Topological rigidity of manifolds and the Farrell—Jones Conjecture

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1 K_1 and h-cobordisms

Recall the projective class group/Grothendieck group of a ring

$$\mathrm{K}_0(R) = \left(\bigoplus_{P \text{ f.g. proj } R\text{-module}} \mathbb{Z} < P > \right) / \text{short exact sequence.}$$

Definition 1.1. Let R be a (not necessarily commutative) ring. Let $\mathrm{GL}(R) = \bigcup_n \mathrm{GL}_n(R)$ (as a group, where each one is embedded into the other by completing with an identity block). We define

$$\mathrm{K}_1(R) := \mathrm{GL}(R)^{\mathrm{ab}} = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)] \simeq \mathrm{GL}(R)/\{\text{elementary matrices}\}.$$

Definition 1.2. Let G be a group. Consider the homomorphism

$$\begin{aligned} f : G \times \{\pm 1\} &\to \mathrm{K}_1(\mathbb{Z}[G]) \\ (g, \pm 1) &\mapsto \pm g \in \mathrm{GL}(\mathbb{Z}[G]) \end{aligned}$$

The Whitehead group of G is

$$\mathrm{Wh}(G) = \mathrm{coker}(f) = \mathrm{GL}(\mathbb{Z}[G])/(G \times \{\pm 1\} \times \{\text{elementary matrices}\}).$$

Definition 1.3. Let M be a topological n-manifold. An h-cobordism over M is an $n+1$-manifold W with boundary such that

- $\partial W \simeq M \coprod N$ where N is an n-manifold
- $M \hookrightarrow W, N \hookrightarrow W$ are homotopy equivalences.

We denote

$$\mathcal{H}(M) = \{\text{h-cobordisms over } M\}/\text{homeomorphisms rel } M.$$

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Theorem 1.4 (s-cobordism)**.** *If M is a closed manifold of dimension ≥ 5, there is a bijection*

$$\mathcal{H}(M) \leftrightarrow \mathrm{Wh}(\pi_1(M)).$$

Corollary 1.5. *Let G be a discrete group. If $\mathrm{Wh}(G) = 0$, then for every closed manifold M of dimension ≥ 5 and $\pi_1(M) = G$, every h-cobordism over M is trivial.*

2 The Borel Conjecture

Definition 2.1. A closed manifold N is called topologically rigid if any homotopy equivalence $M \to N$ with a closed manifold M as source is homotopic to a homeomorphism.

COUNTEREXAMPLE IN NONCLOSED CASES: ANNULUS AND CIRCLE

Definition 2.2. A topological space X is aspherical if $\pi_i(X) = 0$ for all $i \neq 0$.

Conjecture 2.3 (Borel Conjecture for G)**.** *Let G be a discrete finitely presented torsionfree group. Any closed aspherical manifold M with $\pi_1(M) \simeq G$ is topologically rigid.*

The conjecture is true in dimension 1 and 2 by classification, in dimension 3 as a consequence of Perelman's proof of Thurston's Geometrization Conjectures, and open in dimension 4.

It is also open dimension ≥ 5, but proven for many interesting groups by using a K-theoretic criterion, in the same spirit as the vanishing of $\mathrm{Wh}(\pi_1(M))$ implies cobordism-rigidity.

3 K-theory and the Farrell-Jones conjecture

Recall 3.1. There exists a whole algebraic K-theory spectrum of a ring R, which we denote by $\mathrm{K}_*(R)$.

$$\pi_i(K_*(R)) = K_i(R)$$

(this is a theorem for $0, 1$ and a definition for ≥ 2).

There exists also something called L-theory, which we will skip. Instead of parametrizing modules up to short exact sequence, automorphisms, etc., it parametrizes modules with quadratic/symmetric forms up to Lagrangian, etc.

Conjecture 3.2 (K-theoretic Farrell–Jones, simplified)**.** *Slogan: "K-theory of integral group rings is a homology theory!"*

Let G be a finitely presented torsionfree group. Then the ***assembly map***

$$\mathrm{asmb}_{\mathrm{K}} : \Sigma^\infty_+ \mathrm{B}G \wedge \mathrm{K}_*(\mathbb{Z}) \to \mathrm{K}_*(\mathbb{Z}[G])$$

is an equivalence/isomorphism of spectra.

Note: the assembly map exists, the conjecture only says that it is an equivalence.

Theorem 3.3 (Farrell–Jones)**.** *The K-theoretic Farrell–Jones Conjecture, combined with the analogous statement for L-theory, implies the Borel Conjecture in* $\dim \geq 5$*.*

The Farrell–Jones Conjecture has been proven in various cases in works by Lück, Bartels, Reich, et al.: true for

- hyperbolic groups
- finite-dimensional CAT(0) groups
- virtually solvable groups, etc.
- fundamental groups of connected manifolds of dimension ≤ 3
- certain lattices
- $\mathrm{GL}_n(\mathbb{Q}), \mathrm{GL}_n(\mathbb{F}_p(t))$
- $\Gamma^s_{g,r}$, the mapping class group of a closed orientable surface of genus g, r boundary components and s punctures

Also, they show that the class for which the Conjecture is true is closed under:

- subgroups
- overgroups of fte index
- finite direct products,
- directed colimits
- certain extensions
- all free products.

The fact that it is closed under directed colimits makes it harder to build counterexamples.

4 Categorification

K_* is an invariant of categories:

$$R \rightsquigarrow \mathrm{Proj}_R \rightsquigarrow \mathrm{K}(\mathrm{Proj}_R) \simeq \mathrm{K}_*(R)$$

but this can be generalized to stable ∞-categories, that is:

Theorem 4.1. *There is a functor* $\mathrm{K} : \{\text{stable } \infty\text{-categories}\} \to \{\text{spectra}\}$ *s.t.* $\mathrm{K}(\mathrm{Perf}_R) = \mathrm{K}_*(R)$*, sending "fiber-cofiber sequences" to short exact sequences.*

Goal 4.2. Build a fiber-cofiber sequence of stable ∞-categories

$$\mathcal{N} \xrightarrow{\Phi} \mathcal{C} \xrightarrow{\Psi} \mathcal{D}$$

such that $\mathrm{K}(\Phi) = \mathrm{asmb}_K$. Give manageable conditions for which $\mathrm{K}(\mathcal{N}) = 0$ as a spectrum, and hence asmb_K is an equivalence.

The answer turns out to involve a more powerful invariant, namely continuous K-theory.
In work with Levin and Saunier, we achieved the goal for the L-theory version.