boldsymbol{z}_e(\boldsymbol{p})$

Brouwer's fixed-point theorem: if ψ is a continuous mapping from a compact and convex set into itself, then it has a fixed point.

Recall that $\boldsymbol{z}_e(\boldsymbol{p}) = \mathbb{E}([v_e - p_u - p_w]_+) \in [0, \mathbb{E}(v_e)]$

$\Rightarrow$ there are prices $\boldsymbol{p} = \psi(\boldsymbol{p})$

$\blacksquare$

Can we compute $\boldsymbol{p}$? Brouwer's only guarantees existence.

Theorem. For $\varepsilon > 0$, we can compute $\boldsymbol{p}$ in polynomial time s.t.

$$(3+\varepsilon)\cdot\mathbb{E}\big(\boldsymbol{ALG}(\boldsymbol{p})\big) \geq \mathbb{E}(\boldsymbol{OPT})$$

For $\varepsilon > 0$, m edges, n nodes and a bound $B \geq \frac{v_{\max}}{\mathbb{E}(OPT)}$, we can compute $\boldsymbol{p}$ in **time** $poly\left(m, n, \frac{1}{\varepsilon}, B\right)$, using $poly\left(m, n, \frac{1}{\varepsilon}, B\right)$ **samples.**

Sample:

$$\left(v_e^{(s)}\right)_{e\in E}$$

Calculate "empirical" ψ:

$$\hat{\psi}_u^{(s)}(\boldsymbol{p}) = \sum_{e\in\delta(u)} \left[v_e^{(s)} - p_u - p_w\right]_+$$

Repeat for $s = 1, \dots, S$

Average

$$\hat{\psi} = \frac{1}{S}\sum_{s=1}^{S} \hat{\psi}^{(s)}$$

For large S, $\hat{\psi}$ is good approx (concentration bound)

Find

$$\boldsymbol{p} = \hat{\psi}(\boldsymbol{p})$$

convex QP

We want

$$p_u = \frac{1}{S}\sum_{s=1}^{S}\sum_{e\in\delta(u)} \overbrace{\left[v_e^{(s)} - p_u - p_w\right]_+}^{y_{e,s}}, \qquad \text{for all } u \in V$$

convex QP

$$\left\{ \begin{array}{l} \min \sum_{e,s} y_{e,s}\cdot\left(y_{e,s} - \left(v_e^{(s)} - \frac{1}{S}\sum_{s'}\sum_{e'\in\delta(u)\dot\cup\delta(w)} y_{e',s'}\right)\right)\\ s.t.\\ y_{e,s}\geq 0\\ y_{e,s} \geq \left(v_e^{(s)} - \frac{1}{S}\sum_{s'}\sum_{e'\in\delta(u)\dot\cup\delta(w)} y_{e',s'}\right) \end{array}\right.$$

“■”

Hypergraph matching

A hypergraph is a pair (V, E), where $E \subseteq 2^V$

The previous analysis can be extended to hypergraphs

Theorem. [Correa, Cristi, Fielbaum, Pollner, Weinberg, IPCO'22]
If $|e| \leq \boldsymbol{d}$ for all $e \in E$, there is a vector of prices $\boldsymbol{p} \in \mathbb{R}_+^V$ s.t. for any arrival order,

$$\mathbb{E}\big(\boldsymbol{ALG(p)}\big) \geq \frac{1}{\boldsymbol{d}+1} \cdot \mathbb{E}\left(\boldsymbol{OPT}\right)$$

Taking $\boldsymbol{z}_e(\boldsymbol{p}) = \mathbb{E}\left((v_e - \sum_{u \in e} p_u)_+\right)$

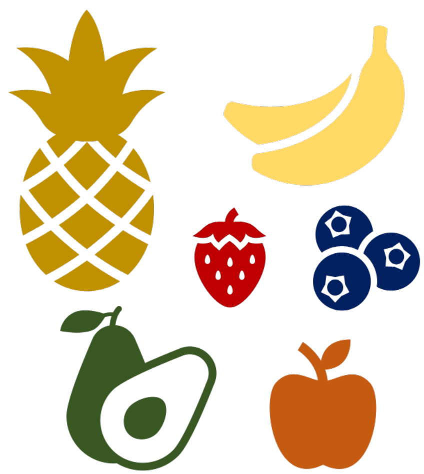
$$\mathbb{E}(\boldsymbol{OPT}) = \mathbb{E}\left(\sum_{u \in V(OPT)} p_u + \sum_{e \in OPT} \left(v_e - \sum_{u \in e} p_u \right) \right)$$
$$\leq \sum_{u \in V} p_u + \sum_{e \in E} \boldsymbol{z}_e(\boldsymbol{p})$$

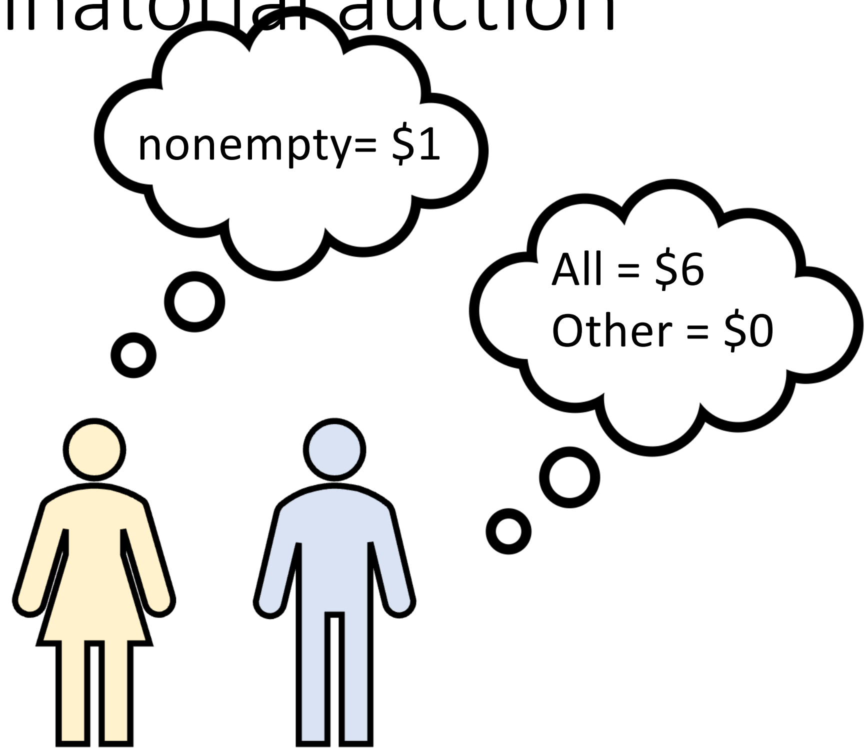
$$\mathbb{E}\left(\boldsymbol{ALG}(\boldsymbol{p})\right) \geq \min_{X \subseteq V} \left\{ \sum_{u \in X} p_u + \sum_{e : e \cap X = \emptyset} \boldsymbol{z}_e(\boldsymbol{p}) \right\}$$

If $p_u = \sum_{e : u \in e} \boldsymbol{z}_e(\boldsymbol{p})$

$$\mathbb{E}(\boldsymbol{OPT}) \leq (d+1) \cdot \sum_{e \in E} \boldsymbol{z}_e(\boldsymbol{p}) \leq (d+1) \cdot \mathbb{E}\left(\boldsymbol{ALG}(\boldsymbol{p})\right)$$

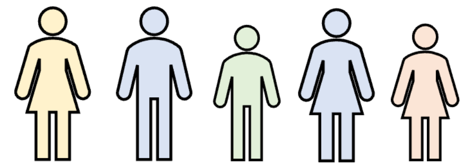
More generally: Combinatorial auction





$v_i: 2^M \to \mathbb{R}_+$

set of items M



buyers come in adversarial order

independent valuations

$v_i \sim F_i$

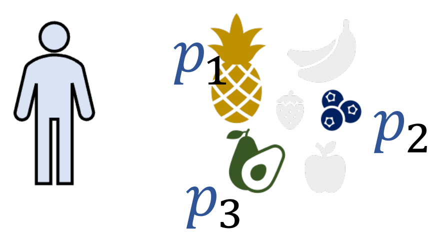
$v_i : 2^M \to \mathbb{R}_+$

parameter $\boldsymbol{d}$

$$v_i(A) = \max_{\{B \subseteq A, |B| \leq d\}} v_i(B)$$

$\boldsymbol{OPT}$ = welfare of optimal allocation

$\boldsymbol{ALG(p)}$ = welfare of resulting allocation



buyers maximize utility

$$\max_{A \subseteq R_i} \left\{ v_i(A) - \sum_{j \in A} p_j \right\}$$

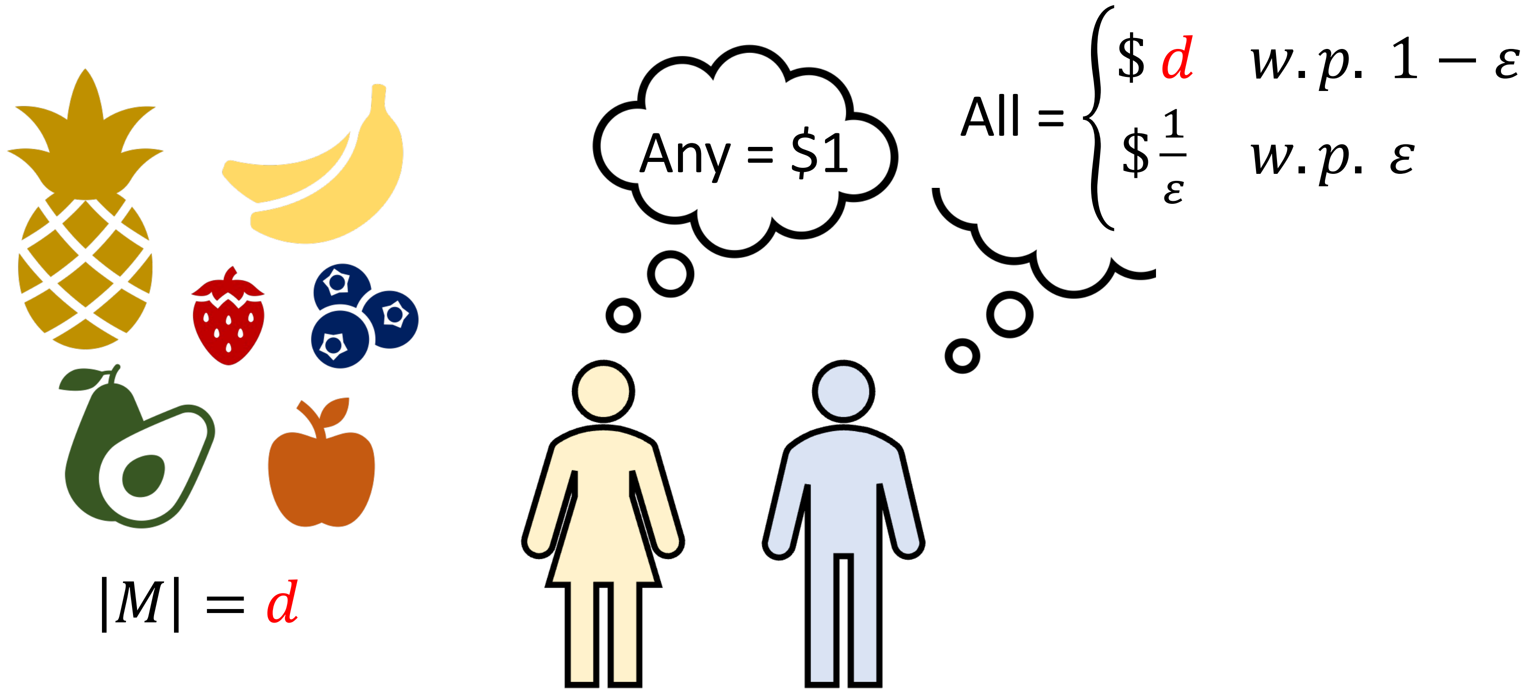
Theorem. There is a vector of prices $\boldsymbol{p} \in \mathbb{R}_+^M$ s.t. for any arrival order,

$$(d + 1) \cdot \mathbb{E}\big(\boldsymbol{ALG}(\boldsymbol{p})\big) \geq \mathbb{E}\ (\boldsymbol{OPT}).$$

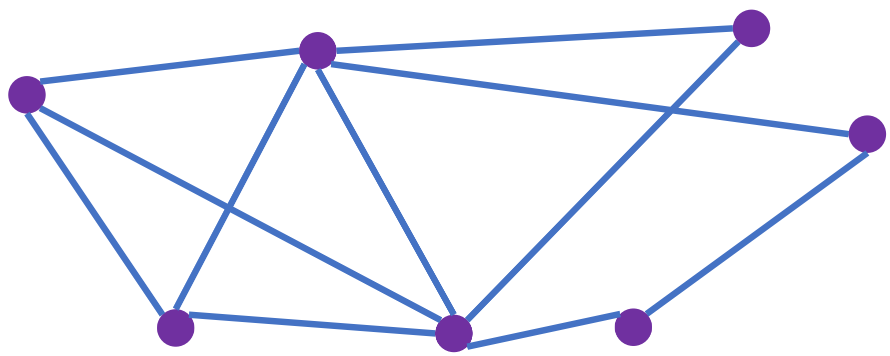
The bound $(d + 1)$ is best possible.

Theorem. These prices can be computed in polynomial time (even for non-constant d).

Tight instance



Matching with vertex arrival



$$(v_{uw})_{w<u} \sim F_u$$

Step: new vertex arrives, together with adjacent edges connected to previous vertices

Weights are independent across steps (but might be correlated within a step)

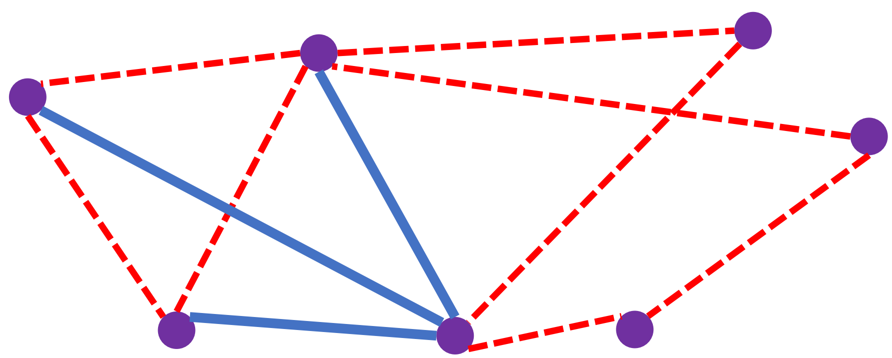
Select **matching** on the fly

Maximize expectation

[Ezra, Feldman, Gravin, Tang, EC 2020]

In each step: should we match u now? to which vertex? We don't know if there will be better edges later.

Idea: "sample" OPT



$(v_{uw})_{w \prec u}$

Sample fresh weights for all other edges

$$(v_e^u)_{e \neq (w \prec u)}$$

Let

$$OPT^u = OPT\left((v_{uw})_{w \prec u}, (v_e^u)_{e \neq (w \prec u)}\right)$$

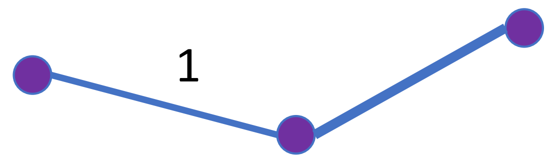
be the optimal solution with these weights.

ALG: try to match u according to OPT^u

Imagine at every vertex u we succeeded with probability β, independently of $(v_{uw})_{w \prec u}$ and OPT^u. Then,

$$\begin{aligned}
\mathbb{E}(ALG) &= \sum_u \beta \cdot \mathbb{E}\left(\sum_{w \prec u} v_{uw} \cdot 1_{\{uw \in OPT^u\}} \right) \\
&= \sum_u \beta \cdot \mathbb{E}\left(\sum_{w \prec u} v_{uw} \cdot 1_{\{uw \in OPT\}} \right) \\
&= \beta \cdot \mathbb{E}\left(\sum_{uw \in OPT} v_{uw} \right) \\
&= \beta \cdot \mathbb{E}(OPT)
\end{aligned}$$

Issue: some edges can be in OPT very often, but carry very little value



$$\begin{cases} \frac{100}{\varepsilon}, & w.p.\ \ \varepsilon \\ 0, & w.p.\ 1-\varepsilon \end{cases}$$

Solution: downplay the decision of OPT^u a bit. When u arrives and we want to match it to w, we toss an independent coin with bias $\alpha_w(u)$

Let $x_{uw} = \mathbb{P}(uw \in OPT)$. We take $\alpha_w(u) = \frac{1}{2\cdot\left(1-\frac{1}{2}\cdot\sum_{z<u} x_{wz}\right)}$ (this is in [0,1]... **Why**?)

This guarantees that we always succeed w.p. at least $1/2$, so $\mathbb{E}(ALG) \geq \mathbb{E}(OPT)$

We take $\alpha_w(u) = \frac{1}{2\cdot\left(1-\frac{1}{2}\cdot\sum_{z\prec u} x_{wz}\right)}$

We prove inductively that $\mathbb{P}(uw \in ALG) = \frac{x_{uw}}{2}$

Assume it's true for all edges (wz), with $w, z \prec u$.

$$
\begin{aligned}
&\mathbb{P}(uw \in ALG) \\
&= \mathbb{P}(w \text{ is free when } u \text{ arrives}) \cdot \mathbb{P}(uw \in OPT^u) \cdot \alpha_w(u) \\
&= \left(1 - \sum_{z\prec u} \frac{x_{wz}}{2}\right) \cdot x_{uw} \cdot \frac{1}{2\cdot\left(1-\frac{1}{2}\cdot\sum_{z\prec u} x_{wz}\right)} \\
&= \frac{x_{uw}}{2}
\end{aligned}
$$

We repeat the argument to conclude:

Upon the arrival of u, imagine $(uw) \in OPT^u$. What is the probability we select it?

$$\begin{aligned}
&\mathbb{P}(w \text{ is free when } u \text{ arrives}) \cdot \alpha_w(u) \\
&= \left(1 - \sum_{z \prec u} \frac{x_{wz}}{2}\right) \cdot \frac{1}{2 \cdot \left(1 - \frac{1}{2} \cdot \sum_{z \prec u} x_{wz}\right)} \\
&= \frac{1}{2}
\end{aligned}$$

Summary

- Best possible PI for selecting k items gets $1 - O\left(\frac{1}{\sqrt{k}}\right)$ [Alaei FOCS 2011]

  → For fixed threshold is degrades to $1 - O\left(\sqrt{\frac{\log k}{k}}\right)$ [Chawla, Devanur, Lykouris WINE 2021]

  → For prophet secretary best possible fixed threshold gives a $1 - O\left(\frac{1}{\sqrt{k}}\right)$ [Arnosti, Ma EC 2022]

- Best possible **prices** for online d -hypergraph matching (or combinatorial auctions with random valuation parametrized by d).

  → 1/3 for bipartite matching [Gravin and Wang, EC'19]
  → Best possible factor is $(1/(d+1)$] [C., Cristi, Fielbaum, Pollner, Weinberg, IPCO 2022]
  → Improves upong $(4d-2)$ [Dütting, Feldman, Kesselheim, Lucier, FOCS'20]
  → For matching ($d = 2$) a 2.96-approx. is possible using adaptive prices [Ezra, Feldman, Gravin, Tang, EC 2020]

- For matching with vertex arrivals ½ is best possible [Ezra, Feldman, Gravin, Tang, EC 2020]